INTUITIONISM AND COMPUTING WITH PARTIAL INFORMATION

HRISTO GANCHEV, PAUL SHAFER, THEODORE A. SLAMAN, ANDREA SORBI,
AND MARIYA I. SOSKOVA

ABSTRACT. There exist initial segments of both the Dyment lattice and the Dyment–Muchnik lattice that yield Brouwer algebras modeling exactly the intuitionistic propositional calculus. For the Dyment–Muchnik lattice, this result is obtained by constructing a splitting class of enumeration degrees. In contrast, the full Dyment lattice and the full Dyment–Muchnik lattice model the intuitionistic propositional calculus plus the weak law of excluded middle. We also observe that certain naturally definable classes of enumeration degrees, which are downwards closed under enumeration reducibility, fail to form splitting classes.

1. Introduction

Mass problems were introduced by Medvedev in [13] to formalize Kolmogorov's proposal of a calculus of problems as a semantics for constructive logics. The central idea is to identify "problems" with mass problems, where a *mass problem* is a set of total functions from ω to ω (with ω denoting the set of natural numbers). The elements of a mass problem are viewed as its solutions. A mass problem is *solvable* if it contains a computable element, i.e. if we can compute a solution to the corresponding problem.

Given two mass problems $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ is *Medvedev reducible* to $\mathcal{B}$ (written $\mathcal{A} \leqslant \mathcal{B}$) if there exists an oracle Turing machine that computes an element of $\mathcal{A}$ whenever it is supplied with an element of $\mathcal{B}$ as an oracle. Intuitively, this implies that we can solve $\mathcal{A}$ if we can solve $\mathcal{B}$. Under this reducibility, the solvable mass problems are the "easiest possible," since they reduce to every other mass problem.

Following Kolmogorov's idea of interpreting logical connectives as operations on problems, observe that we can solve $\mathcal{A}$ *or* $\mathcal{B}$ if and only if we can solve

$$\mathcal{A} \wedge \mathcal{B} = 0^\frown\mathcal{A} \cup 1^\frown\mathcal{B},$$

where $i^\frown\mathcal{C} = \{i^\frown f : f \in \mathcal{C}\}$ and $i^\frown f$ is the function defined by $i^\frown f(0) = i$ and $i^\frown f(x+1) = f(x)$. Likewise, we can solve $\mathcal{A}$ *and* $\mathcal{B}$ if and only if we can solve

$$\mathcal{A} \oplus \mathcal{B} = \{f \oplus g : f \in \mathcal{A},\ g \in \mathcal{B}\},$$

where $f \oplus g$ is defined by $f \oplus g(2x) = f(x)$ and $f \oplus g(2x+1) = g(x)$. One can further define

$$\mathcal{A} \to \mathcal{B} = \{e^\frown f : (\forall g \in \mathcal{A})\,[\,\Phi_e(f \oplus g)\downarrow\in \mathcal{B}\,]\},$$

2010 *Mathematics Subject Classification.* 03B20, 03D30, 03G10, 06D20.

Key words and phrases. Intuitionistic propositional logic, Dyment lattice, Dyment-Muchnik lattice, splitting class.
The second author was supported by EPSRC Overseas Travel Grant EP/R006458/1.
The fourth author is a member of INDAM-GNSAGA.
The fifth author was partially supported by NSF grant DMS-2451099.

where Φ_e is the e-th oracle Turing machine, and $\Phi_e(f \oplus g)\downarrow$ means that $\Phi_e(f \oplus g)$ is a total function. If this mass problem is solvable, then informally the statement "if $\mathcal{A}$ is solvable, then $\mathcal{B}$ is solvable" holds. Finally, one interprets the connective $\neg$ by

$$\neg\mathcal{A} = \mathcal{A} \to \varnothing.$$

Muchnik later proposed in [14] a weakening of Medvedev reducibility, called *weak reducibility*, defined by

$$\mathcal{A} \leqslant_w \mathcal{B} \quad \text{iff} \quad (\forall g \in \mathcal{B})(\exists f \in \mathcal{A})[f \leqslant_T g],$$

where $\leqslant_T$ denotes Turing reducibility. This weaker reducibility induces the same interpretation of the propositional connectives $\wedge$ and $\vee$ on mass problems and the same notion of solvability. However, the Muchnik interpretation of $\to$ is

$$\mathcal{A} \to \mathcal{B} = \{f : (\forall g \in \mathcal{A})(\exists h \in \mathcal{B})\,[h \leqslant_T f \oplus g]\},$$

which need not be equivalent to the Medvedev interpretation. Under these interpretations, a propositional formula is *valid* in either the Medvedev or the Muchnik calculus of problems if, for every substitution of mass problems for propositional atoms, the resulting mass problem is solvable. It turns out that the valid formulas are precisely the theorems of the intermediate logic obtained by adding to the intuitionistic propositional calculus the weak law of excluded middle, i.e. the axiom scheme $\neg\alpha \vee \neg\neg\alpha$ ([13, 21]). This logic is known as the *Jankov logic*, or the *logic of the weak law of the excluded middle*. We will call JAN this logic after Jankov.

For many years it was open whether there exists a mass problem $\mathcal{C}$ such that the mass problems reducible to $\mathcal{C}$ model exactly the intuitionistic propositional logic. For the Medvedev lattice this question was answered positively in [18], and for the Muchnik lattice in [23]. Both results were later improved by Kuyper [9, 10], who explicitly exhibited initial segments arising from "natural" (or at least "more natural") mass problems. By contrast, the proofs of Skvortsova and of Sorbi–Terwijn rely at some point on the classical theorem of Lachlan and Lebeuf [11], which states that every countable upper semilattice with a least element is isomorphic to an initial segment of the Turing degrees.

In both Medvedev's and Muchnik's approaches the emphasis is on total functions as solutions to problems and on Turing computability as the means of reducing mass problems to one another. In this paper we instead adopt the perspective of computability relative to *partial* information. The most general and comprehensive framework for relative computability of partial functions is the model proposed by Myhill [15], according to which a partial function φ is computable relative to a partial function ψ (written $\varphi \leqslant_e \psi$) if there exists an effective enumeration procedure that enumerates the graph of φ from *any* enumeration of the graph of ψ. The precise meaning of "effective enumeration procedure" will be given later, when we introduce enumeration operators.

(An alternative and suggestive formulation of relative computability of partial functions—see, e.g., [12]—states that φ is computable by a nondeterministic partial-function-oracle Turing machine using ψ as oracle; such a machine always loops when it queries an undefined value of the oracle.)

Within this framework it is natural—following Dyment [5]—to identify informal problems with sets of partial functions from ω to ω. Reducibility of mass problems then refers to relative computability of partial functions, and the "easiest" problems (those reducible to every other) are precisely those containing partial computable functions.

1.1. **The results.** In Section 4 we show that there exists an initial segment of the Dyment lattice such that the propositional formulas which are valid in this initial segment are exactly the theorems of the intuitionistic propositional calculus. For this investigation, most of the machinery and of the ingredients used in the proof will turn out to be almost straightforward modifications of the ones employed for the Medvedev lattice in [10], [18], [21]. In addition to some intended simplifications, the most relevant exceptions are perhaps the proofs of Sublemma 1 and Sublemma 2 to Lemma 4.13, which seem to be the only cases where the Dyment lattice demands a proof of its own. In Section 5 we show that there exists an initial segment of the Dyment-Muchnik lattice modeling again exactly the intuitionistic propositional calculus. For this result, the known proof for the Muchnik lattice in [23] is of no help, since it relies on the Lachlan and Lebeuf result on the initial segments of the Turing degrees, which does not hold in the enumeration degrees. We do this by exhibiting a splitting class in the enumeration degrees, not an easy task in fact, since the known examples of splitting classes in the Turing degrees do not seem to be easily exportable to the context of enumeration reducibility, as exemplified in Subsection 5.3. It follows from these results (Section 6) that both the full Dyment lattice and the full Dyment-Muchnik lattice model the Jankov logic JAN.

2. Background and notations

In this section we fix some of the most used notions and notations.

2.1. **Computability theory.** The reader is referred to any standard textbook on computability theory (see, for instance, [3, 17, 19]) for the basic background. Recall that an *enumeration operator* (or *e-operator*) is a mapping $\Phi : 2^\omega \to 2^\omega$ for which there exists a computably enumerable (c.e.) set W (said to *define* Φ) such that, for $A \subseteq \omega$,

$$\Phi(A) = \{x : (\exists u)[\langle x, u\rangle \in W \ \&\ D_u \subseteq A]\},$$

where D_u is the finite set with canonical index u. (In the remainder of the paper we will often identify finite sets with their canonical indices, so we may write $\langle x, D\rangle \in W$ instead of $\langle x, u\rangle \in W$, where $D = D_u$.)

Given $A, B \subseteq \omega$, we say that A is *enumeration reducible* (or *e-reducible*) *to* B (notation: $A \leqslant_e B$) if there exists an enumeration operator Φ such that $A = \Phi(B)$. This reducibility induces in the usual way a degree structure, namely the structure $\mathcal{D}_e^{\mathrm{set}}$ of *e-degrees*. One may refer to the *standard indexing* $(\Phi_e : e \in \omega)$ of the enumeration operators, where Φ_e is the operator defined by the c.e. set W_e.

Let $\mathcal{P}$ denote the set of all partial functions from ω to ω. For $\psi \in \mathcal{P}$, an *enumeration of* ψ means an enumeration of graph(ψ). The definition of e-reducibility transfers in the obvious way from sets to partial functions by declaring that for partial functions φ and ψ, we have $\varphi \leqslant_e \psi$ if graph$(\varphi) \leqslant_e$ graph(ψ). The *partial degree* of φ is the equivalence class of φ under the equivalence relation $\equiv_e$ induced by $\leqslant_e$ on $\mathcal{P}$. The structure of the partial degrees will be denoted by $\mathcal{D}_e^{\mathrm{pf}}$.

If $\varphi, \psi \in \mathcal{P}$ and x is a number, then $\varphi(x) = \psi(x)$ means that either both $\varphi(x)$ and $\psi(x)$ are undefined, or they are both defined and yield the same value.

Any enumeration operator Φ induces a partial mapping $\Phi : \mathcal{P} \to \mathcal{P}$ as follows: for $\psi \in \mathcal{P}$, we say that $\Phi(\psi)\downarrow$ (or $\psi \in$ domain(Φ)) if $\Phi(\text{graph}(\psi))$ is single-valued, i.e. there exists $\chi \in \mathcal{P}$ such that $\Phi(\mathrm{graph}(\psi)) = \mathrm{graph}(\chi)$. In this case we write $\Phi(\psi)\downarrow= \chi$. If $\Phi(\psi)\downarrow$ and $x, y \in \omega$, then $\Phi(\psi)(x)\downarrow= y$ means that $\langle x, y\rangle \in \Phi(\psi)$, and $\Phi(\psi)(x)\downarrow$ means $(\exists z)\,[\langle x, z\rangle \in \Phi(\psi)]$.

2.2. **Lattices and Brouwer algebras.** Our main reference for lattice theory is [1]. A *Brouwer algebra* is a lattice $\mathscr{L} = \langle L, \vee, \wedge, 1, \leqslant\rangle$ (here, $\leqslant$ denotes the partial ordering of the lattice, defined by $a \leqslant b$ iff $a \wedge b = a$ iff $a \vee b = b$; the symbol $\geqslant$ denotes the dual relation of $\leqslant$) with a greatest element such that for every $a, b \in L$, the set $\{c : a \vee c \geqslant b\}$ has a least element (denoted by $a \to b$). An equivalent formulation is that the type of $\mathscr{L}$ can be enriched with a binary operation $a \to b$, such that for all $a, b, c \in L$, $a \vee c \geqslant b \Leftrightarrow a \to b \leqslant c$. The Brouwer algebras form an equational class. It is a routine exercise to verify that every Brouwer algebra has a least element as well, and that every Brouwer algebra is a distributive lattice. Hence we may refer to a Brouwer algebra as an algebra $\mathscr{L} = \langle L, \vee, \wedge, \to, 0, 1\rangle$ where $\langle L, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice, and $a \to b = \min\{c : a \vee c \geqslant b\}$. We will always assume that whether $\mathscr{L}$ is a Brouwer algebra or simply a lattice, it is nontrivial, i.e., $0 \neq 1$. Every finite distributive lattice is easily seen to be a Brouwer algebra. Finally, if $\mathscr{L} = \langle L, \vee, \wedge, \to, 0, 1\rangle$ is a Brouwer algebra and $a \neq 0$ then the interval $[0, a]_{\mathscr{L}} = \langle \{b : b \leqslant a\}, \vee, \wedge, \to, 0, a\rangle$ is a Brouwer algebra as well, with greatest element a, and with the operations obtained by restricting those of $\mathscr{L}$ to the interval.

2.3. **Propositional logics.** Brouwer algebras can be used for a semantics of propositional logics. A propositional formula α is an *identity* of a Brouwer algebra $\mathscr{L}$ if every substitution of elements of $\mathscr{L}$ for the propositional atoms of α always yields the least element 0 of the algebra, when the propositional connectives $\vee$, $\wedge$, and $\to$ are interpreted by the algebraic operations $\wedge$, $\vee$, and $\to$, respectively (note that the connective $\vee$ is interpreted as the meet operation and $\wedge$ as the join operation). Negation $\neg$ is interpreted by the operation $\neg a = a \to 1$, by which we can enrich the type of the Brouwer algebra. The *theory* $\mathrm{Th}(\mathscr{L})$ of a Brouwer algebra $\mathscr{L}$ is the set of its identities. It is easy to see that $\mathrm{IPC} \subseteq \mathrm{Th}(\mathscr{L}) \subseteq \mathrm{CPC}$ for every Brouwer algebra $\mathcal{L}$, where IPC denotes the set of theorems of the intuitionistic propositional calculus, and CPC denotes the set of classically valid propositional formulas. Straightforward verifications show the following.

Fact 2.1. *Let $f : \mathscr{L}_1 \longrightarrow \mathscr{L}_2$ be a homomorphism of Brouwer algebras. If f is one-one then $\mathrm{Th}(\mathscr{L}_2) \subseteq \mathrm{Th}(\mathscr{L}_1)$; if f is onto, then $\mathrm{Th}(\mathscr{L}_1) \subseteq \mathrm{Th}(\mathscr{L}_2)$. Moreover, for every Brouwer algebra $\mathscr{L}$ and elements $x, y, z \in \mathscr{L}$ such that $x < y$ and $y = z \vee x$, then $\mathrm{Th}([0, z]_{\mathscr{L}}) \subseteq \mathrm{Th}([x, y]_{\mathscr{L}})$.*

Proof. The statements concerning the homomorphism $f : \mathscr{L}_1 \longrightarrow \mathscr{L}_2$ follow from straightforward verifications. For the remaining claim, note (see [18, Lemma 4]) that under the given assumptions the mapping $u \mapsto x \vee u$ is a Brouwer algebra homomorphism from $[0, z]_{\mathscr{L}}$ onto $[x, y]_{\mathscr{L}}$. □

For further details on the algebraic semantics of propositional logic the reader is referred to the classic but still valuable text by Rasiowa and Sikorski [16].

2.4. **Notations.** The following notation will be used throughout the paper. The symbols $\mathbb{U}\mathrm{sl}_{\to}$, $\mathbb{L}_d$, and $\mathbb{B}$ denote, respectively, the categories of implicative upper semilattices, distributive lattices, and Brouwer algebras: an upper semilattice (with least upper bound operation $\vee$, and with partial ordering relation $\leqslant$) is *implicative* if it can be equipped with a binary operation $\to$ satisfying that $a \to b = \text{least}\,\{c : b \leqslant a \vee c\}$. The symbol $Fr^{L_d}_{\mathbb{U}\mathrm{sl}_{\to}}(\mathscr{U})$ denotes the free distributive lattice over the implicative upper semilattice $\mathscr{U}$. Given two objects $\mathscr{L}_1$ and $\mathscr{L}_2$ in any of these categories, say $\mathbb{C}$, we write $\mathscr{L}_1 \overset{\mathbb{C}}{\simeq} \mathscr{L}_2$ (or simply $\mathscr{L}_1 \simeq \mathscr{L}_2$ where $\simeq$ denotes order-theoretic isomorphism, since in all of these categories the objects are partially ordered structures and the isomorphisms coincide with the order-theoretic isomorphisms) to indicate that the two objects are isomorphic. We write $f : \mathscr{L}_1 \overset{\mathbb{C}}{\hookrightarrow} \mathscr{L}_2$ to indicate that f is a monomorphism in $\mathbb{C}$ from $\mathscr{L}_1$ to $\mathscr{L}_2$.

3. The Dyment lattice and the Dyment-Muchnik lattice

A *Dyment mass problem* (or a *D-mass problem*) $\mathcal{A}$ is any subset of the collection $\mathcal{P}$ of all partial functions from ω to ω. Given D-mass problems $\mathcal{A}, \mathcal{B}$, we say that $\mathcal{A}$ is *Dyment-reducible* (or, simply, *D-reducible*) to $\mathcal{B}$ (notation $\mathcal{A} \leqslant_D \mathcal{B}$), if there exists an enumeration operator Φ such that $\mathcal{B} \subseteq \text{domain}(\Phi)$, and $\Phi(\psi) \in \mathcal{A}$, for every $\psi \in \mathcal{B}$. The *Dyment degree* (or *D-degree*) of $\mathcal{A}$, denoted by $[\mathcal{A}]_D$, is the equivalence class of $\mathcal{A}$ under the equivalence relation $\equiv_D$ induced by the preordering relation $\leqslant_D$. The set of Dyment degrees is denoted by $\mathbf{M}_D$.

Similar to Muchnik's weakening of Medvedev's reducibility, one can define the weaker non uniform version of D-reducibility on D-mass problems, for which $\mathcal{A}$ is *Dyment-Muchnik reducible to* $\mathcal{B}$ (notation: $\mathcal{A} \leqslant_{D,w} \mathcal{B}$) if $(\forall \psi \in \mathcal{B})(\exists \varphi \in \mathcal{A})[\varphi \leqslant_e \psi]$. The *Dyment-Muchnik degree* of $\mathcal{A}$, denoted by $[\mathcal{A}]_{D,w}$, is the equivalence class of $\mathcal{A}$ under the equivalence $\equiv_{D,w}$ induced by the preordering relation $\leqslant_{D,w}$. The set of Dyment-Muchnik degrees is denoted by $\mathbf{M}_{D,w}$.

Roughly Dyment-Muchnik reducibility is to Dyment reducibility as Muchnik reducibility is to Medvedev reducibility.

We now extend to partial functions and D-mass problems the familiar operations used by Medvedev on total functions and mass problems of total functions. Given partial functions $\varphi, \psi \in \mathcal{P}$ define $\varphi \oplus \psi$ to be the partial function

$$\varphi \oplus \psi(x) = \begin{cases} \varphi(y) & \text{if } x = 2y, \\ \psi(y) & \text{if } x = 2y+1. \end{cases}$$

If $i \in \omega$ and $\varphi \in \mathcal{P}$, define $i\widehat{\ }\varphi$ to be the partial function

$$i\widehat{\ }\varphi(x) = \begin{cases} i & \text{if } x = 0\ , \\ \varphi(x-1) & \text{if } x > 0, \end{cases}$$

and if $\mathcal{A}$ is a D-mass problem then denote $i\widehat{\ }\mathcal{A} = \{i\widehat{\ }\varphi : \varphi \in \mathcal{A}\}$. Finally, let

$$\begin{aligned} &\mathcal{A} \oplus \mathcal{B} = \{\varphi \oplus \psi : \varphi \in \mathcal{A}, \psi \in \mathcal{B}\}, \\ &\mathcal{A} \wedge \mathcal{B} = 0\widehat{\ }\mathcal{A} \cup 1\widehat{\ }\mathcal{B}, \\ &\mathcal{A} \to \mathcal{B} = \{e\widehat{\ }\varphi : (\forall \psi \in \mathcal{A})[\Phi_e(\varphi \oplus \psi) \downarrow \in \mathcal{B}]\}. \end{aligned}$$

The Dyment equivalence relation $\equiv_D$ on D-mass problems is a congruence with respect to the above introduced operations on D-mass problems ([5]). Hence, similarly to the case of Medvedev reducibility, these operations induce corresponding operations (for which we use the same symbols) in the quotient structure. If $\mathbf{0}_D$ denotes the Dyment degree of the D-mass problem containing all partial computable functions, and $\mathbf{1}_D$ denotes the Dyment degree of the empty D-mass problem, then the following holds.

Theorem 3.1. *The algebra $\mathscr{M}_D = \langle \boldsymbol{M}_D, \vee, \wedge, \rightarrow, \mathbf{0}_D, \mathbf{1}_D \rangle$ is a Brouwer algebra, with least element given by $\mathbf{0}_D$, and greatest element given by $\mathbf{1}_D$, and*

$$\begin{aligned}
[\mathcal{A}]_D \vee [\mathcal{B}]_D &= [\mathcal{A} \oplus \mathcal{B}]_D, \\
[\mathcal{A}]_D \wedge [\mathcal{B}]_D &= [\mathcal{A} \wedge \mathcal{B}]_D, \\
[\mathcal{A}]_D \rightarrow [\mathcal{B}]_D &= [\mathcal{A} \rightarrow \mathcal{B}]_D.
\end{aligned}$$

The partial ordering relation $\leqslant_D$ on $\mathscr{M}_D$ (for which we use the same symbol as the preordering relation from which it originates) is given by $[\mathcal{A}]_D \leqslant_D [\mathcal{B}]_D$ if $\mathcal{A} \leqslant_D \mathcal{B}$.

Proof. The proof is essentially the same as the proof that the Medvedev lattice is a Brouwer algebra, see [13] or [17, § 13.7]. □

Definition 3.2. $\mathscr{M}_D$ is called the *Dyment lattice.*

The Dyment-Muchnik equivalence $\equiv_{D,w}$ on D-mass problems is a congruence with respect to the above introduced operations on D-mass problems ([5]), except with $\rightarrow$ now interpreted as

$$\mathcal{A} \rightarrow \mathcal{B} = \{\chi : (\forall \varphi \in \mathcal{A})\, (\exists \psi \in \mathcal{B})\, [\psi \leqslant_e \chi \oplus \varphi]\}.$$

Similarly to the case of Muchnik reducibility, these operations induce corresponding operations (for which we use the same symbols) in the quotient structure. If $\mathbf{0}_{D,w}$ denotes the Dyment-Muchnik degree of the D-mass problem containing all partial computable functions, and $\mathbf{1}_{D,w}$ denotes the Dyment-Muchnik degree of the empty D-problem, then, again, it is straightforward to see that the following holds.

Theorem 3.3. *The algebra $\mathscr{M}_{D,w} = \langle \boldsymbol{M}_{D,w}, \vee, \wedge, \rightarrow, \mathbf{0}_{D,w}, \mathbf{1}_{D,w} \rangle$ is a Brouwer algebra, with least element given by $\mathbf{0}_{D,w}$, and greatest element given by $\mathbf{1}_{D,w}$, and*

$$\begin{aligned}
[\mathcal{A}]_{D,w} \vee [\mathcal{B}]_{D,w} &= [\mathcal{A} \oplus \mathcal{B}]_{D,w}, \\
[\mathcal{A}]_{D,w} \wedge [\mathcal{B}]_{D,w} &= [\mathcal{A} \wedge \mathcal{B}]_{D,w} = [\mathcal{A} \cup \mathcal{B}]_{D,w}, \\
[\mathcal{A}]_{D,w} \rightarrow [\mathcal{B}]_{D,w} &= [\mathcal{A} \rightarrow \mathcal{B}]_{D,w}
\end{aligned}$$

The partial ordering relation $\leqslant_{D,w}$ on $\mathscr{M}_{D,w}$ (for which we use the same symbol as the preordering relation from which it originates) is given by $[\mathcal{A}]_{D,w} \leqslant_{D,w} [\mathcal{B}]_{D,w}$ if $\mathcal{A} \leqslant_{D,w} \mathcal{B}$.

Proof. Again, the proof is essentially the same as the proof that the Muchnik lattice is a Brouwer algebra, see [14] or [20]. □

Definition 3.4. $\mathscr{M}_{D,w}$ is called the *Dyment-Muchnik lattice.*

We will use boldface capital letters as variables for Dyment degrees (or, when clear from context, Dyment–Muchnik degrees). For more information about the Dyment lattice and the Dyment-Muchnik lattice see [5], [20], [22].

4. The Dyment lattice and intermediate logics: An initial segment of the Dyment lattice modeling exactly IPC

We split this section into three subsections. The first subsection reviews some known algebraic facts regarding Brouwer algebras, and known logical ingredients relating Brouwer algebras to the intuitionistic propositional logic. The second subsection shows some useful computability-theoretic facts relative to the Dyment lattice. The third subsection contains the main theorem stating that there is an initial segment of the Dyment lattice modeling IPC.

4.1. **The algebraic ingredients.**

If A is a subset of a preordered set $\mathscr{P} = \langle P, \leqslant \rangle$ then let

$$A_{u\leqslant} = \{y \in P : (\exists x \in A)[y \geqslant x]\},$$
$$A_{d\leqslant} = \{y \in P : (\exists x \in A)[y \leqslant x]\}.$$

We call $A_{u\leqslant}$ the *upwards $\leqslant$-closure of A in $\mathscr{P}$*, and we call $A_{d\leqslant}$ the *downwards $\leqslant$-closure of A in $\mathscr{P}$*. If $A = A_{u\leqslant}$ then we say that A is *upwards $\leqslant$-closed in $\mathscr{P}$*, and if $A = A_{d\leqslant}$ then we say that A is *downwards $\leqslant$-closed in $\mathscr{P}$*.

If $\mathscr{P}$ is a preorder as above, then let $\mathrm{Up}(\mathscr{P}) = \langle \mathrm{Up}(P), \supseteq \rangle$ be the poset with universe

$$\mathrm{Up}(P) = \{X \subseteq P : \ X \text{ is upwards } \leqslant\text{-closed in } \mathscr{P}\},$$

and partially ordered by reverse inclusion. Simple calculations show that $\mathrm{Up}(\mathscr{P})$ is a Brouwer algebra, with $\cup$ giving the operation $\wedge$ of greatest lower bound, $\cap$ giving the operation $\vee$ of least upper bound, $0 = U$, $1 = \varnothing$, and finally,

$$X \to Y = \bigcup \{Z \in \mathrm{Up}(P) : Y \supseteq Z \cap X\}.$$

Starting from a pre-upper semilattice $\mathscr{U} = \langle U, +, \leqslant \rangle$ (see next Remark) one can carry out a similar construction, and get a Brouwer algebra $\mathrm{Up}(\mathscr{U})$. In this case we can also describe $\vee$ as $X \vee Y = \{x + y : x \in X, \, y \in Y\}$, and $\to$ as $X \to Y = \{z : (\forall x \in X)\,[z + x \in Y]\}$.

Remark 4.1. By a *pre-upper semilattice* $\mathscr{U} = \langle U, +, \leqslant \rangle$ we mean that $\leqslant$ is a preordering relation and the equivalence relation originated by $\leqslant$ is a congruence with respect to $+$, so that the quotient structure $\mathscr{U}_{/\equiv} = \langle U_{/\equiv}, +_{/\equiv}, \leqslant_{/\equiv} \rangle$ is a an upper semilattice. A *pre-upper semilattice with a least element* is defined similarly. It is easy to see that if $\mathscr{P}$ is a preorder then $\mathrm{Up}(\mathscr{P}) \simeq \mathrm{Up}(\mathscr{P}_{/\equiv})$ via the isomorphism $\mathcal{A} \mapsto \{a_{/\equiv} : a \in \mathcal{A}\}$, and consequently if $\mathscr{U}$ is a pre-upper semilattice, then $\mathrm{Up}(\mathscr{U}) \simeq \mathrm{Up}(\mathscr{U}_{/\equiv})$.

Recall from Subsection 2.1 that $\mathscr{D}_e^{pf}$ is the upper semilattice of the partial degrees. Let $\mathscr{P}_e = \langle \mathcal{P}, \oplus, \leqslant_e \rangle$ be the pre-upper semilattice of the partial functions under enumeration reducibility $\leqslant_e$. Then

Theorem 4.2. *We have*

$$\mathscr{M}_{D,w} \simeq \mathrm{Up}(\mathscr{D}_e^{pf}) \simeq \mathrm{Up}(\mathscr{P}_e).$$

Proof. For every pair of D-mass problems $\mathcal{A}$ and $\mathcal{B}$, we have

$$\mathcal{A} \leqslant_{D,w} \mathcal{B} \Leftrightarrow \mathcal{B} \subseteq \mathcal{A}_{u\leqslant_e} \Leftrightarrow \mathcal{B}_{u\leqslant_e} \subseteq \mathcal{A}_{u\leqslant_e}.$$

Hence the mapping $[\mathcal{A}]_{D,w} \mapsto \mathcal{A}_{u\leqslant_e}$ provides an isomorphism between $\mathscr{M}_{D,w}$ and $\mathrm{Up}(\mathscr{P}_e)$. □

Definition 4.3. [10, Definition 3.1] Given a downwards $\leqslant$-closed set A in a pre-upper semilattice $\mathscr{U} = \langle U, +, \leqslant \rangle$, by a *strong upwards antichain in A* (also, abbreviated as *strong u–antichain in A*) we mean any collection of elements $\{x_i : i \in I\} \subseteq A$ such that

$$(\forall i, j \in I)[i \neq j \Rightarrow x_i + x_j \notin A].$$

An obvious instance of this definition is when $\{x_i : i \in I\}$ is an antichain, and $A = \{x_i : i \in I\}_{d\leqslant}$.

Definition 4.4. If $\mathscr{U}$ and A are as above, and $\{x_i : i \in I\}$ is a strong u–antichain in A, then for every $X \subseteq I$ define

$$\alpha(X) = A^c \cup \{x_i : i \in X\}_{u\leqslant},$$

where $A^c = U \smallsetminus A$.

Lemma 4.5. *Notice that $\alpha(X)$ is upwards $\leqslant$-closed. Moreover, $\alpha(I) = A^c \cup \{x_i : i \in I\}_{u\leqslant}$ and $\alpha(\varnothing) = A^c$.*

Proof. Immediate. ☐

Lemma 4.6. [10, Theorem 3.3] *If $\mathscr{U}$, A, $\{x_i : i \in I\}$, α are as in Definitions 4.3 and 4.4, and $\mathscr{P}(I) = \langle \mathcal{P}(I), \supseteq\rangle$ is the Boolean algebra of the power set of I ordered by reverse inclusion, then $\alpha : \mathscr{P}(I) \xhookrightarrow{\mathbb{U}sl} [\alpha(I), \alpha(\varnothing)]_{\mathrm{Up}(\mathscr{U})}$ is an embedding which preserves both the least and the greatest elements.*

Proof. The proof follows from easy calculations: the lazy reader can look at [10, Theorem 3.3], where these calculations refer to the case when $\mathscr{U}$ is the pre-upper semilattice $\langle \omega^\omega, \oplus, \leqslant_T\rangle$ of total functions under Turing reducibility. ☐

Recall that an element x of a lattice is *meet-reducible* if there are $y, z > x$ such that $x = y \wedge z$. Dually, x is *join-reducible* if there are $y, z < x$ such that $x = y \vee z$. Elements that are not meet-reducible are called *meet-irreducible*, and elements that are not join-reducible are called *join-irreducible*.

Definition 4.7. [18] A subset $\mathscr{C}$ of a Brouwer algebra $\mathscr{B}$ is called *canonical in $\mathscr{B}$* if

(1) if $a \in \mathscr{C}$ and $a, b \in \mathscr{B}$ then $a \to b \wedge c = (a \to b) \wedge (a \to c)$;
(2) the elements of $\mathscr{C}$ are meet-irreducible;
(3) $\mathscr{C}$ is a sub-implicative upper semilattice of $\mathscr{B}$.

Lemma 4.8. *Suppose that $\mathscr{C}$ is a canonical set in a Brouwer algebra $\mathscr{B}$. If $\{a_i : i \in I\}$ and $\{b_j : j \in J\}$ are elements of $\mathscr{C}$ then we have:*

(1) $\bigwedge_{i\in I} a_i \leqslant \bigwedge_{j\in J} b_j \Leftrightarrow (\forall j \in J)\,(\exists i \in I)\,[a_i \leqslant b_j]$;
(2) $\bigwedge_{i\in I} a_i \to \bigwedge_{j\in J} b_j = \bigvee_{i\in I} \Big(\bigwedge_{j\in J}(a_i \to b_j)\Big)$.

Proof. Item (1) is obvious by meet-irreducibility of the elements of $\mathscr{C}$. As to item (2), it is known (see for instance [1, IX §2. Theorem 3(ix)] in the dual context of Heyting algebras) that in any Brouwer algebra we have that $a \wedge b \to c = (a \to c) \vee (b \to c)$. Indeed, by taking infima and distributivity, from $a \vee (a \to c) \vee (b \to c) \geqslant c$ and $b \vee (a \to c) \vee (b \to c) \geqslant c$ one gets $(a \wedge b) \vee ((a \to c) \vee (b \to c)) \geqslant c$, thus $a \wedge b \to c \leqslant (a \to c) \vee (b \to c)$; on the other hand, $a \vee (a \wedge b \to c) \geqslant (a \wedge b) \vee (a \wedge b \to c) \geqslant c$, from which $a \to c \leqslant a \wedge b \to c$, and similarly $b \to c \leqslant a \wedge b \to c$, which gives $(a \to c) \vee (b \to c) \leqslant a \wedge b \to c$.

Putting things together, and using that $a \to b \wedge c = (a \to b) \wedge (a \to c)$ if $a \in \mathscr{C}$, we conclude

$$\bigwedge_{i\in I} a_i \to \bigwedge_{j\in J} b_j = \bigvee_{i\in I}\left(a_i \to \bigwedge_{j\in J} b_i\right) = \bigvee_{i\in I}\bigwedge_{j\in J}(a_i \to b_j).$$

This shows item (2). ☐

Lemma 4.9. *Let $\mathscr{U}$ be a finite implicative upper semilattice. Then the elements of $\mathscr{U}$ that freely generate $Fr^{L_d}_{\mathbb{U}sl_\to}(\mathscr{U})$ correspond exactly to the meet-irreducible elements of $Fr^{L_d}_{\mathbb{U}sl_\to}(\mathscr{U})$, and they form a canonical set in $Fr^{L_d}_{\mathbb{U}sl_\to}(\mathscr{U})$.*

Proof. We hope that a short description of $Fr^{L_d}_{\mathbb{U}\mathrm{sl}_\to}(\mathscr{U})$ is in order. Let $\mathscr{U} = \langle U, \vee_U, \to_U\rangle$ be an implicative upper semilattice with partial ordering relation $\leqslant_U$. For a subset $X \subseteq U$, denote for simplicity $[X) = X_{u\leqslant_U}$, i.e. the upwards $\leqslant_U$-closure of X in $\mathscr{U}$, which is a generic element of $\mathrm{Up}(U)$. The lowercase Latin letters x, y, z will vary through the elements of U, and we will write $[x)$ for $[\{x\})$.

Let Fr_U be the set of the formal expressions that one can obtain from the operation symbol $\wedge$, and the elements of $\mathscr{U}$ of the form $[x)$ for $x \in U$, i.e.,

$$Fr_U = \{\bigwedge_{x\in X}[x) : X \subseteq U,\ X \text{ finite}\},$$

Using the $\leqslant_U$ of $\mathscr{U}$, define on Fr_U the following preordering relation for finite subsets $X, Y \subseteq U$:

$$\bigwedge_{x\in X}[x) \leqslant \bigwedge_{y\in Y}[y) \Leftrightarrow (\forall y \in Y)\,(\exists x \in X)\,\big[x \leqslant_U y\big].$$

It is not difficult to see that the equivalence relation induced by this preordering partitions Fr_U into equivalence classes that form a distributive lattice with 0 and 1, where the lattice operations of meet and join are defined unambiguously on representatives of the equivalence classes by

$$\begin{aligned}\bigwedge_{x\in X}[x) \wedge \bigwedge_{y\in Y}[y) &= \bigwedge_{z\in X\cup Y}[z),\\ \bigwedge_{x\in X}[x) \vee \bigwedge_{y\in Y}[y) &= \bigwedge_{x\in X, y\in Y}[x \vee_U y).\end{aligned}$$

where $\vee_U$ is the join operation of $\mathscr{U}$.

We claim that this lattice (a sub-upper semilattice of $\mathrm{Up}(\mathscr{U})$) is the desired free distributive lattice $Fr^{L_d}_{\mathbb{U}\mathrm{sl}_\to}(\mathscr{U})$. To verify this, as well as the additional claims of the lemma, in order to simplify notation we will work directly with the elements of Fr_U, rather than with their equivalence classes. One can easily see that the function $\iota(x) = [x)$ is an upper semilattice embedding of $\mathscr{U}$ into $Fr^{L_d}_{\mathbb{U}\mathrm{sl}_\to}(\mathscr{U})$. The elements of Fr_U are of the form $\bigwedge_{x\in X}[x)$ with X finite, so that the meet-irreducible elements of $Fr^{L_d}_{\mathbb{U}\mathrm{sl}_\to}(\mathscr{U})$ are exactly the elements in the image of the embedding ι. With their meets they generate $Fr^{L_d}_{\mathbb{U}\mathrm{sl}_\to}(\mathscr{U})$. Moreover, $Fr^{L_d}_{\mathbb{U}\mathrm{sl}_\to}(\mathscr{U})$ and ι satisfy the universal property of free objects: if $\mathcal{G} = \langle G, \wedge_G, \vee_G\rangle$ is a distributive lattice and $g : \mathscr{U} \to \mathcal{G}$ is an upper semilattice homomorphism, then there exists a unique homomorphism $g' : Fr^{L_d}_{\mathbb{U}\mathrm{sl}_\to}(\mathscr{U}) \to \mathcal{G}$ such that $g' \circ \iota = g$, namely $g'(\bigwedge_{x\in X}[x)) = \bigwedge_G\{g(x) : x \in X\}$. To see for instance that g' preserves least upper bounds, we observe

$$\begin{aligned} g'\left(\bigwedge_{x\in X}[x) \vee \bigwedge_{y\in Y}[y)\right) &= g'\left(\bigwedge_{x\in X, y\in Y}[x_i \vee_U y)\right) \\ &= \bigwedge_G \{g(x \vee_U y) : x \in X, y \in Y\} \\ &= \bigwedge_G \{g(x) \vee_G g(y) : x \in X, y \in Y\} \\ &= \bigwedge_G \{g(x) : x \in X\} \vee_G \bigwedge_G \{g(y) : y \in Y\} \\ &= g'\left(\bigwedge_{x\in X}[x)\right) \vee_G g'\left(\bigwedge_{y\in Y}[y)\right). \end{aligned}$$

Although the operation of implication plays no role in the definition of g', note that if $Fr^{L_d}_{\mathbb{U}\mathrm{sl}_\rightarrow}(\mathscr{U})$ is finite, then it is a Brouwer algebra since it is a finite distributive lattice. A closer inspection shows that in this case ι preserves the operation of implication as well. Indeed, if $\bigwedge_{z\in Z}[z) \in Fr_U$ then as $[y)$ is meet irreducible and $\bigwedge_{z\in Z}[x) \vee [z) = \bigwedge_{z\in Z}[x \vee_U z)$ we have

$$\begin{aligned} [y) \leqslant [x) \vee \bigwedge_{z\in Z}[z) &\Leftrightarrow (\forall z \in Z)\,[y \leqslant_U x \vee_U z] \\ &\Leftrightarrow (\forall z \in Z)\,[x \rightarrow_U y \leqslant_U z] \\ &\Leftrightarrow [x \rightarrow_U y) \leqslant \bigwedge_{z\in Z}[z). \end{aligned}$$

This shows that $\iota(x \rightarrow_U y) = \iota(x) \rightarrow \iota(y)$, where the implication on the right side denotes the arrow operation of $Fr^{L_d}_{\mathbb{U}\mathrm{sl}_\rightarrow}(\mathscr{U})$. Therefore the generators of $Fr^{L_d}_{\mathbb{U}\mathrm{sl}_\rightarrow}(\mathscr{U})$ (the elements of the form $\iota(x)$, for $x \in U$) satisfy properties (2) and (3) of the definition of a canonical set.

We must show at this point that (1) of the definition of a canonical set is satisfied. For the sake of further simplifying notation, in the rest of the proof we identify the elements of $\mathscr{U}$ with their images in $Fr^{L_d}_{\mathbb{U}\mathrm{sl}_\rightarrow}(\mathscr{U})$ under the embedding i, and we use lowercase Latin letters as variables for the elements of $Fr^{L_d}_{\mathbb{U}\mathrm{sl}_\rightarrow}(\mathscr{U})$. Property (1) amounts to showing that if $a \in \mathscr{U}$, then for every b, c we have $a \rightarrow b \wedge c = (a \rightarrow b) \wedge (a \rightarrow c)$. Suppose that $a \rightarrow b \wedge c = \bigwedge_{i\in I} d_i$ where each d_i is meet-irreducible (it is well known that in a finite distributive lattice any element can be uniquely represented as the meet of an antichain of meet-irreducible elements). We have

$$b \wedge c \leqslant a \vee \bigwedge_{i\in I} d_i = \bigwedge_{i\in I}(a \vee d_i).$$

Now, each $a \vee d_i$ is meet-irreducible, being an element of $\mathscr{U}$, hence $b \leqslant a \vee d_i$ or $c \leqslant a \vee d_i$. Therefore, we can pick two subsets $J, K \subseteq I$ such that $b \leqslant \bigwedge_{j\in J}(a \vee d_j) = a \vee \bigwedge_{j\in J} d_j$ and $c \leqslant \bigwedge_{k\in K}(a \vee d_k) = a \vee \bigwedge_{k\in K} d_k$. Hence $a \rightarrow b \leqslant \bigwedge_{j\in J} d_j$ and $a \rightarrow c \leqslant \bigwedge_{k\in K} d_k$, giving $(a \rightarrow b) \wedge (a \rightarrow c) \leqslant \bigwedge_{i\in I}(a \vee d_i) = a \rightarrow b \wedge c$. On the other hand, by distributivity and easy calculations we get $b \wedge c \leqslant a \vee ((a \rightarrow b) \wedge (a \rightarrow c))$, which gives $a \rightarrow b \wedge c \leqslant (a \rightarrow b) \wedge (a \rightarrow c)$, thus showing property (1) of the definition of a canonical set. ☐

Let now $I_n = \{1, 2, \ldots, n\}$, and, for any given set X, let $\mathcal{P}(X)$ denote the power set of X, and let $\mathscr{P}(X) = \langle \mathcal{P}(X), \supseteq \rangle$.

Definition 4.10. Let $\mathscr{B}_n$ denote the Brouwer algebra $Fr^{L_d}_{\mathbb{U}\mathrm{sl}_\to}(\mathscr{P}(I_n))$.

The intermediate propositional logic given by $\bigcap_{n>0} \mathrm{Th}(\mathcal{B}_n)$ is known as the *Medvedev logic*. The following holds:

Lemma 4.11. $\bigcap_{n>0}\bigcap_{x\in\mathscr{B}_n} \mathrm{Th}([0,x]_{\mathscr{B}_n}) = \mathrm{IPC}$.

Proof. See [18, Lemma 3], the remark following it, and [10, Propositions 4.6 and 4.7]. □

4.2. **The computability-theoretic ingredients.** We now isolate an important class of Dyment degrees, which constitute s canonical set in $\mathscr{M}_D$.

Definition 4.12. A D-mass problem $\mathcal{A} \subseteq \mathcal{P}$ is said to be *Dyment-Muchnik* if $\mathcal{A}$ is nonempty and $\mathcal{A}$ is upwards $\leqslant_e$-closed. A *Dyment-Muchnik* degree is the degree of some Dyment-Muchnik mass problem.

Lemma 4.13. *The Dyment-Muchnik degrees form a canonical set in $\mathscr{M}_D$.*

Proof. With three sublemmata we will verify the three conditions in the definition of a canonical set. The analogue of Sublemma 1 for the Medvedev lattice was independently proved by Skvortsova [18] and Sorbi [21]. Throughout the proofs of Sublemma 1 and Sublemma 2, we will denote by $\mathcal{P}_F$ the set of *finite* partial functions.

Sublemma 1. *Let $\mathcal{A}, \mathcal{B}_0, \mathcal{B}_1, \mathcal{Z}$ be D-mass problems, with $\mathcal{A}$ being Dyment-Muchnik. If $\mathcal{A} \oplus \mathcal{Z} \geqslant_D \mathcal{B}_0 \wedge \mathcal{B}_1$, then there are $\mathcal{Z}_0, \mathcal{Z}_1 \subseteq \mathcal{Z}$ such that*

- *$\mathcal{Z} \equiv_D \mathcal{Z}_0 \wedge \mathcal{Z}_1$ and*
- *$\mathcal{A} \oplus \mathcal{Z}_i \geqslant_D \mathcal{B}_i$ for $i = 0, 1$.*

Hence, we have

$$\mathcal{A} \to (\mathcal{B}_0 \wedge \mathcal{B}_1) \equiv_D (\mathcal{A} \to \mathcal{B}_0) \wedge (\mathcal{A} \to \mathcal{B}_1).$$

Proof of Sublemma 1. Let Ψ be an enumeration operator such that $\mathcal{A} \oplus \mathcal{Z} \subseteq \mathrm{domain}(\Psi)$, and $\Psi(\mathcal{A} \oplus \mathcal{Z}) \subseteq 0^\frown\mathcal{B}_0 \cup 1^\frown\mathcal{B}_1$.

For $i = 0, 1$, let

$$\mathcal{Z}_i = \{\chi \in \mathcal{Z} : (\exists\alpha \in \mathcal{P}_F)[\langle 0, i\rangle \in \Psi(\alpha \oplus \chi)]\}.$$

Notice that $\mathcal{Z} \leqslant_D \mathcal{Z}_0 \wedge \mathcal{Z}_1$ since $\mathcal{Z}_0, \mathcal{Z}_1 \subseteq \mathcal{Z}$.

Fix $\chi \in \mathcal{Z}$, and let $\psi \in \mathcal{A}$. Then $\Psi(\psi\oplus\chi)\downarrow$ and $\Psi(\psi\oplus\chi)(0)\downarrow\in \{0, 1\}$. Suppose that $\Psi(\psi\oplus\chi)(0)\downarrow= 0$. We will show in this case that for every $\alpha \in \mathcal{P}_F$, and every y, if $\langle 0, y\rangle \in \Psi(\alpha \oplus \chi)$ then $y = 0$. Given $\alpha, \beta \in \mathcal{P}$, let $\psi^\alpha = \psi \smallsetminus \{\langle x, y\rangle : x \in \mathrm{domain}(\alpha)\}$ and $\psi^{\alpha,\beta} = (\psi^\alpha)^\beta$. By our assumptions there exists $\alpha_0 \in \mathcal{P}_F$ such that $\langle 0, 0\rangle \in \Psi(\alpha_0 \oplus \chi)$. Assume, by contradiction, that there exist $\alpha_y \in \mathcal{P}_F$ and $y \neq 0$ such that $\langle 0, y\rangle \in \Psi(\alpha_y \oplus \chi)$. Now, $\psi^{\alpha_0,\alpha_y} \equiv_e \psi$, and since $\mathcal{A}$ is Dyment-Muchnik, we have that $\psi^{\alpha_0,\alpha_y} \in \mathcal{A}$, thus $\Psi(\psi^{\alpha_0,\alpha_y} \oplus \chi)\downarrow$, and thus for some $z \in \{0, 1\}$ there exists $\beta_z \in \mathcal{P}_F$ such that $\beta_z \subseteq \psi^{\alpha_0,\alpha_y}$ and $\langle 0, z\rangle \in \Psi(\beta_z \oplus \chi)$. But this yields a contradiction: if $z = 0$ then $\psi^{\alpha_0,\alpha_y} \cup \alpha_y$ is a partial function (since $\mathrm{domain}(\psi^{\alpha_0,\alpha_y}) \cap \mathrm{domain}(\alpha_y) = \varnothing$) which lies in $\mathcal{A}$ (since $\psi^{\alpha_0,\alpha_y} \cup \alpha_y \equiv_e \psi^{\alpha_0,\alpha_y}$), but $\Psi((\psi^{\alpha_0,\alpha_y} \cup \alpha_y) \oplus \chi)$ is not single-valued (since both $\langle 0, 0\rangle$ and $\langle 0, y\rangle$ lie in $\Psi((\psi^{\alpha_0,\alpha_y} \cup \alpha_y) \oplus \chi)$, being $\beta_z, \alpha_y \subseteq \psi^{\alpha_0,\alpha_y} \cup \alpha_y$), contradicting

$(\psi^{\alpha_0,\alpha_y} \cup \alpha_y) \oplus \chi \in \text{domain}(\Psi)$. If $z = 1$ then a similar argument leads to showing that $\psi^{\alpha_0,\alpha_y} \cup \alpha_0$ lies in $\mathcal{A}$ but $(\psi^{\alpha_0,\alpha_y} \cup \alpha_0) \oplus \chi \notin \text{domain}(\Psi)$.

By symmetry, the previous argument shows that if we initially choose $\psi \in \mathcal{A}$ such that $\Psi(\psi \oplus \chi)(0) \downarrow= 1$, then for every $\alpha \in \mathcal{P}_F$, and every y, if $\langle 0, y\rangle \in \Psi(\alpha \oplus \chi)$ then $y = 1$.

Let Γ be the following enumeration operator: for $\chi \in \mathcal{P}$,

$$\Gamma(\chi) = \{\langle i, y\rangle : i > 0 \,\&\, \langle i-1, y\rangle \in \chi\} \cup \{\langle 0, y\rangle : (\exists \alpha \in \mathcal{P}_F)\,[\langle 0, y\rangle \in \Psi(\alpha \oplus \chi)]\}.$$

By the previous remarks it is clear that $\mathcal{Z} \subseteq \text{domain}(\Gamma)$, and for every $\chi \in \mathcal{Z}$, we have that $\Gamma(\chi)(0) \downarrow\in \{0,1\}$. Therefore $\mathcal{Z} \geqslant_D \mathcal{Z}_0 \wedge \mathcal{Z}_1$ via Γ. The e-operator $\Phi(\psi\oplus\chi)$ such that $\Phi(\psi\oplus\chi)(y) = \Psi(\psi \oplus \chi)(y+1)$ for each y provides a reduction $\mathcal{A} \oplus \mathcal{Z}_i \geqslant_D \mathcal{B}_i$ for both $i \in \{0,1\}$.

To finish off, we must show that $\mathcal{A} \to \mathcal{B}_0 \wedge \mathcal{B}_1 \equiv_D (\mathcal{A} \to \mathcal{B}_0) \wedge (\mathcal{A} \to \mathcal{B}_1)$. Well, in every Brouwer algebra we have $a \to b_0 \wedge b_1 \leqslant (a \to b_0) \wedge (a \to b_1)$ since $a \vee ((a \to b_0) \wedge (a \to b_1)) \geqslant b_0 \wedge b_1$. Therefore, $\mathcal{A} \to \mathcal{B}_0 \wedge \mathcal{B}_1 \leqslant_D (\mathcal{A} \to \mathcal{B}_0) \wedge (\mathcal{A} \to \mathcal{B}_1)$. On the other hand, by what proved in the first part of the sublemma taking $\mathcal{Z}$ to be $\mathcal{A} \to \mathcal{B}_0 \wedge \mathcal{B}_1$, from

$$\mathcal{B}_0 \wedge \mathcal{B}_1 \leqslant_D \mathcal{A} \oplus (\mathcal{A} \to \mathcal{B}_0 \wedge \mathcal{B}_1)$$

one gets that there exist $\mathcal{Z}_0, \mathcal{Z}_1$ such that $\mathcal{A} \to \mathcal{B}_0 \wedge \mathcal{B}_1 \equiv_D \mathcal{Z}_0 \wedge \mathcal{Z}_1$, and $\mathcal{A} \to \mathcal{B}_i \leqslant_D \mathcal{Z}_i$ for $i = 0, 1$. Hence

$$\mathcal{A} \to \mathcal{B}_0 \wedge \mathcal{B}_1 \geqslant_D (\mathcal{A} \to \mathcal{B}_0) \wedge (\mathcal{A} \to \mathcal{B}_1). \qquad \square$$

Sublemma 2. *If $\mathcal{A}$ is a Dyment-Muchnik mass problem then its Dyment degree is meet-irreducible.*

Proof of Sublemma 2. Suppose that $\mathcal{A}$ is a Dyment-Muchnik mass problem and $\mathcal{A} \equiv_D \mathcal{B}_0 \wedge \mathcal{B}_1$. Let $\mathcal{A} \geqslant_D \mathcal{B}_0 \wedge \mathcal{B}_1$ via the enumeration operator Γ. We want to show that either $\mathcal{B}_0 \leqslant_D \mathcal{A}$ or $\mathcal{B}_1 \leqslant_D \mathcal{A}$. Let $\varphi \in \mathcal{A}$. We claim that if $\Gamma(\varphi)(0) \downarrow= 0$ then $\Gamma(\mathcal{A}) \subseteq 0^\frown\mathcal{B}_0$, and thus $\mathcal{B}_0 \leqslant_D \mathcal{A}$; and if $\Gamma(\varphi)(0) \downarrow= 1$ then $\Gamma(\mathcal{A}) \subseteq 1^\frown\mathcal{B}_1$, and thus $\mathcal{B}_1 \leqslant_D \mathcal{A}$.

So, let us assume that $\Gamma(\varphi)(0) \downarrow= 0$: the case $\Gamma(\varphi)(0) \downarrow= 1$ is symmetric. So there exists $\alpha_0 \in \mathcal{P}_F$ with $\alpha_0 \subseteq \varphi$ and $\langle 0, 0\rangle \in \Gamma(\alpha_0)$. Suppose now that $\psi \in \mathcal{A}$ is another partial function such that $\Gamma(\psi)(0) \downarrow= 1$, and let $\alpha_1 \in \mathcal{P}_F$ with $\alpha_1 \subseteq \psi$ and $\langle 0, 1\rangle \in \Gamma(\alpha_1)$. Arguing as in the previous sublemma, we have that $\varphi^{\alpha_0,\alpha_1} \in \mathcal{A}$, and thus $\Gamma(\varphi^{\alpha_0,\alpha_1})(0) \downarrow\in \{0,1\}$. We may suppose that $\Gamma(\varphi^{\alpha_0,\alpha_1})(0) \downarrow= 0$, the other case being symmetric. But then $\{\langle 0,0\rangle, \langle 0,1\rangle\} \subseteq \Gamma(\varphi^{\alpha_0,\alpha_1} \cup \alpha_1)$, which is a contradiction since $\varphi^{\alpha_0,\alpha_1} \cup \alpha_1 \in \mathcal{A}$, and thus $\varphi^{\alpha_0,\alpha_1} \cup \alpha_1 \in \text{domain}(\Gamma)$.

It follows that $\Gamma(\psi) \in 0^\frown\mathcal{B}_0 \equiv_D \mathcal{B}_0$ for every $\psi \in \mathcal{A}$, as desired. $\square$

Sublemma 3. *On Dyment-Muchnik mass problems, the operation of least upper bound is given by intersection, which still yields a Dyment-Muchnik mass problem. If $\mathcal{B}$ is Dyment-Muchnik, then*

$$\mathcal{A} \to \mathcal{B} = \{\psi : (\forall \varphi \in \mathcal{A})\,[\varphi \oplus \psi \in \mathcal{B}]\},$$

which still is a Dyment-Muchnik mass problem. Hence the Dyment-Muchnik degrees form an sub-implicative upper semilattice of $\mathscr{M}_D$.

Proof of Sublemma 3. To show the claim about $\to$, let $\mathcal{D} = \{\psi : (\forall \varphi \in \mathcal{A})\,[\varphi \oplus \psi \in \mathcal{B}]\}$. Clearly, $\mathcal{B} \leqslant_D \mathcal{A} \oplus \mathcal{D}$ since $\mathcal{A} \oplus \mathcal{D} \subseteq \mathcal{B}$. To show minimality of $\mathcal{D}$, assume that Φ is an enumeration operator reducing $\mathcal{B}$ to $\mathcal{A} \oplus \mathcal{Z}$. If $\psi \in \mathcal{Z}$ then for every $\varphi \in \mathcal{A}$ we have $\Phi(\varphi \oplus \psi)$ is defined and $\Phi(\varphi \oplus \psi) \in \mathcal{B}$. But since $\mathcal{B}$ is Dyment-Muchnik, this implies that $\varphi \oplus \psi \in \mathcal{B}$, and thus $\psi \in \mathcal{D}$, giving $\mathcal{Z} \subseteq \mathcal{D}$, and therefore $\mathcal{D} \leqslant_D \mathcal{Z}$. The claim about $\oplus$ being intersection is trivial. From this it follows that $\mathcal{A} \to \mathcal{B}$ and $\mathcal{A} \oplus \mathcal{B}$ are Dyment-Muchnik mass problems, if so are $\mathcal{A}$ and $\mathcal{B}$. $\square$

Putting the three sublemmata together we conclude that the Dyment-Muchnik degrees form a canonical set in $\mathscr{M}_D$. Indeed, Sublemma 1 shows that they satisfy item (1) of Definition 4.7; Sublemma 2 shows that they satisfy item (2) of Definition 4.7, and Sublemma 3 shows that they satisfy item (3) of Definition 4.7. □

Corollary 4.14. *If $\{\mathbf{A}_i : i \in I\}$ and $\{\mathbf{B}_j : j \in J\}$ are Dyment-Muchnik degrees then, in $\mathscr{M}_D$,*

$$\bigwedge_{i\in I} \mathbf{A}_i \to \bigwedge_{j\in J} \mathbf{B}_j = \bigvee_{i\in I} \bigwedge_{j\in J} (\mathbf{A}_i \to \mathbf{B}_j).$$

Proof. By Lemma 4.8 and Lemma 4.13. □

Corollary 4.15. *Let A, $\{x_i : i \in I\}$, and α be as in Definition 4.4 relative to the pre-upper semilattice $\mathscr{U} = \mathscr{P}_e$. Then each D-mass problem $\alpha(X)$ is Dyment-Muchnik.*

Proof. By Lemma 4.5, the D-mass problem $\alpha(X)$ is upwards $\leqslant_e$-closed. □

Corollary 4.16. *For every $n > 0$, if $\{\psi_i : i \in I_n\}$ is a strong u–antichain in some downwards $\leqslant_e$-closed subset $\mathcal{A}$ of $\mathcal{P}$ and $\alpha : \mathcal{P}(I_n) \longrightarrow \mathrm{Up}(\mathscr{P}_e)$ is as in Lemma 4.6, then there exists an embedding $\gamma : \mathscr{B}_n \overset{\mathbb{B}}{\hookrightarrow} \Big[[\alpha(I_n)]_D, [\alpha(\varnothing)]_D\Big]_{\mathscr{M}_D}$.*

Proof. Let $n > 0$ be given. Let $\iota : \mathscr{P}(I_n) \xhookrightarrow{\mathbb{U}\mathrm{sl}_\to} \mathscr{B}_n$ be the $\mathbb{U}\mathrm{sl}_\to$-embedding described in the proof of Lemma 4.9, i.e. $\iota(x) = [x]$ for $x \in \mathscr{P}(I_n)$. In analogy with the proof of Lemma 4.9 we use the letters x, y as variables for the elements of $\mathscr{P}(I_n)$, and X, Y as variables for subsets of $\mathscr{P}(I_n)$. Then, consider the embedding $\alpha : \mathscr{P}(I_n) \xhookrightarrow{\mathbb{U}\mathrm{sl}_\to} \Big[\alpha(I_n), \alpha(\varnothing)\Big]_{\mathrm{Up}(\mathscr{P}_e)}$, as in Lemma 4.6. Finally, let $d : \mathrm{Up}(\mathscr{P}_e) \to \mathscr{M}_D$ be $d(\mathcal{A}) = [\mathcal{A}]_D$. By the fact that the Dyment-Muchnik degrees form a canonical set in $\mathscr{M}_D$, the composition $d \circ \alpha$ is an $\mathbb{U}\mathrm{sl}_\to$-embedding $d \circ \alpha : \mathscr{P}(I_n) \xhookrightarrow{\mathbb{U}\mathrm{sl}_\to} \Big[[\alpha(I_n)]_D, [\alpha(\varnothing)]_D\Big]_{\mathscr{M}_D}$. Note that both ι and $d \circ \alpha$ preserve $0, 1$. By the freeness properties of $\mathscr{B}_n$ there is a unique morphism $\gamma : \mathscr{B}_n \xrightarrow{\mathbb{L}_d} \Big[[\alpha(I)]_D, [\alpha(\varnothing)]_D\Big]_{\mathscr{M}_D}$, preserving $0, 1$, and such that $\gamma([x]) = d(\alpha(x)) = [\alpha(x)]_D$, for every $x \in \mathscr{P}(I_n)$.

It is only left to show that γ preserves $\to$ and is an order-theoretic embedding. In the rest of proof, for notational simplicity we use the same symbols for the operations of $\mathscr{P}(I_n)$ and the corresponding operations of $\mathscr{M}_D$, leaving it to the reader to easily distinguish which denotes which. By Lemma 4.13, every element of $\mathscr{B}_n$ has the form $\bigwedge_{x\in X}[x]$ for some $X \subseteq \mathscr{P}(I_n)$. Moreover, on generators $[x], [y]$ we have $\gamma([x] \to [y]) = \gamma([x]) \to \gamma([y])$ since $\gamma([x] \to [y])) = d(\alpha(x \to y))$, but $x \to y$ is an element of $\mathscr{P}(I_n)$ and thus $\alpha(x \to y) = \alpha(x) \to \alpha(y)$, therefore $d(\alpha(x \to y)) = d(\alpha(x) \to \alpha(y)) = d(\alpha(x)) \to d(\alpha(y))$ since $\alpha(x)$ and $\alpha(y)$ are Dyment-Muchnik mass problems.

Therefore, using that γ preserves $\wedge$ and $\vee$, we have, for subsets $X, Y \subseteq \mathscr{P}(I_n)$:

$$\begin{aligned}
\gamma\left(\bigwedge_{x\in X}[x) \to \bigwedge_{y\in Y}[y)\right) &= \gamma\left(\bigvee_{x\in X}\bigwedge_{y\in Y}([x) \to [y))\right) \qquad \text{(by Lemma 4.8)},\\
&= \bigvee_{x\in X}\bigwedge_{y\in Y}\gamma([x) \to [y))\\
&= \bigvee_{x\in X}\bigwedge_{y\in Y}(\gamma([x)) \to \gamma([y)))\\
&= \bigwedge_{x\in X}\gamma([x)) \to \bigwedge_{y\in Y}\gamma([y)) \qquad \text{(by Corollary 4.14)}\\
&= \gamma\left(\bigwedge_{x\in X}[x)\right) \to \gamma\left(\bigwedge_{j\in J}[y)\right).
\end{aligned}$$

It follows that γ preserves the arrow-operation, since every element in $\mathscr{B}_n$ is a meet of meet-irreducible elements.

Finally, γ is an order isomorphism. Indeed, if $X = \bigwedge_{x\in X}[x)$ and $Y = \bigwedge_{y\in Y}[y)$ are elements of $\mathscr{B}_n$ with $X, Y \subseteq \mathscr{P}(I_n)$, then by the meet-irreducibility of each $[x), [y)$ in $\mathscr{B}_n$, the meet-irreducibility in $\mathscr{M}_D$ of every $\gamma([x))$ (see property (2) of the Dyment-Muchnik degrees), and since γ is an order-theoretic embedding on generators, we have

$$\begin{aligned}
\bigwedge_{x\in X}[x) \leqslant \bigwedge_{y\in Y}[y) &\Leftrightarrow (\forall y \in Y)\,(\exists x \in X)\ [[x) \leqslant [y)]\\
&\Leftrightarrow (\forall y \in Y)\,(\exists x \in X)\ [\gamma([x)) \leqslant_D \gamma([y))]\\
&\Leftrightarrow \bigwedge_{x\in X}\gamma([x)) \leqslant_D \bigwedge_{y\in Y}\gamma([y))\\
&\Leftrightarrow \gamma(\bigwedge_{x\in X}[x)) \leqslant_D \gamma(\bigwedge_{y\in Y}[y)).
\end{aligned}$$

□

4.3. **There is a Dyment degree Z such that** $\mathrm{Th}([\mathbf{0}_D, \mathbf{Z}]_{\mathscr{M}_D}) = \mathrm{IPC}$**.** Our $\mathbf{Z} = [\mathcal{Z}]_D$ will be chosen so that it is possible to implement the following plan. Fix $n > 0$. In view of Corollary 4.16, we will look for a suitable strong u–antichain in some downwards $\leqslant_e$-closed set of partial functions so that, by Corollary 4.16, there is an embedding $\gamma : \mathscr{B}_n \overset{\mathbb{B}}{\hookrightarrow} [[\alpha(I_n)]_D, [\alpha(\varnothing)]_D]_{\mathscr{M}_D}$. Let us denote $\alpha(I_n) = \mathcal{X}_n$ and $\alpha(\varnothing) = \mathcal{Y}_n$. The strong u–antichain and therefore the embedding will be arranged so that for $X \in \mathscr{B}_n$ (say $X = \bigwedge_{k\in K}[x_k)$, where each $x_k \in \mathscr{P}(I_n)$), in view of Fact 2.1, we will have

$$\mathcal{Y}_{n,X} \equiv_D \mathcal{Z} \oplus \mathcal{X}_n,$$

where $\mathcal{Y}_{n,X} = \bigwedge_{k\in K}\alpha(x_k)$ and so $\gamma(X) = [\mathcal{Y}_{n,X}]_D$. Letting $\mathbf{X}_n = [\mathcal{X}_n]_D$ and $\mathbf{Y}_{n,X} = [\mathcal{Y}_{n,X}]_D$, we have $\mathbf{Y}_{n,X} = \mathbf{Z} \vee \mathbf{X}_n$, and from Fact 2.1 it follows that $\mathrm{Th}([\mathbf{X}_n, \mathbf{Y}_{n,X}]_{\mathscr{M}_D}) \subseteq \mathrm{Th}([0, X]_{\mathscr{B}_n})$, and thus by Lemma 4.11

$$\begin{aligned}
\mathrm{Th}([\mathbf{0}_D, \mathbf{Z}]_{\mathscr{M}_D}) &\subseteq \bigcap_{n>0}\bigcap_{X\in\mathscr{B}_n}\mathrm{Th}\left([\mathbf{X}_n, \mathbf{Y}_{n,X}]_{\mathscr{M}_D}\right)\\
&\subseteq \bigcap_{n>0}\bigcap_{X\in\mathscr{B}_n}\mathrm{Th}\left([0, X]_{\mathscr{B}_n}\right)\\
&= \mathrm{IPC}.
\end{aligned}$$

In order to find a D-mass problem $\mathcal{Z}$ as required above, we still need a couple of computability-theoretic results due to Kuyper, which we review in the next two lemmas. In the rest of this section, if g is a total function, then by g' we mean a total function which is $\equiv_T$ to the Turing jump of g.

Lemma 4.17. [10, Theorem 4.10] *Suppose that p, q are total functions such that $p' \leqslant_e q$, and let $k_0, k_1, \dots$ be total functions which are uniformly e-reducible to q and $k_i \nleqslant_e p$. Then there exists a total function f such that $p \leqslant_e f$, $f' \leqslant_e q$ and for every i, $q \leqslant_e f \oplus k_i$.*

Proof. See [10] where the statement is proved for sets and Turing reducibility. Our claim then follows from the fact that on total functions $\leqslant_T$ can be uniformly copied to $\leqslant_e$. ☐

If $\varphi \in \mathcal{P}$ then by $\varphi^{[i]}$ we mean the partial function $\varphi^{[i]}(x) = \varphi(\langle i, x\rangle)$; notice that if f is a total function then $f^{[i]}$ is total for every i. Furthermore, a partial function φ is called *computably e-independent* if for every i, $\varphi^{[i]} \nleqslant_e \bigoplus_{j \neq i} \varphi^{[j]}$, where we understand that if $I \subseteq \omega$ is a set of indices then $\bigoplus_{i \in I} \varphi^{[i]}$ is the partial function such that $\bigoplus_{i\in I} \varphi^{[i]}(\langle j, x\rangle) = \varphi_j(x)$ if $j \in I$, and $\bigoplus_{i\in I} \varphi^{[i]}(\langle j, x\rangle) = 0$ if $j \notin I$.

Lemma 4.18. [10, Lemma 4.11] *Let g be a computably e-independent total function. Then there exists a total function h such that $g \oplus h$ is computably e-independent, where we understand that the $2i$-th column of $g \oplus h$ is $g^{[i]}$ and the $2i+1$-th column is $h^{[i]}$.*

Proof. See [10] where a full proof is given for sets and Turing reducibility. ☐

Theorem 4.19. *If g is a computably e-independent total function, then $\mathrm{Th}([\mathbf{0}_D, \mathbf{Z}]_{\mathscr{M}_D}) = \mathrm{IPC}$, where $\mathbf{Z} = [\mathcal{Z}]_D$ and $\mathcal{Z} = \{i^\frown\varphi : i \in \omega, g^{[i]} \leqslant_e \varphi\}$.*

Proof. The proof is as in [10, Theorem 1.1], but replacing $\leqslant_T$ with $\leqslant_e$. For the sake of completeness and the ease of the reader we give the details as follows. Suppose that g is total and computably e-independent. Given $n > 0$ and $x_1, \dots, x_k \in \mathscr{P}(I_n)$, let us show first that there exist a downwards $\leqslant_e$-closed D-mass problem $\mathcal{A}$ and a strong u–antichain of total functions $f_1, \dots, f_n \in \mathcal{A}$ such that (in the notation of the proof of Corollary 4.16 but taking $I = I_n$),

$$(\mathcal{A}^c \cup \{f_1, \dots, f_n\}_{u\leqslant_e}) \oplus \left\{i^\frown\varphi : i \in \omega, g^{[i]} \leqslant_e \varphi\right\} \equiv_D \bigwedge_{1\leqslant j\leqslant k} (\mathcal{A}^c \cup \{f_i : i \in x_j\}_{u\leqslant_e}), \tag{1}$$

i.e.

$$\alpha(I_n) \oplus \mathcal{Z} \equiv_D \bigwedge_{1\leqslant j\leqslant k} \alpha(x_j).$$

By fixing $n > 0$ and $x_1, \dots, x_k \in \mathscr{P}(I_n)$ we have just fixed a pair n, X, with $n > 0$ and $X \in \mathscr{B}_n$, where $X = \bigwedge_{1\leqslant j\leqslant k}[x_j)$, so that

$$\begin{aligned}
\mathcal{Z} &= \left\{i^\frown\varphi : i \in \omega, g^{[i]} \leqslant_e \varphi\right\} \\
\mathcal{X}_n &= \mathcal{A}^c \cup \{f_1, \dots, f_n\}_{u\leqslant_e} \\
\mathcal{Y}_{n,X} &= \bigwedge_{1\leqslant j\leqslant k} \alpha(x_j),
\end{aligned}$$

obtaining $\mathcal{X}_n \oplus \mathcal{Z} \equiv_D \mathcal{Y}_{n,X}$.

Since $[0, X]_{\mathscr{B}_n} \overset{\mathbb{B}}{\hookrightarrow} [\mathbf{X}_n, \mathbf{Y}_{n,X}]_{\mathscr{M}_D}$, we have $\mathrm{Th}\left([\mathbf{X}_n, \mathbf{Y}_{n,X}]_{\mathscr{M}_D}\right) \subseteq \mathrm{Th}([0, X])_{\mathscr{B}_n})$.

To show the existence of $\mathcal{A}$, by Lemma 4.18 let h be a total function such that $g \oplus h$ is computably e-independent. Define

$$\mathcal{A} = \left(\{(g \oplus h)'\}_{u\leqslant_e}\right)^c$$

(where complementation is taken with respect to the set of all partial functions) so that $\mathcal{A}$ is clearly downwards $\leqslant_e$-closed. Given i with $1 \leqslant i \leqslant n$, apply Lemma 4.17 with $p = \left(\bigoplus_{1\leqslant j\leqslant k, i\in x_j} g^{[j]}\right) \oplus h^{[i]}$, $q = (g \oplus h)'$, and the sequence of k's listing the $g^{[j]}$ with either $i \notin x_j$ or $j > k$ together with the $h^{[j]}$ with $j \neq i$. The result is a total function f_i such that the following hold:

$$\begin{aligned}
&\Big(\bigoplus_{1\leqslant j\leqslant k, i\in x_j} g^{[j]}\Big) \oplus h^{[i]} \leqslant_e f_i,\\
&f_i' \leqslant_e (g \oplus h)',\\
&(\forall j)\Big[\big[[1 \leqslant j \leqslant k \,\&\, i \notin x_j] \vee j > k\big] \Rightarrow (g \oplus h)' \leqslant_e f_i \oplus g^{[j]}\Big],\\
&(\forall j \neq i)\big[(g \oplus h)' \leqslant_e f_i \oplus h^{[j]}\big].
\end{aligned}$$

We now show that $\{f_1, \dots, f_n\}$ is a strong u–antichain in $\mathcal{A}$. It is immediate to see that $f_i \in \mathcal{A}$ for every $1 \leqslant i \leqslant n$. If $1 \leqslant i < j \leqslant n$ then

$$(g \oplus h)' \leqslant_e h^{[i]} \oplus f_j \leqslant_e f_i \oplus f_j,$$

so that $f_i \oplus f_j \notin \mathcal{A}$.

Finally, we show equation (1). To see $\geqslant_D$, assume that

$$\psi \oplus \chi \in (\mathcal{A}^c \cup \{f_1, \dots, f_n\}_{u\leqslant_e}) \oplus \left\{i\hat{}\varphi : g^{[i]} \leqslant_e \varphi\right\} :$$

in particular let j be such that $\chi = j\hat{}\varphi$ and $g^{[j]} \leqslant_e \varphi$. We split the verification into cases:

(1) Assume first that $j = 0$ or $j > k$. We show in this case that $\psi \oplus \varphi \in \mathcal{A}^c \subseteq \alpha(x_1)$. Indeed, if $\psi \geqslant_e f_i$ for some $1 \leqslant i \leqslant n$ then

$$(g \oplus h)' \leqslant_e f_i \oplus g^{[j]} \leqslant_e \psi \oplus \varphi.$$

On the other hand, if $\psi \in \mathcal{A}^c$, i.e. $(g \oplus h)' \leqslant_e \psi$, then $(g \oplus h)' \leqslant_e \psi \oplus \varphi$.

In either case we have that $\psi \oplus \varphi \in \mathcal{A}^c$ and therefore the mapping $\psi \oplus (j\hat{}\varphi) \mapsto 1\hat{}(\psi \oplus \varphi)$ maps $\psi \oplus (j\hat{}\varphi)$ to an element in $1\hat{}(\alpha(x_1))$.

(2) Assume now that $1 \leqslant j \leqslant k$. In this case we show that $\psi \oplus \varphi \in \alpha(x_j)$. Indeed, if $\psi \geqslant_e f_i$ for some $i \in x_j$ then $f_i \leqslant_e \psi \oplus \varphi$, but then we can use the fact that $\{f_i\}_{u\leqslant_e} \subseteq \alpha(x_j)$. On the other hand, if $\psi \geqslant_e f_i$ for some $i \notin x_j$ then

$$(g \oplus h)' \leqslant_e f_i \oplus g^{[j]} \leqslant_e \psi \oplus \varphi \in \alpha(x_j).$$

In either case, we have that the mapping $\psi \oplus (j\hat{}\varphi) \mapsto j\hat{}(\psi \oplus \varphi)$ maps $\psi \oplus (j\hat{}\varphi)$ to an element in $j\hat{}(\alpha(x_j))$.

Finally, if $(g \oplus h)' \leqslant_e \psi$ then trivially $(g \oplus h)' \leqslant_e \psi \oplus \varphi$, and thus again $\psi \oplus \varphi \in \alpha(x_j)$. Even in this case, we have that the mapping $\psi \oplus (j\hat{}\varphi) \mapsto j\hat{}(\psi \oplus \varphi)$ maps $\psi \oplus (j\hat{}\varphi)$ to an element of $j\hat{}(\alpha(x_j))$.

So, uniformly in j such that $\chi(0) = j$ (if $\chi(0)\downarrow$ and this is certainly the case if $\chi \in \{i\hat{}\varphi : g^{[i]} \leqslant_e \varphi\}$) we can map, via an enumeration operator, the partial function $\psi \oplus \chi$ to some partial function in $r\hat{}(\alpha(x_r))$, for some $r \in \{1, \dots, k\}$.

We still have the prove $\leqslant_D$, i.e

$$(\mathcal{A}^c \cup \{f_1, \dots, f_n\}_{u \leqslant_e}) \oplus \left\{ i\hat{\ }\varphi : i \in \omega, g^{[i]} \leqslant_e \varphi \right\} \leqslant_D \bigwedge_{1 \leqslant j \leqslant k} \alpha(x_j).$$

But this is obvious as $\{f_r : r \in x_j\}_{u \leqslant_e} \subseteq \{f_1, \dots, f_n\}_{u \leqslant_e}$ and $\{f_r : r \in x_j\}_{u \leqslant_e} \subseteq \left\{\varphi : g^{[j]} \leqslant_e \varphi\right\}$, being $g^{[j]} \leqslant_e f_r$ for every $r \in x_j$. So the mapping $j\hat{\ }\varphi \mapsto \varphi \oplus (j\hat{\ }\varphi)$ induces an enumeration operator yielding the desired reduction. □

5. The Dyment-Muchnik lattice and intermediate logics: An initial segment of the Dyment-Muchnik lattice modeling exactly IPC

We begin by giving a convenient characterization of $\mathscr{M}_{D,w}$. A *mass problem of sets* is a subset $\mathcal{A} \subseteq 2^\omega$. On mass problems $\mathcal{A}$ and $\mathcal{B}$ of sets consider the relations $\leqslant_D$ and $\leqslant_{D,w}$ (for which we use the same symbols as for Dyment reducibility and Dyment-Muchnik reducibility on mass problems of partial functions) given respectively by

$$\mathcal{A} \leqslant_D \mathcal{B} \Leftrightarrow (\exists e\text{-operator } \Phi)\, [\Phi(\mathcal{B}) \subseteq \mathcal{A}],$$

and

$$\mathcal{A} \leqslant_{D,w} \mathcal{B} \Leftrightarrow (\forall B \in \mathcal{B})\, (\exists A \in \mathcal{A})\, [A \leqslant_e B].$$

These relations induce preordering relations, and thus, degree structures $\mathscr{M}_D^{set}$ and $\mathscr{M}_{D,w}^{set}$, on the collection of mass problems of sets. Now, the functions induced on e-degrees by the two mappings graph : $\mathcal{P} \longrightarrow 2^\omega$ and $s : 2^\omega \longrightarrow \mathcal{P}$, given by, respectively, $\varphi \mapsto \mathrm{graph}(\varphi)$ and $A \mapsto A \times \{0\}$, are easily seen to be order-theoretic homomorphisms which invert each other up to degree and therefore they yield an isomorphism $F : \mathscr{D}_e^{pf} \simeq \mathscr{D}_e^{set}$, showing that $\mathscr{M}_D \simeq \mathscr{M}_D^{set}$ and $\mathscr{M}_{D,w} \simeq \mathscr{M}_{D,w}^{set}$ via the isomorphisms induced in both cases by the assignment $\mathcal{A} \mapsto \{\mathrm{graph}(\varphi) : \varphi \in \mathcal{A}\}$, for every mass problem $\mathcal{A} \subseteq \mathcal{P}$. In view of this circumstance, hereafter in our investigation of the Dyment-Muchnik lattice as a Brouwer algebra we shall work with mass problems of sets of numbers[1], viewing $\mathscr{M}_{D,w}$ as $\mathscr{M}_{D,w}^{set}$. In this characterization of $\mathscr{M}_{D,w}$, the least element is the degree of any mass problem containing some c.e. set, the greatest element is the degree of the empty set, and the operations $\vee$, $\wedge$ and $\to$ are as follows: if $\mathcal{A}, \mathcal{B}$ are mass problems of sets, then $[\mathcal{A}]_{D,w} \vee [\mathcal{B}]_{D,w} = [\{A \oplus B : A \in \mathcal{A}, B \in \mathcal{B}]_{D,w}$, $[\mathcal{A}]_{D,w} \wedge [\mathcal{B}]_{D,w} = [\mathcal{A} \cup \mathcal{B}]_{D,w}$, and $[\mathcal{A}]_{D,w} \to [\mathcal{B}]_{D,w} = [\{C : (\forall A \in \mathcal{A})\, (\exists B \in \mathcal{B})\, [B \leqslant_e C \oplus A]\}]_{D,w}$. On the other hand, we know by Theorem 4.2 that $\mathscr{M}_{D,w} \simeq \mathrm{Up}(\mathscr{D}_e^{pf}) \simeq \mathrm{Up}(\mathcal{P}, \leqslant_e\rangle)$. The same argument is easily seen to show that $\mathscr{M}_{D,w}^{set} \simeq \mathrm{Up}(\mathscr{D}_e^{set}) \simeq \mathrm{Up}(\langle 2^\omega, \leqslant_e\rangle)$. Indeed, the mapping which sends an upwards $\leqslant_e$-closed subset $A \subseteq \mathscr{D}_e^{pf}$ to $\{F(\mathbf{a}) : \mathbf{a} \in A\}$ is an isomorphism of $\mathrm{Up}(\mathscr{D}_e^{pf})$ onto $\mathrm{Up}(\mathscr{D}_e^{set})$.

5.1. **More on Brouwer algebras and intermediate logics.** As is well known, every partial order $\mathscr{P}$ set can be viewed as a *Kripke frame*, and as such one can talk about $\mathrm{Th}_K(\mathscr{P})$, i.e. the set of propositional formulas that are valid in $\mathscr{P}$ according to Kripke semantics. We do not go into details here: see [2, 6] for suitable references. Suffice it to say that one can prove

Fact 5.1. $\mathrm{Th}_K(\mathscr{P}) = \mathrm{Th}(\mathrm{Up}(\mathscr{P}))$.

Proof. See for instance [2, Theorem 7.20]. □

[1] A similar approach, which considers mass problems of sets rather than mass problems of total functions, has been widely used in the literature on the Medvedev lattice and the Muchnik lattice, the two "Turing reducibility" counterparts of $\mathscr{M}_D$ and $\mathscr{M}_{D,w}$.

Now, given partial orders $\langle P_1, \leqslant_1\rangle$, $\langle P_2, \leqslant_2\rangle$, a function $f : P_1 \longrightarrow P_2$ is called a *p-morphism* (see [4]) if

- $x \leqslant_1 y$ implies $f(x) \leqslant_2 f(y)$, for all $x, y \in P_1$;
- for every $x \in P_1$, and $y \in P_2$, if $f(x) \leqslant_2 y$ then there exists $z \in P_1$ such that $x \leqslant_1 z$ and $f(z) = y$.

If there exists a p-morphism from $\mathscr{P}_1$ to $\mathscr{P}_2$ then $\mathrm{Th}_K(\mathscr{P}_1) \subseteq \mathrm{Th}_K(\mathscr{P}_2)$: see [2, Corollary 2.17].

Furthermore, let $\langle 2^{<\omega}, \preceq\rangle$ be the poset where $2^{<\omega}$ is the set of $0,1$-valued finite strings and $\preceq$ is the partial ordering on strings so that $x \preceq y$ if x is a prefix of y. It is known:

Lemma 5.2. [6, Theorem 4.12] $\mathrm{IPC} = \mathrm{Th}_K\left(\langle 2^{<\omega}, \preceq\rangle\right)$.

The following definition comes from [9]: in fact it is one of the equivalent characterizations proved by Kuyper of [9, Definition 3.1].

Definition 5.3. Let $\mathscr{U} = \langle U, +, \leqslant\rangle$ be an upper semilattice. A nonempty and countable downwards $\leqslant$-closed $A \subseteq U$ is a *splitting class* if and only if for every $a \in A$ and every finite $B \subseteq \{b \in A : b \nleqslant a\}$ there is $c \in A$, $c > a$, such that for all $b \in B$, $b + c \notin A$.

If $\mathscr{U} = \langle U, +, \leqslant\rangle$ is a pre-upper semilattice, then we say that a nonempty and countable downwards $\leqslant$-closed $A \subseteq U$ is a splitting class in $\mathscr{U}$ if the set of $\equiv$-classes of the elements lying in A form a splitting class in $\mathscr{U}_{/\equiv}$, see Remark 4.1. Of course, in this case, A is a splitting class in $\mathscr{U}$ if and only if $A_{/\equiv}$ is a splitting class in $\mathscr{U}_{/\equiv}$.

The following results (along the lines of [9]) constitute the final ingredients needed to reach our goal.

Lemma 5.4. [9, Proposition 5.6] *If A is a splitting class in an upper semilattice $\mathscr{U}$ with a least element, then there exists a p-morphism $f : \langle A, \leqslant\rangle \longrightarrow \langle 2^{<\omega}, \preceq\rangle$. Hence*

$$\mathrm{Th}_K\left(\langle A, \leqslant\rangle\right) \subseteq \mathrm{Th}_K\left(\langle 2^{<\omega}, \preceq\rangle\right) = \mathrm{IPC}\,.$$

Proof. See [9, Proposition 5.6] where the proof is given when $\mathscr{U}$ is the upper semilattice of Turing degrees, but it works for every upper semilattice with a least element. □

From these lemmata we get:

Theorem 5.5. [9, Proposition 5.2 and Theorem 5.7] *If A is a splitting class in $\mathscr{D}_e^{set}$ and $\mathcal{A} = \{B \in 2^\omega : \deg_e(B) \in A\}$ then*

$$\mathrm{Th}\left([\mathbf{0}_{D,w}, [\mathcal{A}^c]_{D,w}]_{\mathscr{M}_{D,w}}\right) = \mathrm{IPC},$$

where $\mathcal{A}^c = 2^\omega \smallsetminus \mathcal{A}$.

Proof. $\mathcal{A}$ is downwards $\leqslant_e$-closed in $\langle 2^\omega, \leqslant_e\rangle$, so

$$\mathrm{Up}\left(\langle \mathcal{A}, \leqslant_e\rangle\right) \simeq [2^\omega, \mathcal{A}^c]_{\mathrm{Up}(\langle 2^\omega, \leqslant_e\rangle)}$$

via the assignment $B \mapsto B \cup \mathcal{A}^c$, for every $B \in \mathrm{Up}(\langle \mathcal{A}, \leqslant_e\rangle)$. Since A is a splitting class in $\mathscr{D}_e^{set}$, then from Lemma 5.4, and the earlier observation in Fact 5.1 that for every partial order $\mathscr{P}$ we have $\mathrm{Th}_K(\mathscr{P}) = \mathrm{Th}(\mathrm{Up}(\mathscr{P}))$, it follows that

$$\mathrm{Th}_K\left(\langle A, \leqslant_e\rangle\right) = \mathrm{Th}\left(\mathrm{Up}(\langle A, \leqslant_e\rangle)\right) = \mathrm{Th}\left(\mathrm{Up}(\langle \mathcal{A}, \leqslant_e\rangle)\right) = \mathrm{Th}\left([2^\omega, \mathcal{A}^c]_{\mathrm{Up}(\langle 2^\omega, \leqslant_e\rangle)}\right) \subseteq \mathrm{IPC}\,.$$

But, $\mathscr{M}_{D,w} \simeq \mathrm{Up}(\langle 2^\omega, \leqslant_e\rangle)$ and, under this isomorphism, $2^\omega \mapsto \mathbf{0}_{D,w}$ and $\mathcal{A}^c \mapsto [\mathcal{A}^c]_{D,w}$. Hence

$$\mathrm{Th}\left([\mathbf{0}_{D,w}, [\mathcal{A}^c]_{D,w}]_{\mathscr{M}_{D,w}}\right) = \mathrm{IPC}\,. \qquad \square$$

We have just proved that if there exists a splitting class of e-degrees, then there exists an initial segment of $\mathscr{M}_{D,w}$ which models exactly IPC. It is only left to show that splitting classes of e-degrees do exist. This will be done in the next subsection.

5.2. **A splitting class in the e-degrees.** We will show that every countable downwards $\leqslant_e$-closed class $\mathcal{I}$ of sets can be extended to a countable splitting class. We will do this by inductively adding to $\mathcal{I}$ more and more sets, eventually getting a countable downwards $\leqslant_e$-closed set $\mathcal{I}_f$, with enough added elements to witness that for every pair $(A, \mathcal{B})$ where $A \in \mathcal{I}_f$ and $\mathcal{B} \subseteq \mathcal{I}_f$ is a finite set such that $B \nleqslant_e A$ for every $B \in \mathcal{B}$, there exists $C \in \mathcal{I}_f$ with $C >_e A$ and $C \oplus B \notin \mathcal{I}_f$ for every $B \in \mathcal{B}$. The following lemma helps describe the inductive steps in the construction of the splitting class.

Throughout this subsection we will treat a given set Γ of numbers as a *generalized enumeration operator*, i.e. as a mapping from 2^ω to 2^ω, such that

$$\Gamma(A) = \{x : (\exists u)\,[\langle x, u\rangle \in \Gamma \,\&\, D_u \subseteq A]\}.$$

If $\mathcal{A} \subseteq 2^\omega$ then, again, we denote $\mathcal{A}^c = 2^\omega \smallsetminus \mathcal{A}$.

Lemma 5.6. *Suppose that $\mathcal{I}, \mathcal{G}$ are sets of sets of numbers and that A, B are sets such that*

- *(a) $\mathcal{I}$ is countable and downwards $\leqslant_e$-closed;*
- *(b) $\mathcal{G} \subseteq \mathcal{I}^c$ is countable and closed under $\equiv_e$;*
- *(c) $A, B \in \mathcal{I}$ and $B \nleqslant_e A$.*

Then there exists a generalized enumeration operator Γ satisfying

- *(0) $\Gamma \oplus A \nleqslant_e A$*
- *(1) if $X \in \mathcal{I}$ then $\Gamma \oplus A \geqslant_e X$ if and only if $A \geqslant_e X$;*
- *(2) if $X \in \mathcal{G}$ then $\Gamma \oplus A \ngeqslant_e X$;*
- *(3) $\Gamma(B) \notin \mathcal{I}$;*
- *(4) $\Gamma \oplus A \ngeqslant_e \Gamma(B)$.*

Proof. Obtain Γ by forcing with $\langle P, \leqslant_P\rangle$ where

I. each condition $p \in P$ is a triple $p = \langle \Gamma_p, B_p, N_p\rangle$, where
   - Γ_p is a finite set of axioms $\langle n, u\rangle$ where $n \in \omega$ and u is the canonical index of the finite set D_u;
   - B_p is a finite subset of B;
   - N_p is a finite subset of ω.

II. $p \leqslant_P q$ if
   - $\Gamma_q \subseteq \Gamma_p$ (each axiom of Γ_q is an axiom of Γ_p);
   - $B_q \subseteq B_p$ and $N_q \subseteq N_p$;
   - $(\forall \langle n, u\rangle \in \Gamma_p \smallsetminus \Gamma_q)\ [B_p \subseteq D_u]$ (the new axioms must use B_p);
   - $(\forall \langle n, u\rangle \in \Gamma_p \smallsetminus \Gamma_q)\ [n \notin N_p]$ (no new axiom enumerates elements of N_p).

We will exhibit appropriate dense sets, so that if Γ comes from a sufficiently generic filter then it satisfies the claims of the lemma.

The following sublemma addresses item (0) of Lemma 5.6.

Sublemma 4. *$\Gamma \not\leqslant_e A$.*

Proof of Sublemma 4. Let Θ be an e-operator. The set of conditions

$$\mathcal{C} = \{p : (\exists m)\ [m \in \Theta(A) \,\&\, p \Vdash m \notin \Gamma]\} \cup \{(\exists m)\ [m \notin \Theta(A) \,\&\, p \Vdash m \in \Gamma]\}$$

is dense. To see this, suppose $p = \langle \Gamma_p, B_p, N_p \rangle$ is a condition. Let $m = \langle m_0, m_1 \rangle$ be such that m, m_0, m_1 are larger than any number mentioned in p, with $D_{m_1} \supseteq B_p$. If $m \in \Theta(A)$, then let $q = \langle \Gamma_p, B_p, N_p \cup \{m_0\} \rangle$ so that $q \Vdash \langle m_0, m_1 \rangle \notin \Gamma$. If $m \notin \Theta(A)$, then let $q = \langle \Gamma_p \cup \{m\}, B_p, N_p \rangle$. In either case there is a condition $q \leqslant_P p$ such that $q \in \mathcal{C}$. ☐

The following sublemma addresses items (1) and (2) of Lemma 5.6.

Sublemma 5. *For $X \in \mathcal{I}$, $\Gamma \oplus A \geqslant_e X$ if and only if $A \geqslant_e X$. For $X \in \mathcal{G}$, $\Gamma \oplus A \not\geqslant_e X$.*

Proof of Sublemma 5. Let $X \in \mathcal{I} \cup \mathcal{G}$. Suppose that Θ is an e-operator and that $\Theta(\Gamma \oplus A) = X$. This implies that the set of conditions

$$\begin{aligned} \mathcal{C} &= \{p : p \Vdash \Theta(\Gamma \oplus A) \neq X\} \\ &= \{p : (\exists m)\ [m \notin X \,\&\, p \Vdash m \in \Theta(\Gamma \oplus A)]\} \cup \{p : (\exists m)\ [m \in X \,\&\, p \Vdash m \notin \Theta(\Gamma \oplus A)]\} \end{aligned}$$

is not dense. (Here, if $p = \langle \Gamma_p, B_p, N_p \rangle$ is a condition, then $p \Vdash m \in \Theta(\Gamma \oplus A)$ means $m \in \Theta(\Gamma_p \oplus A)$, and $p \Vdash m \notin \Theta(\Gamma \oplus A)$ means that $(\forall q \leqslant_P p)\,[m \notin \Theta(\Gamma_q \oplus A)]$. Analogous definitions will be tacitly assumed throughout the proof of Lemma 5.6.) Hence there exists a condition p with no extension in this set, which implies

$$(\forall m)\,\big[m \in X \Leftrightarrow (\exists q \leqslant_P p)\,[m \in \Theta(\Gamma_q \oplus A)]\big].$$

Then $A \geqslant_e X$: indeed, A enumerates $m \in X$ by searching for a condition $q \leqslant_P p$ forcing $m \in \Theta(\Gamma_q \oplus A)$. The set of such conditions q is positively c.e. in A. So we conclude that

$$(\forall X \in \mathcal{I} \cup \mathcal{G})\,[\Gamma \oplus A \geqslant_e X \Leftrightarrow A \geqslant_e X]. \qquad ☐$$

The next sublemma addresses item (3) of Lemma 5.6.

Sublemma 6. *$\Gamma(B) \notin \mathcal{I}$.*

Proof of Sublemma 6. For every condition $p = \langle \Gamma_p, B_p, N_p \rangle$ and $X \in \mathcal{I}$, condition p has an extension in the set $\mathcal{C}$ of conditions

$$\mathcal{C} = \{q : (\exists m)\ [m \in X \,\&\, q \Vdash m \notin \Gamma(B)]\} \cup \{q : (\exists m)\ [m \notin X \,\&\, q \Vdash m \in \Gamma(B)]\},$$

which therefore is dense, and thus $X \neq \Gamma(B)$. Indeed, to see density, let $p = \langle \Gamma_p, B_p, N_p \rangle$ be a condition, and let m be larger than any number mentioned in p. If $m \in X$, then the condition $q = \langle \Gamma_p, B_p, N_p \cup \{m\} \rangle$ extends p and forces $m \notin \Gamma(B)$. If $m \notin X$, then the condition $q = \langle \Gamma_p \cup \{\langle m, u \rangle\}, B_p, N_p \rangle$ where $D_u = B_p$ extends p with $q \Vdash m \in \Gamma(B)$. ☐

The next sublemma addresses item (4) of Lemma 5.6.

Sublemma 7. *$\Gamma \oplus A \not\geqslant_e \Gamma(B)$.*

Proof of Sublemma 7. Suppose that $p = \langle \Gamma_p, B_p, N_p \rangle$ is a condition, and let Θ be an e-operator. Let $\mathcal{C}$ be the set of conditions

$$\mathcal{C} = \{q : (\exists m)\ [q \Vdash m \in \Theta(\Gamma \oplus A) \,\&\, q \Vdash m \notin \Gamma(B)]\}, \\ \cup \{q : (\exists m)\ [q \Vdash m \in \Gamma(B) \,\&\, q \Vdash m \notin \Theta(\Gamma \oplus A)]\}.$$

In Case 1 and Case 2 below, m and d vary through the numbers greater than any number mentioned in p.

<u>Case 1.</u> $(\exists m)\,(\exists d \notin B)\,(\exists q \leqslant_P p)\big[m \in \Theta(\Gamma_q \oplus A) \,\&\, (\forall \langle n, u\rangle \in \Gamma_q \smallsetminus \Gamma_p)\,[d \in D_u]\big]$.

Let

$$r = \langle \Gamma_q, B_q, N_q \cup \{m\}\rangle.$$

Then $r <_P p$, $r \Vdash m \notin \Gamma(B)$ since Γ_p does not mention m and the axioms in $\Gamma_q \smallsetminus \Gamma_p$ do not apply to B, and $r \Vdash m \in \Theta(\Gamma \oplus A)$. Hence p has an extension $r \in \mathcal{C}$.

<u>Case 2.</u> $(\exists m)\,(\exists d \in B)\,[\langle \Gamma_p, B_p \cup \{d\}, N_p\rangle \Vdash m \notin \Theta(\Gamma \oplus A)]$ (thus requiring that using d in all new axioms makes it impossible for $\Theta(\Gamma \oplus A)$ to enumerate m).

Let $r = \langle \Gamma_p \cup \{\langle m, u\rangle\}, B_p \cup \{d\}, N_p\rangle$, where $D_u = B_p \cup \{d\}$. Then $r <_P p$, $r \Vdash m \notin \Theta(\Gamma \oplus A)$, and $r \Vdash m \in \Gamma(B)$.

Again, p has an extension $r \in \mathcal{C}$.

<u>Case 3.</u> Otherwise.

Then, for all sufficiently large d,

$$d \in B \Leftrightarrow (\exists m)\,(\exists q \leqslant_P p)\big[m \in \Theta(\Gamma_q \oplus A) \ \&\ (\forall \langle n, u\rangle \in \Gamma_q \smallsetminus \Gamma_p)\,[d \in D_u]\big]. \tag{2}$$

Indeed, the righthand side of (2) implies that $d \in B$, since otherwise Case 1 would hold. On the other hand, if $d \in B$, then $\langle \Gamma_p, B_p \cup \{d\}, N_p\rangle \nVdash m \notin \Theta(\Gamma \oplus A)$ for every sufficiently large m since Case 2 does not hold. This means that there is a $q \leqslant_P \langle \Gamma_p, B_p \cup \{d\}, N_p\rangle \leqslant_P p$ with $m \in \Theta(\Gamma_q \oplus A)$, and $(\forall \langle n, u\rangle \in \Gamma_q \smallsetminus \Gamma_p)\,[d \in D_u]$ by the definition of extension. Thus the righthand side of (2) holds.

But then, contradicting the assumption, it follows that $B \leqslant_e A$ since the description in the right-hand side of (2) is positive in A.

It follows that every p has an extension in $\mathcal{C}$, so $\mathcal{C}$ is a dense set of conditions. ☐

In conclusion, if Γ is sufficiently generic and intersects the dense sets of conditions exhibited in the proof then the generalized enumeration operator Γ has the desired properties, which satisfy Lemma 5.6. ☐

Corollary 5.7. *Let $\mathcal{I}$, $\mathcal{G}$, A, B be as in the hypotheses of Lemma 5.6, and let Γ be the generalized enumeration operator built in the proof of the lemma. Define $\mathcal{I}_+ = \mathcal{I} \cup \{X : X \leqslant_e C\}$ where $C = \Gamma \oplus A$, and define $\mathcal{G}_+ = \mathcal{G} \cup \{X : X \equiv_e \Gamma(B)\}$. Then the following hold: $\mathcal{I}_+$ and $\mathcal{G}_+$ are countable, $\mathcal{I}_+ \supseteq \mathcal{I}$, $\mathcal{G}_+ \supseteq \mathcal{G}$, $\mathcal{I}_+$ is downwards $\leqslant_e$-closed, $\mathcal{G}_+$ is closed under $\equiv_e$, $\mathcal{G}_+ \subseteq (\mathcal{I}_+)^c$, $\Gamma(B) \in \mathcal{G}_+$ (hence $\Gamma(B) \notin \mathcal{I}_+$). Finally, $C >_e A$, $C \in \mathcal{I}_+$ (hence $C \notin \mathcal{G}_+$), and $B \oplus C \notin \mathcal{I}_+$.*

Proof. By construction $\mathcal{I}_+$ and $\mathcal{G}_+$ are countable, $\mathcal{I}_+$ is downwards $\leqslant_e$-closed, and $\mathcal{G}_+$ is closed under $\equiv_e$. To show that $\mathcal{G}_+ \subseteq (\mathcal{I}_+)^c$, it is enough to show that $\Gamma(B) \notin \mathcal{I}_+$: this follows from the fact that $\Gamma(B) \notin \mathcal{I}$ by Lemma 5.6(3), and $\Gamma(B) \nleqslant_e \Gamma \oplus A$ by Lemma 5.6(4). By Lemma 5.6(0) we have that $C >_e A$. By construction we have $C \in \mathcal{I}_+$. Finally, notice that $\Gamma(B) \leqslant_e B \oplus C$: thus, if $B \oplus C \in \mathcal{I}_+$ then $\Gamma(B) \in \mathcal{I}$ or $\Gamma(B) \leqslant_e \Gamma \oplus A$, but both of these two cases have already been excluded. In fact, $B \oplus C \leqslant_e \Gamma \oplus A$ can be excluded also for the fact that it would give $B \leqslant_e \Gamma \oplus A$ and thus, by Sublemma 5, $B \leqslant_e A$. □

Remark 5.8. We can look at the construction described in the proof of Lemma 5.6 as the partial assignment s_+, defined on all quadruples $(\mathcal{I}, \mathcal{G}, A, B)$ as in the hypotheses of Lemma 5.6, such that

$$s_+(\mathcal{I}, \mathcal{G}, A, B) = (\mathcal{I}_+, \mathcal{G}_+, \Gamma),$$

where

- $\mathcal{I}_+$ is the set constructed in Corollary 5.7 starting from $\mathcal{I}, \mathcal{G}, A, B$;
- $\mathcal{G}_+$ is the set constructed in Corollary 5.7 starting from $\mathcal{I}, \mathcal{G}, A, B$;
- Γ is the generalized enumeration operator constructed in Corollary 5.7 from $\mathcal{I}, \mathcal{G}, A, B$.

Lemma 5.9. *Suppose that $\mathcal{I}, \mathcal{G}$ are sets of sets of numbers, A is a set, and $\mathcal{B} = \{B_1, \ldots, B_k\}$ is a finite set of sets such that*

(a) *$\mathcal{I}$ is countable and downwards $\leqslant_e$-closed;*
(b) *$\mathcal{G} \subseteq \mathcal{I}^c$ is countable and closed under $\equiv_e$;*
(c) *$A \in \mathcal{I}$ and, for every $1 \leqslant i \leqslant k$, $B_i \in \mathcal{I}$ and $B_i \nleqslant_e A$.*

Then there exist sets $\mathcal{I}^+$, $\mathcal{G}^+$, and a set C such that $\mathcal{I}^+$ and $\mathcal{G}^+$ are countable, $\mathcal{I}^+ \supseteq \mathcal{I}$, $\mathcal{G}^+ \supseteq \mathcal{G}$, $\mathcal{I}^+$ is downwards $\leqslant_e$-closed, $\mathcal{G}^+$ is closed under $\equiv_e$, $\mathcal{G}^+ \subseteq (\mathcal{I}_+)^c$, $C >_e A$, $C \in \mathcal{I}^+$ (hence $C \notin \mathcal{G}^+$), and for every $1 \leqslant i \leqslant k$, there is a $D \leqslant_e B_i \oplus C$ with $D \in \mathcal{G}^+$ (hence $D \notin \mathcal{I}^+$).

Proof. Let us start with $\mathcal{I}, \mathcal{G}$ and a pair $(A, \mathcal{B})$ as in the statement of this lemma. Define a sequence $\{(\mathcal{I}_i, \mathcal{G}_i, \Gamma_i) : 1 \leqslant i \leqslant k\}$ as follows, where s_+ is the partial function defined in Remark 5.8:

- $(\mathcal{I}_1, \mathcal{G}_1, \Gamma_1) = s_+(\mathcal{I}_0, \mathcal{G}_0, A, B_1)$
- $(\mathcal{I}_{i+1}, \mathcal{G}_{i+1}, \Gamma_{i+1}) = s_+(\mathcal{I}_i, \mathcal{G}_i, \hat{\Gamma}_i \oplus A, B_{i+1})$ for $1 \leqslant i < k$,

where we understand that $\mathcal{I}_0 = \mathcal{I}$, $\mathcal{G}_0 = \mathcal{G}$, $\hat{\Gamma}_1 = \Gamma$, and $\hat{\Gamma}_{i+1} = \Gamma_{i+1} \oplus \hat{\Gamma}_i$.

The following claims are proved by induction on i with $1 \leqslant i \leqslant k$, using Lemma 5.6 and Corollary 5.7.

(0) $\hat{\Gamma}_i \oplus A \nleqslant_e A$ if $i = 1$, and $\hat{\Gamma}_i \oplus A \nleqslant_e \hat{\Gamma}_{i-1} \oplus A$ if $i > 1$;
(1) if $X \in \mathcal{I}_{i-1}$, then $\hat{\Gamma}_i \oplus A \geqslant_e X$ if and only if $\hat{\Gamma}_{i-1} \oplus A \geqslant_e X$;
(2) if $X \in \mathcal{G}_{i-1}$, then $\hat{\Gamma}_i \oplus A \ngeqslant_e X$;
(3) $\Gamma_i(B_i) \notin \mathcal{I}_{i-1}$;
(4) $\hat{\Gamma}_i \oplus A \ngeqslant_e \Gamma_i(B_i)$.

Moreover, it immediately follows from the construction and Corollary 5.7 that for every $1 \leqslant i \leqslant k$, we have $\mathcal{I}_{i-1} \subseteq \mathcal{I}_i$, $\mathcal{G}_{i-1} \subseteq \mathcal{G}_i$, $\hat{\Gamma}_i \oplus A \in \mathcal{I}_i$, $\Gamma_i(B_i) \in \mathcal{G}_i$, and for every $i \leqslant k$, $\mathcal{G}_i \subseteq (\mathcal{I}_i)^c$.

Define $C = \hat{\Gamma}_k \oplus A$, $\mathcal{I}^+ = \mathcal{I}_k$, and $\mathcal{G}^+ = \mathcal{G}_k$. For each $1 \leqslant i \leqslant k$, define $D_i = \Gamma_i(B_i)$. Then $\mathcal{I}^+$ and $\mathcal{G}^+$ are countable, extend $\mathcal{I}$ and $\mathcal{G}$ respectively, $\mathcal{I}^+$ is downwards $\leqslant_e$-closed, $\mathcal{G}^+$ is closed under $\equiv_e$, $\mathcal{G}^+ \subseteq (\mathcal{I}^+)^c$, $C >_e A$, and $C \in \mathcal{I}^+$. Furthermore, for each $1 \leqslant i \leqslant k$, $D_i \in \mathcal{G}_i \subseteq \mathcal{G}^+$ and $D_i \leqslant_e B_i \oplus \Gamma_i \leqslant_e B_i \oplus C$. □

Remark 5.10. We can look at the construction described in the proof of Lemma 5.9 as the (partial) assignment

$$s_{++}(\mathcal{I}, \mathcal{G}, A, \mathcal{B}) = (\mathcal{I}^+, \mathcal{G}^+, C)$$

where $\mathcal{I}^+$, $\mathcal{G}^+$, and C are as in the conclusion of Lemma 5.9 when starting from $\mathcal{I}$, $\mathcal{G}$, A, and $\mathcal{B}$.

Finally,

Theorem 5.11. *For any pair $\mathcal{I}, \mathcal{G}$ of sets of numbers such that*

(a) $\mathcal{I}$ is countable and downwards $\leqslant_e$-closed;
(b) $\mathcal{G} \subseteq \mathcal{I}^c$ is countable and closed under $\equiv_e$,

there exists a pair of countable sets $(\mathcal{I}_f, \mathcal{G}_f)$ such that $(\mathcal{I}_f, \mathcal{G}_f) \supseteq (\mathcal{I}, \mathcal{G})$ (meaning that $\mathcal{I}_f \supseteq \mathcal{I}$ and $\mathcal{G}_f \supseteq \mathcal{G}$), $\mathcal{I}_f$ is downwards $\leqslant_e$-closed, $\mathcal{G}_f$ is closed under $\equiv_e$, $\mathcal{G}_f \subseteq (\mathcal{I}_f)^c$, and $\mathcal{I}_f$ forms a splitting class.

Proof. Let $\mathcal{I}, \mathcal{G}$ be as in the statement of the theorem. We define by transfinite induction on $\beta \leqslant \omega^2$, a sequence of pairs $(\mathcal{I}_\beta, \mathcal{G}_\beta)$ such that $(\mathcal{I}_\beta, \mathcal{G}_\beta) \subseteq (\mathcal{I}_\gamma, \mathcal{G}_\gamma)$ for every $\beta < \gamma \leqslant \omega^2$. Our desired final pair $(\mathcal{I}_f, \mathcal{G}_f)$ will be $(\mathcal{I}_{\omega^2}, \mathcal{G}_{\omega^2})$.

$\beta = 0$. Let $(\mathcal{I}_0, \mathcal{G}_0) = (\mathcal{I}, \mathcal{G})$.

β *successor.* Suppose that $\beta = \omega \cdot j + i$ where i, j are natural numbers such that $j \geqslant 0$ and $i > 0$. Let $\{(A_r^{\omega\cdot j}, \mathcal{B}_r^{\omega\cdot j}) : r > 0\}$ be an enumeration of all pairs such that $A_r^{\omega\cdot j} \in \mathcal{I}_{\omega\cdot j}$ and $\mathcal{B}_r^{\omega\cdot j}$ is a finite subset of $\mathcal{I}_{\omega\cdot j}$ such that $B \nleqslant_e A_r^{\omega\cdot j}$ for all $B \in \mathcal{B}_r^{\omega\cdot j}$.

Let s_{++} be the partial function defined in Remark 5.10, and suppose

$$s_{++}(\mathcal{I}_{\omega\cdot j+i-1}, \mathcal{G}_{\omega\cdot j+i-1}, A_i^{\omega\cdot j}, \mathcal{B}_i^{\omega\cdot j}) = (\mathcal{I}^+, \mathcal{G}^+, C).$$

Define $\mathcal{I}_\beta = \mathcal{I}^+$ and $\mathcal{G}_\beta = \mathcal{G}^+$, and also record that $C_\beta = C$.

β *limit.* Define $\mathcal{I}_\beta = \bigcup_{\gamma<\beta} \mathcal{I}_\gamma$ and $\mathcal{G}_\beta = \bigcup_{\gamma<\beta} \mathcal{G}_\gamma$.

It is easy to see that each of $\mathcal{I}_f$ and $\mathcal{G}_f$ is union of a chain of sets. Each of the addenda in the union yielding $\mathcal{I}_f$ is downwards $\leqslant_e$-closed, and so is $\mathcal{I}_f$. Each of the addenda in the union yielding $\mathcal{G}_f$ is $\equiv_e$-closed, and so is $\mathcal{G}_f$.

In order to show that $\mathcal{G}_f \subseteq (\mathcal{I}_f)^c$, let us prove by ordinal induction on $\beta \leqslant \omega^2$ that $\mathcal{G}_\beta \subseteq (\mathcal{I}_\beta)^c$. The claim is straightforward if $\beta = 0$ or β is limit. So, suppose that β is a successor and has the form $\beta = \omega \cdot j + i$ with $j \geqslant 0$ and $i > 0$. We are assuming that $\mathcal{I}_{\omega\cdot j}$, $\mathcal{G}_{\omega\cdot j}$, $A_i^{\omega\cdot j}$, and $\mathcal{B}_i^{\omega\cdot j}$ satisfy the hypotheses of Lemma 5.9, so $\mathcal{I}_{\omega\cdot j+i-1}$, $\mathcal{G}_{\omega\cdot j+i-1}$, $A_i^{\omega\cdot j}$, and $\mathcal{B}_i^{\omega\cdot j}$ also satisfy these hypotheses because $\mathcal{I}_{\omega\cdot j} \subseteq \mathcal{I}_{\omega\cdot j+i-1}$ and $\mathcal{G}_{\omega\cdot j+i-1} \subseteq (\mathcal{I}_{\omega\cdot j+i-1})^c$. Thus $s_{++}(\mathcal{I}_{\omega\cdot j+i-1}, \mathcal{G}_{\omega\cdot j+i-1}, A_i^{\omega\cdot j}, \mathcal{B}_i^{\omega\cdot j})$ produces $\mathcal{I}_\beta$ and $\mathcal{G}_\beta$ with $\mathcal{G}_\beta \subseteq (\mathcal{I}_\beta)^c$.

Finally, let us show that $\mathcal{I}_f$ is a splitting class. Suppose that $A \in \mathcal{I}_f$ and that $\mathcal{B} = \{B_1, \ldots, B_k\} \subseteq \mathcal{I}_f$ is a finite set such that $B_r \nleqslant_e A$ for every $1 \leqslant r \leqslant k$. Suppose that $j \geqslant 0$ is least such that $A \in \mathcal{I}_{\omega\cdot j}$ and $\mathcal{B} \subseteq \mathcal{I}_{\omega\cdot j}$. Hence, for some $i > 0$, $A = A_i^{\omega\cdot j}$ and $\mathcal{B} = \mathcal{B}_i^{\omega\cdot j}$. At stage $\beta = \omega \cdot j + i$, we produced $(\mathcal{I}_\beta, \mathcal{G}_\beta, C_\beta) = s_{++}(\mathcal{I}_{\beta-1}, \mathcal{G}_{\beta-1}, A_i^{\omega\cdot j}, \mathcal{B}_i^{\omega\cdot j})$ via Lemma 5.9. Thus $C_\beta >_e A_i^{\omega\cdot j}$ and $C_\beta \in \mathcal{I}_\beta \subseteq \mathcal{I}_f$. Furthermore, $B_r \oplus C_\beta \notin \mathcal{I}_f$ for every $1 \leqslant r \leqslant k$. This is because $B_r \oplus C_\beta \geqslant_e D$ for some $D \in \mathcal{G}_\beta \subseteq \mathcal{G}_f$. So if $B_r \oplus C_\beta \in \mathcal{I}_f$, then D would be in $\mathcal{I}_f$ too because $\mathcal{I}_f$ is downwards $\leqslant_e$-closed, which would contradict that $\mathcal{G}_f \subseteq (\mathcal{I}_f)^c$. Thus C_β is the required witness for A and $\mathcal{B}$. $\square$

5.3. **More on splitting classes in the e-degrees.** Kuyper in [9] exhibited several "natural" classes of sets which form splitting classes in the Turing degrees, including the low Turing degrees, the 1-generic Turing degrees below the halting set K, and the Turing degrees of low_2 hyperimmune-free functions. He also generalized the notion to splitting classes of cardinality $\aleph_1$ (see [9, Definition 6.2]) for which Theorem 5.5 is still valid, and he showed that under CH the full class of Turing degrees of hyperimmune-free functions form an $\aleph_1$ splitting class.

We have exhibited a splitting class in the e-degrees, but one might object that this is, after all, only an ad hoc example. On the other hand, we have tested several natural downward $\leqslant_e$-closed classes of sets, but unfortunately they all turned out not to be splitting classes because of properties of $\mathcal{K}$-pairs. To illustrate these unsuccessful attempts, we need to recall some basic facts about $\mathcal{K}$-pairs in the e-degrees.

A pair $\{B_1, B_2\}$ of sets form a *$\mathcal{K}$-pair relative to a set C* if there exists a *$\mathcal{K}$-pair-defining* set $M \leqslant_e C$ for the pair, meaning that $B_1 \times B_2 \subseteq M$ and for every $\langle x, y\rangle \in M$, either $x \in B_1$ or $y \in B_2$. A $\mathcal{K}$-pair relative to C is *nontrivial* if $B_1 \nleqslant_e C$ and $B_2 \nleqslant_e C$. Clearly if $C \leqslant_e D$, then $\{B_1, B_2\}$ forms also a $\mathcal{K}$-pair relative to D. A pair $\{B_1, B_2\}$ is a *$\mathcal{K}$-pair* if it is a $\mathcal{K}$-pair relative to $\varnothing$. Consequently, a $\mathcal{K}$-pair is nontrivial if neither of the two sets is c.e.

In the following, for a given set X, we denote $K_X = \{e : e \in \Phi_e(X)\}$ and $J_e(X) = K_X \oplus (K_X)^c$. The set $J_e(X)$ is the *enumeration jump* of X. A set X is *e-low* if $J_e(X) \equiv_e J_e(\varnothing)$, or equivalently $J_e(X) \equiv_e K^c$. A set A is *quasi-minimal* if A is not c.e., and $f \leqslant_e A$ implies f is computable.

The following lemma collects several facts on $\mathcal{K}$-pairs, due to Kalimullin [8], used in the next four corollaries.

Lemma 5.12. *Let $\{B_1, B_2\}$ form a $\mathcal{K}$-pair relative to a set C. Then*

(1) $\{B_1 \oplus C, B_2 \oplus C\}$ form a $\mathcal{K}$-pair relative to C;
(2) If $X_1 \leqslant_e B_1$ and $X_2 \leqslant_e B_2$, then $\{X_1, X_2\}$ form a $\mathcal{K}$-pair relative to C;
(3) If $B_2 \nleqslant_e C$, then $B_1 \leqslant_e C \oplus B_2^c$ and $B_1^c \leqslant_e J_e(C) \oplus B_2$;
(4) If $\{B_1, B_2\}$ is a nontrivial $\mathcal{K}$-pair relative to C, then the e-degree of C is the meet of the e-degrees of $B_1 \oplus C$ and $B_2 \oplus C$. (In fact, for every $X \geqslant_e C$, the e-degree of X is the meet of the e-degrees of $B_1 \oplus X$ and $B_2 \oplus X$.)
(5) If $\{B_1, B_2\}$ is a nontrivial $\mathcal{K}$-pair, then both B_1 and B_2 are quasi-minimal.

Proof. See [8]. ☐

Corollary 5.13. *Let $\{B_1, B_2\}$ form a $\mathcal{K}$-pair below K^c. Then*

$$(\forall C \ngeqslant_e B_2)\,[J_e(B_1 \oplus C) \equiv_e J_e(C)].$$

Proof. Let $\{B_1, B_2\}$ be a $\mathcal{K}$-pair such that $B_1, B_2 \leqslant_e K^c$. Then $\{B_1, B_2\}$ is also a $\mathcal{K}$-pair relative to any set C. Hence by Lemma 5.12(1), $\{B_1 \oplus C, B_2 \oplus C\}$ is a $\mathcal{K}$-pair relative to C, and thus by Lemma 5.12(2) $\{K_{B_1 \oplus C}, B_2\}$ is a $\mathcal{K}$-pair relative to C. If $C \ngeqslant_e B_2$, by Lemma 5.12(3) applied to this $\mathcal{K}$-pair, we have that $(K_{B_1\oplus C})^c \leqslant_e J_e(C) \oplus B_2$. But $B_2 \leqslant J_e(\varnothing) \leqslant_e J_e(C)$, so $(K_{B_1\oplus C})^c \leqslant_e J_e(C)$. On the other hand, $K_{B_1 \oplus C} \leqslant_e B_1 \oplus C \leqslant_e B_1 \oplus J_e(C) \leqslant_e J_e(C)$ since $B_1 \leqslant_e J_e(\varnothing) \leqslant_e J_e(C)$. It follows that $J_e(B_1 \oplus C) = K_{B_1\oplus C} \oplus (K_{B_1 \oplus C})^c \leqslant_e J_e(C)$, from which we get $J_e(B_1 \oplus C) \equiv_e J_e(C)$. ☐

Let $\mathcal{QM}$ be the class of quasi-minimal sets e-below $J_e(\varnothing)$, plus the c.e. sets.

Corollary 5.14. *$\mathcal{QM}$ is not a splitting class.*

Proof. We show that $\mathcal{QM}$ is not a splitting class by showing that there exist quasi-minimal sets B_1, B_2 for which there is no set $C \in \mathcal{QM}$ such that both $B_1 \oplus C$ and $B_2 \oplus C$ lie outside of $\mathcal{QM}$. Thus $\mathcal{QM}$ is not a splitting class since Definition 5.3 is falsified by taking for instance a to be the set $A = \varnothing$ and the finite family $\mathcal{B} = \{b_1, b_2\}$ to be the family consisting of the above two sets $\{B_1, B_2\}$, for which $B_1 \nleqslant_e A, B_2 \nleqslant_e A$, but there is no $C \in \mathcal{QM}$ such that both $B_1 \oplus C$ and $B_2 \oplus C$ lie outside of $\mathcal{QM}$. To show this, take a nontrivial $\mathcal{K}$-pair $\{B_1, B_2\}$. By Lemma 5.12(5), both B_1 and B_2 are quasi-minimal. Towards a contradiction, assume C to be a quasi-minimal set such that both $B_1 \oplus C$ and $B_2 \oplus C$ are not quasi-minimal. Then B_1 and B_2 are not e-reducible to C, so $\{B_1 \oplus C,\ B_2 \oplus C\}$ is a nontrivial $\mathcal{K}$-pair relative to C. Let D_1, D_2 be two total non c.e. sets e-reducible to $B_1 \oplus C$ and $B_2 \oplus C$ respectively. Then D_1, D_2 are not e-reducible to C (as C is quasi-minimal), so by Lemma 5.12(2) they are a nontrivial $\mathcal{K}$-pair relative to C. From here (again by Lemma 5.12(3) and by the totality of D_2), we have $D_1 \leqslant_e C \oplus D_2^c \leqslant_e C \oplus D_2$, which, together with $D_1 \leqslant_e C \oplus D_1$, by Lemma 5.12(4) implies $D_1 \leqslant_e C$, contradicting that C is quasi-minimal. □

Contrary to what happens in the Turing degrees, where the sets of low Turing degree form a splitting class (see [10]) we have the following.

Corollary 5.15. *The sets with low e-degree do not form a splitting class in the e-degrees.*

Proof. The argument is similar to the one given in the previous proof. Take a nontrivial $\mathcal{K}$-pair $\{B_1, B_2\}$ below K^c (see [8]). By Corollary 5.13 both B_1 and B_2 are e-low. Assume C to be an e-low set such that both $B_1 \oplus C$ and $B_2 \oplus C$ are not e-low. Then by Corollary 5.13 we have $J_e(B_1 \oplus C) \equiv_e J_e(C)$, which implies that $B_1 \oplus C$ is low, a contradiction. Therefore the pair $A = \varnothing$ and $\mathcal{B} = \{B_1, B_2\}$ witnesses that the e-low sets do not form a splitting class. □

Corollary 5.16. *The class of arithmetical sets $A \ngeqslant_e K^c$ (i.e. the arithmetical sets with e-degree not above the first e-jump) is not a splitting class.*

Proof. Take a $\mathcal{K}$-pair $\{B_1, B_2\}$ with B_1, B_2 in the class. Let C be in the class (in particular $K^c \nleqslant_e C$), and assume that $B_1 \oplus C$ and $B_2 \oplus C$ are not in the class. Then since $K^c \leqslant_e B_1 \oplus C$ and $K^c \leqslant_e B_2 \oplus C$, by Lemma 5.12(4) we have $K^c \leqslant_e C$, contradiction. Again, the pair $A = \varnothing$ and $\mathcal{B} = \{B_1, B_2\}$ witnesses that the arithmetical sets $\ngeqslant_e K^c$ do not form a splitting class. □

In view of these negative results we ask:

Question 5.17. *Are there natural splitting classes in the e-degrees?*

6. Evaluating $\mathrm{Th}(\mathscr{M}_D)$ and $\mathrm{Th}(\mathscr{M}_{D,w})$

Having shown that there exist initial segments of the Dyment lattice $\mathscr{M}_D$ and of the Dyment-Muchnik lattice $\mathscr{M}_{D,w}$ whose logics coincide with the theorems of the intuitionistic propositional logic, we may also infer:

Theorem 6.1. $\mathrm{Th}(\mathscr{M}_D) = \mathrm{Th}(\mathscr{M}_{D,w}) = \mathrm{JAN}$.

Proof. If $\mathscr{A}$ is a Brouwer algebra, and $\mathscr{B}$ is the Brouwer algebra obtained by adding to $\mathscr{A}$ an element on top of all the elements of $\mathscr{A}$, then for every *positive* propositional formula σ (i.e. a formula not containing $\neg$) we have that if σ is true in $\mathscr{B}$ then is true in $\mathscr{A}$ as well. This follows from the following observation: assume that σ is true in $\mathscr{B}$, and consider any evaluation of σ in $\mathscr{A}$. But this is also an evaluation of σ in $\mathscr{B}$, and gives 0 in $\mathscr{B}$. But then it gives 0 also in $\mathscr{A}$ as the only operation that can take elements of $\mathscr{A}$ (viewed as elements of $\mathscr{B}$) to 1 in $\mathscr{B}$ is $\neg$. It follows that the positive propositional formulas that are valid in $\mathscr{M}_D$ are exactly the positive formulas in IPC. On the other hand JAN $\subseteq$ Th($\mathscr{M}_D$), as it is easy to see that the greatest element of $\mathscr{M}_D$ is join-irreducible and this implies that $\neg\alpha \vee \neg\neg\alpha$ is valid in $\mathscr{M}_D$, as for every Dyment degree $\mathbf{A}$ we have that $\neg\mathbf{A} = \mathbf{1}_D$ if $\mathbf{A} \neq \mathbf{1}_D$, otherwise $\neg\mathbf{A} = \mathbf{0}_D$. The result follows by maximality of JAN, among the intermediate logics whose positive fragment coincides with that of IPC: see [7].

By a similar argument, having shown that there exists an initial segment of the Dyment-Muchnik lattice $\mathscr{M}_{D,w}$ whose logic coincides with the theorems of the intuitionistic propositional logic, we also conclude that Th($\mathscr{M}_{D,w}$) = JAN. □

The identity Th($\mathscr{M}_D$) = JAN also follows from the following theorem (as transfers to the Dyment lattice a result already proved in [21] for the Medvedev lattice), characterizing the finite Brouwer algebras that are embeddable in the Dyment lattice.

Theorem 6.2. *The finite Brouwer algebras that are embeddable into $\mathscr{M}_D$ coincide with the finite Brouwer algebras in which* 0 *is meet-irreducible and* 1 *is join-irreducible.*

Proof. (Sketch) In one direction (i.e., to show that every finite Brouwer algebra embeddable in $\mathscr{M}_D$ has meet-irreducible 0 and join-irreducible 1) the claim is trivial since $\mathscr{M}_D$ has meet-irreducible 0 and join-irreducible 1. For the other direction, we observe that the proof given in [21] for the Medvedev lattice makes use of Medvedev mass problems of the form $\mathcal{B}_f = \{g : g \nleqslant_T f\}$ and is based on the fact that the degrees of these mass problems are join-irreducible and form a canonical set. But this is true also for the Dyment mass problems of the form $\mathcal{B}_\varphi = \{\varphi : \psi \nleqslant_e \varphi\}$. Indeed, checking join-irreducibility in this case is straightforward, and these mass problems are Dyment–Muchnik mass problems, so Lemma 4.13 applies to them. The remainder of the proof in [21] is purely algebraic and carries over unchanged to the setting of the Dyment lattice. □

References

[1] R. Balbes and P. Dwinger. *Distributive Lattices.* University of Missouri Press, Columbia, Missouri, 1974.
[2] M. Chagrov and A. Zakharyashev. *Modal Logic.* Clarendon Press, Oxford, 1997.
[3] S. B. Cooper. *Computability Theory.* Chapman & Hall/CRC Mathematics, Boca Raton, London, New York, Washington, DC, 2003.
[4] D.H.J. de Jongh and A.S. Troelstra. On the connection of partially ordered sets with some pseudo-Boolean algebras. *Indag. Math.*, 28:317–329, 1966.
[5] E. Z. Dyment. Certain properties of the Medvedev lattice. *Mathematics of the USSR Sbornik*, 30:321–340, 1976. English Translation.
[6] D. M. Gabbay. *Semantical Investigations In Heyting's Intuitionistic Logic.* Number 148 in Synthese Library. Springer-Science+Business Media, B.V., Dordrecht, 1981.
[7] V. A. Jankov. The calculus of the weak law of excluded middle. *Math. USSR Izvestija*, 2:997–1004, 1968.
[8] I. Sh. Kalimullin. Definability of the jump operator in the enumeration degrees. *J. Math. Log.*, 3(2):257–267, 2003.
[9] R. Kuyper. Natural factors of the Muchnik lattice capturing IPC. *Ann. Pure Appl. Logic*, 164(10):1025–1036, 2013.
[10] R. Kuyper. Natural factors of the Medvedev lattice capturing IPC. *Arch. Math. Logic*, 53(7):865–879, 2014.
[11] A. H. Lachlan and R. Lebeuf. Countable initial segments of the degrees. *J. Symbolic Logic*, 41:289–300, 1976.
[12] K. McEvoy. *The Structure of the Enumeration Degrees.* PhD thesis, School of Mathematics, University of Leeds, 1984.
[13] Y. T. Medevdev. Degrees of difficulty of the mass problems. *Dokl. Nauk. SSSR*, 104(4):501–504, 1955.
[14] A. A. Muchnik. On strong and weak reducibility of algorithmic problems. *Sibirskii Matematicheskii Zhurnal*, 4:1328–1341, 1963. Russian.

[15] J. Myhill. A note on degrees of partial functions. *Proc. Amer. Math. Soc.*, 12:519–521, 1961.
[16] H. Rasiowa and R. Sikorski. *The Mathematics of Metamathematics*. Panstowe Wydawnictwo Naukowe, Warszawa, 1963.
[17] H. Rogers, Jr. *Theory of Recursive Functions and Effective Computability*. McGraw-Hill, New York, 1967.
[18] E. Z. Skvortsova. Faithful interpretation of the intuitionistic propositional calculus by an initial segment of the Medvedev lattice. *Sib. Math. J.*, 29(1):133–139, 1988.
[19] R. I. Soare. *Recursively Enumerable Sets and Degrees*. Perspectives in Mathematical Logic, Omega Series. Springer-Verlag, Heidelberg, 1987.
[20] A. Sorbi. Some remarks on the algebraic structure of the Medvedev lattice. *J. Symbolic Logic*, 55(2):831–853, 1990.
[21] A. Sorbi. Embedding Brouwer algebras in the Medvedev lattice. *Notre Dame J. Formal Logic*, 32(2):266–275, 1991.
[22] A. Sorbi. The Medvedev lattice of degrees of difficulty. In S. B. Cooper, T. A. Slaman, and S. S. Wainer, editors, *Computability, Enumerability, Unsolvability - Directions in Recursion Theory*, London Mathematical Society Lecture Notes Series, pages 289–312. Cambridge University Press, New York, 1996.
[23] A. Sorbi and S. Terwijn. Intuitionistic logic and Muchnik degrees. *Algebra Universalis*, 67(2):175–188, 2012.

Department of Mathematical Logic, Faculty of Mathematics and Computer Science, Sofia University, Sofia

Email address: ganchev@fmi.uni-sofia.bg

School of Mathematics, University of Leeds, Leeds, LS2 9JT, United Kingdom

Email address: p.e.shafer@leeds.ac.uk

Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720, USA

Email address: slaman@math.berkeley.edu

Dipartimento di Ingegneria Informatica e Scienze Matematiche, Università Degli Studi di Siena, I-53100 Siena, Italy

Email address: andrea.sorbi@unisi.it

Department of Mathematics, University of Wisconsin, Madison, USA

Email address: msoskova@math.wisc.edu