TENNENBAUM-LIKE THEOREMS FOR COHESIVE POWERS

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ABSTRACT. We investigate the encoding ability of the cohesive power construction. We compute a graph $\mathcal{G}$ where the cohesive power $\prod_C \mathcal{G}$ of $\mathcal{G}$ by any Δ_2 cohesive set C has degree $0''$. That is, $0''$ computes a presentation of $\prod_C \mathcal{G}$, and every presentation of $\prod_C \mathcal{G}$ computes $0''$. We also compute a linear order $\mathcal{L}$ where no cohesive power of $\mathcal{L}$ has a computable presentation. We accomplish this by ensuring that if $\mathcal{P}$ is a presentation of a cohesive power of $\mathcal{L}$, then $\mathcal{P}''$ has PA-degree relative to $0''$.

1. INTRODUCTION

We investigate the encoding ability of the cohesive power construction. A *cohesive power* is a miniature power of a computable structure $\mathcal{A}$ built from a cohesive subset of $\mathbb{N}$ and partial computable functions $\mathbb{N} \to |\mathcal{A}|$ instead of from an ultrafilter on $\mathbb{N}$ and all functions $\mathbb{N} \to |\mathcal{A}|$. These powers are countable structures because there are only countably many partial computable functions. For this reason, they can be analyzed by traditional techniques from computable structure theory that focus on countable structures, as we do in this article.

Close relatives of cohesive powers of the standard model of arithmetic $\mathcal{N} = (\mathbb{N}; 0, 1, +, \times, <)$ have been widely studied for a long time. Even Skolem's original construction of a countable non-standard model of arithmetic [29] may be seen as a miniature ultrapower. Skolem's model, and indeed all countable non-standard models of arithmetic, were shown to have no computable copies by Tennenbaum [31]. Even more closely related to cohesive powers are the *recursive ultrapowers* (aka *Nerode semi-rings*), *r.e. ultrapowers*, and similar constructions studied in works including [5, 9, 10, 14, 16–21, 28]. Shavrukov [28] observes that cohesive powers as studied here coincide with his *r.e. prime powers*. The idea to study powers of arbitrary computable structures via cohesive sets is more recent, originating with Dimitrov [3].

Łoś's theorem for cohesive powers (see Theorem 2.5 below) only holds for sentences that are Δ_3-expressible, and a cohesive power of $\mathcal{N}$ will not be a model of Peano arithmetic. In fact, $\mathsf{B}\Sigma_1$ fails in cohesive powers of $\mathcal{N}$ [28, Proposition 2.11]. Nevertheless, preserving Δ_3 facts is enough for Tennenbaum's no-computable-copy theorem to apply. To wit, Shavrukov [28] shows that cohesive powers of $\mathcal{N}$ model the Π_2 theory of true arithmetic, and it follows that Tennenbaum's theorem applies. We give the details using Łoś's theorem for cohesive powers, which we explain more fully later on.

Theorem 1.1 (essentially [28, Corollary 2.8]). *Let $\mathcal{N} = (\mathbb{N}; 0, 1, +, \times, <)$ be the standard model of arithmetic, and let $C \subseteq \mathbb{N}$ be a cohesive set. Then the cohesive power $\prod_C \mathcal{N}$ of $\mathcal{N}$ over C does not have a computable presentation.*

Proof. A priori, when discussing arithmetic in this context and intending to apply a version of Łoś's theorem for cohesive powers, one must take care to distinguish between the formula classes Σ_n and Π_n, where the matrix formula is allowed bounded quantifiers, and the formula classes $\exists_n$ and $\forall_n$, where the matrix formula is quantifier-free. However, Δ_0 formulas in arithmetic are decidable in $\mathcal{N}$, and therefore Lemma 2.3 applies to them. Specifically, Lemma 2.3 item (2) implies that $\prod_C \mathcal{N}$ satisfies the Π_2 theory of true arithmetic. It follows that $\prod_C \mathcal{N} \models \mathsf{I}\Delta_0$. Tennenbaum's theorem applies to non-standard models of $\mathsf{I}\Delta_0$ (see, for example, [12, Theorem 11.8]), and $\prod_C \mathcal{N}$ is non-standard because the identity function represents a non-standard element. Therefore $\prod_C \mathcal{N}$ does not have a computable presentation. □

The present article is motivated by Tennenbaum's theorem and the investigations in [4, 27] concerning cohesive powers of computable copies of the linear order $(\mathbb{N}; <)$. These works examine how the theory of cohesive powers deviates from the classical theory of ultrapowers, even for basic structures like $(\mathbb{N}; <)$. For the usual presentation of $(\mathbb{N}; <)$, where the successor function is computable, and any cohesive set C, the cohesive power $\prod_C(\mathbb{N}; <)$ has order-type $\omega + \zeta \cdot \eta$. Here ω, ζ, and η denote the order-types of

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$(\mathbb{N}; <)$, $(\mathbb{Z}; <)$, and $(\mathbb{Q}; <)$. This is expected, as $\omega + \zeta \cdot \eta$ is the order type of any non-standard model of arithmetic and is elementary equivalent to ω. However, it is possible to compute a linear order $\mathcal{L}$ that is classically (but not computably) isomorphic to $(\mathbb{N}; <)$ where for some cohesive sets C, the cohesive power $\prod_C \mathcal{L}$ does not have order-type or even the same elementary class as $\omega + \zeta \cdot \eta$. For example, there is a fixed computable copy $\mathcal{L}$ of ω such that $\prod_C \mathcal{L} \cong \omega + \eta$ for every Δ_2 cohesive set C. While this is perhaps surprising, it is not Tennenbaum-like; even though $\omega + \eta$ is quite different from ω, it still has a computable copy. More than just $\omega + \eta$ can be produced in this way, however. The most general theorem known in this direction is the following.

Theorem 1.2 ([27, Theorem 4.5])**.** *Let $X \subseteq \mathbb{N} \setminus \{0\}$ be a Boolean combination of Σ_2 sets, thought of as a set of finite order-types. Then there is a computable copy $\mathcal{L}$ of ω such that for every Δ_2 cohesive set C, the cohesive power $\prod_C \mathcal{L}$ has order-type $\omega + \boldsymbol{\sigma}(X \cup \{\omega + \zeta \cdot \eta + \omega^*\})$. Moreover, if X is finite and non-empty, then there is also a computable copy $\mathcal{L}$ of ω such that for every Δ_2 cohesive set C, the cohesive power $\prod_C \mathcal{L}$ has order-type $\omega + \boldsymbol{\sigma}(X)$.*

Here, $\boldsymbol{\sigma}$ denotes the *shuffle sum*, and ω^* denotes the reverse of ω. The shuffle sum $\boldsymbol{\sigma}(\{\mathcal{L}_i\}_{i\in\mathbb{N}})$ of a countable collection of linear orders is obtained by coloring $(\mathbb{Q}; <)$ with countably many colors in such a way that every color is dense and then replacing each point of color i by a copy of order $\mathcal{L}_i$.

It is natural to wonder if any of these constructions are Tennenbaum-like, in other words, if any of these linear orders fail to have computable copies. A *block* in a linear order $\mathcal{L}$ is a maximal set $B \subseteq |\mathcal{L}|$ such that whenever a and b are in B with $a \prec_{\mathcal{L}} b$, the interval $[a, b]$ is finite. For a computable linear order $\mathcal{L}$, the set $\{n : \mathcal{L} \text{ has a block of size } n\}$ is Σ_3. If we identify $X \subseteq \mathbb{N} \setminus \{0\}$ with the corresponding set of finite order-types, then X is exactly the sizes of the finite blocks of $\omega + \boldsymbol{\sigma}(X \cup \{\omega + \zeta \cdot \eta + \omega^*\})$. If X is a Boolean combination of Σ_2 sets as in Theorem 1.2, or indeed even a Σ_3 set, then $\boldsymbol{\sigma}(X \cup \{\omega + \zeta \cdot \eta + \omega^*\})$ and hence $\omega + \boldsymbol{\sigma}(X \cup \{\omega + \zeta \cdot \eta + \omega^*\})$ has a computable copy by (a slight modification of) [1, Lemma 2.1]. Indeed, all of the order-types produced by Theorem 1.2 have computable copies, so are not Tennenbaum-like. On the other hand, if X is Π_3 but not Σ_3, then $\omega + \boldsymbol{\sigma}(X \cup \{\omega + \zeta \cdot \eta + \omega^*\})$ is not a computable order-type. This situation raises several questions.

Questions.

(1) In Theorem 1.2, can 'a Boolean combination of Σ_2 sets' be improved to 'a Π_3 set?'

(2) Is there a computable copy $\mathcal{L}$ of ω where the cohesive power $\prod_C \mathcal{L}$ has no computable presentation for some cohesive set C? For all cohesive sets C?

(3) Is there a computable linear order $\mathcal{L}$ (not necessarily of order-type ω) where the cohesive power $\prod_C \mathcal{L}$ has no computable presentation for some cohesive set C? For all cohesive sets C?

(4) More generally, what is the encoding strength of the cohesive power construction?

We answer Question (3) and address the open-ended Question (4). Questions (1) and (2) remain open.

These questions are of particular interest because of the special place that linear orders have in computable structure theory. Linear orders are a rich class of structures where a lot of complicated constructions are possible. More formally, linear orders are on top for Borel reducibility, so any other structure can be transformed into a linear order in a concrete way that preserves isomorphism [6]. In contrast, unlike ultra-rich structures like graphs or groups, linear orders are unable to computably interpret every other type of structure [7,8]. This means that any transformation of a structure into a linear order necessarily destroys some computability-theoretic information about the structure in the process. Linear orders are complicated enough to be interesting but not so complicated that you can get away with an argument that focuses on coding a different type of structure. They merit individual consideration. Linear orders are an ideal place to test the understanding of new ideas and push the sophistication of the known constructions.

The study of what degrees can compute a structure, or the degree spectrum of a structure, is quite important in computable structure theory (see [23, Chapter V]). Unsurprisingly, the degree spectra of linear orders have also been the specific subject of study in works such as [1, 11, 13, 22, 25]. By addressing the questions regarding cohesive powers of linear orders, this article falls into this tradition. In particular, we demonstrate a new way to construct linear orders that do not have computable copies and therefore have interesting degree spectra.

This article is organized as follows. Section 2 presents some background material on computable structures and cohesive powers. Section 3 addresses Question (4). We show that there is a computable structure that can smuggle arbitrary information into its cohesive powers via the cohesive sets (Proposition 3.1). However, if one restricts to Δ_n cohesive sets (for $n \geq 2$), then the cohesive power of a computable structure over a Δ_n cohesive set always has a Δ_{n+1} presentation (Proposition 3.2). Conversely, there is a computable graph $\mathcal{G}$ such that for every cohesive set C, every presentation of $\prod_C \mathcal{G}$ computes $0''$ (Theorem 3.3). For this graph $\mathcal{G}$ and any Δ_2 cohesive set C, the cohesive power $\prod_C \mathcal{G}$ therefore has degree $0''$: $0''$ computes a presentation of $\prod_C \mathcal{G}$, and every presentation of $\prod_C \mathcal{G}$ computes $0''$. Section 4 answers Question (3). We compute a linear order $\mathcal{L}$ such that $\prod_C \mathcal{L}$ has no computable presentation for any cohesive set C. In fact, $\mathcal{L}$ has the property that for every cohesive set C, the double-jump $\mathcal{P}''$ of every presentation $\mathcal{P}$ of $\prod_C \mathcal{L}$ has PA-degree relative to $0''$ (Theorem 4.3). Finally, Section 5 raises a few further questions.

2. Background

We refer the reader to [15, 30] for computability theory in general and to [2, 23, 24] for computable structure theory in particular. We also refer the reader to [26] for the theory of linear orders.

Computability notation and definitions. Partial computable (aka partial recursive) functions $\mathbb{N} \to \mathbb{N}$ are denoted by φ, ψ, etc. For a partial computable $\varphi : \mathbb{N} \to \mathbb{N}$, $\varphi(n)\downarrow$ means that φ halts on input n, and $\varphi(n)\uparrow$ means that φ does not halt on input n. Let $\mathrm{dom}(\varphi) = \{n : \varphi(n)\downarrow\}$. A set $W \subseteq \mathbb{N}$ is *computably enumerable* (aka *recursively enumerable*) if $W = \mathrm{dom}(\varphi)$ for some partial computable φ. The c.e. sets are also exactly the sets that are Σ_1-definable in arithmetic. Let $(\varphi_e)_{e \in \mathbb{N}}$ denote the usual effective enumeration of all partial computable functions $\mathbb{N} \to \mathbb{N}$. Sometimes we also write $\varphi_0, \dots, \varphi_{m-1}$ to denote a sequence of m partial computable functions; the usage will be clear from context. For each $n \geq 2$, $\langle x_0, \dots, x_{n-1} \rangle \colon \mathbb{N}^n \to \mathbb{N}$ denotes the usual computable bijective n-tupling function, and for each $i < n$, π_i denotes the projection function onto coordinate i. For $\sigma, \tau \in \mathbb{N}^{<\mathbb{N}}$ and $f \in \mathbb{N}^{\mathbb{N}}$, $\sigma \sqsubseteq \tau$ denotes that σ is an initial segment of τ, and $\sigma \sqsubseteq f$ denotes that σ is an initial segment of f. For $f \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$, $f{\restriction}n = (f(0), f(1), \dots, f(n-1))$ denotes the initial segment of f of length n. We identify $\mathcal{P}(\mathbb{N})$ with $2^{\mathbb{N}}$ and often view their elements as infinite binary sequences.

For sets A and B, $A \subseteq^* B$ means that $A \setminus B$ is finite: A is a subset of B except for finitely many exceptions. A set $C \subseteq \mathbb{N}$ is *cohesive* if for every c.e. set W, either $C \subseteq^* W$ or $C \subseteq^* \mathbb{N} \setminus W$. A cohesive set cannot be Σ_1, but a famous theorem of Friedberg (see [30, Theorem X.3.3]) guarantees that there are Π_1 cohesive sets.

Section 4 makes use of PA degrees and separating sets. A function $f \colon \mathbb{N} \to 2$ is DNC_2 (aka DNR_2) relative to a function $g \colon \mathbb{N} \to \mathbb{N}$ if $\forall e\,(\varphi_e^g(e)\downarrow \;\to\; f(e) \neq \varphi_e^g(e))$. A function $f \colon \mathbb{N} \to \mathbb{N}$ is said to have PA-*degree* relative to $g \colon \mathbb{N} \to \mathbb{N}$ if f computes a function that is DNC_2 relative to g. The terminology is explained by the classical fact that DNC_2 functions compute complete consistent extensions of Peano arithmetic and *vice versa.* If $A, B \subseteq \mathbb{N}$ are disjoint sets, then an *AB-separator* is a set $S \subseteq \mathbb{N}$ such that $A \subseteq S \subseteq \mathbb{N} \setminus B$. For any $g \colon \mathbb{N} \to \mathbb{N}$, let $A = \{e : \varphi_e^g(e) = 0\}$ and $B = \{e : \varphi_e^g(e) = 1\}$. Then A and B are disjoint sets that are c.e. relative to g, and the (characteristic functions of the) AB-separators are exactly the functions that are DNC_2 relative to g. Thus a function has PA-degree relative to g if and only if it computes an AB-separator for this A and B.

Computable structure theory notation and definitions. Fix a computable language $\mathfrak{L}$. Following [23, 24], a *presentation* or *copy* of a countable $\mathfrak{L}$-structure $\mathcal{M}$ is an isomorphic copy $\mathcal{A}$ of $\mathcal{M}$ whose domain is a (non-empty) subset of $\mathbb{N}$. In this case, $\mathcal{A}$ can be encoded as the object

$$\mathcal{P} \;=\; |\mathcal{A}| \;\oplus\; \bigoplus_{R \in \mathfrak{L}_{\mathsf{rel}}} R^{\mathcal{A}} \;\oplus\; \bigoplus_{F \in \mathfrak{L}_{\mathsf{func}}} F^{\mathcal{A}} \;\oplus\; \bigoplus_{R \in \mathfrak{L}_{\mathsf{const}}} c^{\mathcal{A}}$$

where $|\mathcal{A}| \subseteq \mathbb{N}$ is the domain of $\mathcal{A}$, and $\mathfrak{L}_{\mathsf{rel}}$, $\mathfrak{L}_{\mathsf{func}}$, and $\mathfrak{L}_{\mathsf{const}}$ are the relation, function, and constant symbols of $\mathfrak{L}$.

If $\mathcal{A}$ is an $\mathfrak{L}$-structure with $|\mathcal{A}| \subseteq \mathbb{N}$, then we typically identify $\mathcal{A}$ with its encoding $\mathcal{P}$, which itself can be identified with a subset of the natural numbers using the n-tupling function. Thus we say that $\mathcal{A}$ is *computable* if $\mathcal{P}$ is computable. So a computable $\mathfrak{L}$-structure consists of a non-empty computable domain and uniformly computable interpretations of the relation, function, and constant symbols of $\mathfrak{L}$. Similarly, a *uniformly computable sequence of $\mathfrak{L}$-structures* $(\mathcal{A}_n : n \in \mathbb{N})$ consists of a uniformly computable sequence $(|\mathcal{A}_n| : n \in \mathbb{N})$ of non-empty domains, along with uniformly computable interpretations of all the symbols of $\mathfrak{L}$ in the structures $(\mathcal{A}_n : n \in \mathbb{N})$. All of these concepts readily relativize.

For a countable structure $\mathcal{A}$ in a computable language, the *degree spectrum* of $\mathcal{A}$, denoted $\mathrm{DgSp}(\mathcal{A})$, is the collection of Turing degrees that can compute a copy of $\mathcal{A}$. If the degree spectrum of $\mathcal{A}$ has a least element, then that is called the *degree* of $\mathcal{A}$. So $\mathcal{A}$ has degree $\deg_{\mathrm{T}}(X)$ if $\mathcal{A}$ has an X-computable presentation and every presentation of $\mathcal{A}$ computes X. If no such degree exists, we say that $\mathcal{A}$ has no degree. It is a result of Richter [25] that if $\mathcal{L}$ is a linear order, then the only possible degree of $\mathcal{L}$ is $\mathbf{0}$.

Theorem 2.1 ([25]). *If $\mathcal{L}$ is a linear order, then $\mathrm{DgSp}(\mathcal{L})$ has a minimum element if and only if $\mathcal{L}$ has a computable copy. In particular, $\mathcal{L}$ has a degree if and only if it has a computable copy.*

Cohesive powers and products. Cohesive products and cohesive powers of computable structures are introduced by Dimitrov [3]. We give the definition as it is presented in [4].

Definition 2.2. Let $\mathfrak{L}$ be a computable language. Let $(\mathcal{A}_n : n \in \mathbb{N})$ be a uniformly computable sequence of $\mathfrak{L}$-structures with corresponding uniformly computable sequence of non-empty domains $(|\mathcal{A}_n| : n \in \mathbb{N})$. Let $C \subseteq \mathbb{N}$ be a cohesive set. The *cohesive product of* $(\mathcal{A}_n : n \in \mathbb{N})$ *over* C is the $\mathfrak{L}$-structure $\prod_C \mathcal{A}_n$ defined as follows.

- Let D be the set of partial computable functions φ such that $\forall n\,(\varphi(n)\downarrow \to \varphi(n) \in |\mathcal{A}_n|)$ and $C \subseteq^* \mathrm{dom}(\varphi)$.
- For $\varphi, \psi \in D$, let $\varphi =_C \psi$ denote $C \subseteq^* \{n : \varphi(n)\downarrow = \psi(n)\downarrow\}$. The relation $=_C$ is an equivalence relation on D. Let $[\varphi]$ denote the equivalence class of $\varphi \in D$ with respect to $=_C$.
- The domain of $\prod_C \mathcal{A}_n$ is the set $|\prod_C \mathcal{A}_n| = \{[\varphi] : \varphi \in D\}$.
- Let R be an m-ary relation symbol of $\mathfrak{L}$. For each $[\varphi_0], \dots, [\varphi_{m-1}] \in |\prod_C \mathcal{A}_n|$, define $R^{\prod_C \mathcal{A}_n}([\varphi_0], \dots, [\varphi_{m-1}])$ by
$$R^{\prod_C \mathcal{A}_n}([\varphi_0], \dots, [\varphi_{m-1}]) \;\Leftrightarrow\; C \subseteq^* \big\{n : R^{\mathcal{A}_n}(\varphi_0(n), \dots, \varphi_{m-1}(n))\big\}.$$
Here, $R^{\mathcal{A}_n}(\varphi_0(n), \dots, \varphi_{m-1}(n))$ includes the condition that $\varphi_i(n)\downarrow$ for each $i < m$.
- Let f be an m-ary function symbol of $\mathfrak{L}$. For each $[\varphi_0], \dots, [\varphi_{m-1}] \in |\prod_C \mathcal{A}_n|$, let ψ be the partial computable function defined by
$$\psi(n) \simeq f^{\mathcal{A}_n}(\varphi_0(n), \dots, \varphi_{m-1}(n)),$$
and notice that $C \subseteq^* \mathrm{dom}(\psi)$ because $C \subseteq^* \mathrm{dom}(\varphi_i)$ for each $i < m$. Define $f^{\prod_C \mathcal{A}_n}$ by
$$f^{\prod_C \mathcal{A}_n}([\varphi_0], \dots, [\varphi_{m-1}]) = [\psi].$$
- Let c be a constant symbol of $\mathfrak{L}$. Let ψ be the total computable function defined by $\psi(n) = c^{\mathcal{A}_n}$, and define $c^{\prod_C \mathcal{A}_n} = [\psi]$.

In the case where $\mathcal{A}_n$ is the same fixed computable structure $\mathcal{A}$ for every n, the cohesive product $\prod_C \mathcal{A}_n$ is called the *cohesive power of* $\mathcal{A}$ *over* C and is denoted $\prod_C \mathcal{A}$.

We present versions of Łoś's theorem for cohesive products and cohesive powers as in [4]. The results of [4] also show that these versions of Łoś's theorem are best possible. Analogs of Łoś's theorem for cohesive powers of arbitrary computable structures originate with Dimitrov's *fundamental theorem of cohesive powers* [3]. Analogs of Łoś's theorem for various flavors of effective ultrapowers of the particular structure $\mathcal{N}$ extend back to work of Hirschfeld [9] and of McLaughlin [18]. See also [28].

A formula Φ is *uniformly decidable* in a uniformly computable sequence of $\mathfrak{L}$-structures $(\mathcal{A}_n : n \in \mathbb{N})$ if there is an algorithm that determines whether or not $\mathcal{A}_n \models \Psi(\vec{a})$ given n, a subformula $\Psi(\vec{v})$ of Φ, and a sequence of elements $\vec{a}$ of $|\mathcal{A}_n|$ of the appropriate length. The presence of more uniform decidability extends the sorts of formulas that Łoś's theorem for cohesive products and cohesive powers applies to. We abuse terminology by saying that a formula is Δ_n if it is equivalent to both a Σ_n formula and a Π_n formula.

Lemma 2.3 ([4, Lemma 2.5]). *Let $\mathfrak{L}$ be a computable language, let $(\mathcal{A}_n : n \in \mathbb{N})$ be a uniformly computable sequence of $\mathfrak{L}$-structures, and let C be a cohesive set. Let $\Phi(\vec{x}, \vec{y}, v_0, \dots, v_{m-1})$ be a formula that is uniformly decidable in $(\mathcal{A}_n : n \in \mathbb{N})$.*

(1) For any $[\varphi_0], \dots, [\varphi_{m-1}] \in |\prod_C \mathcal{A}_n|$,

$$\prod_C \mathcal{A}_n \models \exists \vec{x} \forall \vec{y}\, \Phi(\vec{x}, \vec{y}, [\varphi_0], \dots, [\varphi_{m-1}]) \;\Rightarrow\; C \subseteq^* \big\{n : \mathcal{A}_n \models \exists \vec{x} \forall \vec{y}\, \Phi(\vec{x}, \vec{y}, \varphi_0(n), \dots, \varphi_{m-1}(n))\big\}.$$

(2) For any $[\varphi_0], \dots, [\varphi_{m-1}] \in |\prod_C \mathcal{A}_n|$,

$$C \subseteq^* \big\{n : \mathcal{A}_n \models \forall \vec{x} \exists \vec{y}\, \Phi(\vec{x}, \vec{y}, \varphi_0(n), \dots, \varphi_{m-1}(n))\big\} \;\Rightarrow\; \prod_C \mathcal{A}_n \models \forall \vec{x} \exists \vec{y}\, \Phi(\vec{x}, \vec{y}, [\varphi_0], \dots, [\varphi_{m-1}]).$$

Theorem 2.4 ([4, Theorem 2.7])**.** *Let $\mathfrak{L}$ be a computable language, let $(\mathcal{A}_n : n \in \mathbb{N})$ be a uniformly computable sequence of $\mathfrak{L}$-structures, and let C be a cohesive set.*

(1) Let $\Phi(v_0, \dots, v_{m-1})$ be a Σ_2 formula. Then for any $[\varphi_0], \dots, [\varphi_{m-1}] \in |\prod_C \mathcal{A}_n|$,

$$\prod_C \mathcal{A}_n \models \Phi([\varphi_0], \dots, [\varphi_{m-1}]) \;\Rightarrow\; C \subseteq^* \big\{n : \mathcal{A}_n \models \Phi(\varphi_0(n), \dots, \varphi_{m-1}(n))\big\}.$$

(2) Let $\Phi(v_0, \dots, v_{m-1})$ be a Π_2 formula. Then for any $[\varphi_0], \dots, [\varphi_{m-1}] \in |\prod_C \mathcal{A}_n|$,

$$C \subseteq^* \big\{n : \mathcal{A}_n \models \Phi(\varphi_0(n), \dots, \varphi_{m-1}(n))\big\} \;\Rightarrow\; \prod_C \mathcal{A}_n \models \Phi([\varphi_0], \dots, [\varphi_{m-1}]).$$

(3) Let $\Phi(v_0, \dots, v_{m-1})$ be a Δ_2 formula. Then for any $[\varphi_0], \dots, [\varphi_{m-1}] \in |\prod_C \mathcal{A}_n|$,

$$\prod_C \mathcal{A}_n \models \Phi([\varphi_0], \dots, [\varphi_{m-1}]) \;\Leftrightarrow\; C \subseteq^* \big\{n : \mathcal{A}_n \models \Phi(\varphi_0(n), \dots, \varphi_{m-1}(n))\big\}.$$

Theorem 2.5 ([4, Theorem 2.9])**.** *Let $\mathfrak{L}$ be a computable language, let $\mathcal{A}$ be a computable $\mathfrak{L}$-structure, and let C be a cohesive set.*

(1) Let Φ be a Δ_3 sentence. Then $\mathcal{A} \models \Phi$ if and only if $\prod_C \mathcal{A} \models \Phi$.

(2) Let Φ be a Σ_3 sentence. If $\mathcal{A} \models \Phi$, then $\prod_C \mathcal{A} \models \Phi$.

By Theorem 2.5, a cohesive product of (simple) graphs is again a graph, and a cohesive product of linear orders is again a linear order. We need Theorem 2.7 below in Section 4, which is a slight generalization of [4, Theorem 6.3] explaining how cohesive products commute with generalized sums of linear orders. We include the proof for completeness.

Definition 2.6 (see [26, Definition 1.38])**.** Let $\mathcal{K}$ be a linear order, and let $(\mathcal{M}_k : k \in |\mathcal{K}|)$ be a sequence of linear orders indexed by $|\mathcal{K}|$. The *generalized sum* $\sum_{k \in |\mathcal{K}|} \mathcal{M}_k$ of $(\mathcal{M}_k : k \in |\mathcal{K}|)$ over $\mathcal{K}$ is the linear order $\mathcal{S} = (S; \prec_{\mathcal{S}})$ defined as follows. Write $\mathcal{K} = (K; \prec_{\mathcal{K}})$, and write $\mathcal{M}_k = (M_k; \prec_{\mathcal{M}_k})$ for each $k \in K$. Define $S = \{(k, m) : k \in K \wedge m \in M_k\}$, and define

$$(k_0, m_0) \prec_{\mathcal{S}} (k_1, m_1) \quad \text{if and only if} \quad (k_0 \prec_{\mathcal{K}} k_1) \vee (k_0 = k_1 \wedge m_0 \prec_{\mathcal{M}_{k_0}} m_1).$$

Note that if $\mathcal{K}$ is computable and $(\mathcal{M}_k : k \in |\mathcal{K}|)$ is uniformly computable, then $\sum_{k \in |\mathcal{K}|} \mathcal{M}_k$ is also computable. Theorem 2.7 says that if $(\mathcal{K}_n : n \in \mathbb{N})$ and $(\mathcal{M}_{n,k} : n \in \mathbb{N}, k \in |\mathcal{K}_n|)$ are uniformly computable sequences of linear orders and C is cohesive, then

$$\prod_C \sum_{k \in |\mathcal{K}_n|} \mathcal{M}_{n,k} \;\cong\; \sum_{[\theta] \in |\prod_C \mathcal{K}_n|} \prod_C \mathcal{M}_{n,\theta(n)}.$$

The left-hand side is a cohesive product of generalized sums, where the n^{th} sum is of $(\mathcal{M}_{n,k} : k \in |\mathcal{K}_n|)$ over $\mathcal{K}_n$. The right-hand side is a generalized sum of cohesive products over a linear order that is itself a cohesive product and thus requires further explanation. The sum is over the linear order $\prod_C \mathcal{K}_n$. Term $[\theta]$ in the sum is the cohesive product $\prod_C \mathcal{M}_{n,\theta(n)}$ of the sequence $\mathcal{M}_{0,\theta(0)}, \mathcal{M}_{1,\theta(1)}, \mathcal{M}_{2,\theta(2)}, \dots$. Of course, $\theta(n)$ and hence $\mathcal{M}_{n,\theta(n)}$ may be undefined for some n, but $C \subseteq^* \operatorname{dom}(\theta)$ by virtue of $[\theta] \in |\prod_C \mathcal{K}_n|$, so $\mathcal{M}_{n,\theta(n)}$ is defined for almost every $n \in C$. Thus by $\prod_C \mathcal{M}_{n,\theta(n)}$, we mean the product constructed from partial computable functions φ with $C \subseteq^* \operatorname{dom}(\varphi) \subseteq \operatorname{dom}(\theta)$ and $\forall n\, (\varphi(n)\downarrow \,\rightarrow\, \varphi(n) \in |\mathcal{M}_{n,\theta(n)}|)$. See [4, Section 6] for further details.

Theorem 2.7 (Generalizing [4, Theorem 6.3])**.** *Let $(\mathcal{K}_n : n \in \mathbb{N})$ and $(\mathcal{M}_{n,k} : n \in \mathbb{N}, k \in |\mathcal{K}_n|)$ be uniformly computable sequences of linear orders. Let C be a cohesive set. Then*

$$\prod_C \sum_{k \in |\mathcal{K}_n|} \mathcal{M}_{n,k} \;\cong\; \sum_{[\theta] \in |\prod_C \mathcal{K}_n|} \prod_C \mathcal{M}_{n,\theta(n)}.$$

Proof. To ease notation, let

$$\mathcal{L}_n = \sum_{k \in |\mathcal{K}_n|} \mathcal{M}_{n,k} \qquad \text{for each } n,$$

$$\mathcal{X} = \prod\nolimits_C \mathcal{K}_n,$$

$$\mathcal{Y}_{[\theta]_\mathcal{X}} = \prod\nolimits_C \mathcal{M}_{n,\theta(n)} \qquad \text{for each } [\theta]_\mathcal{X} \in |\mathcal{X}|,$$

$$\mathcal{A} = \prod\nolimits_C \mathcal{L}_n,$$

$$\mathcal{B} = \sum_{[\theta]_\mathcal{X} \in |\mathcal{X}|} \mathcal{Y}_{[\theta]_\mathcal{X}}.$$

The goal is to show that $\mathcal{A} \cong \mathcal{B}$. The elements of $\mathcal{A}$ are of the form $[\varphi]_\mathcal{A}$ for partial computable φ with $\forall n\,(\varphi(n)\downarrow \,\rightarrow\, \varphi(n) \in |\mathcal{L}_n|)$ and $C \subseteq^* \mathrm{dom}(\varphi)$. The elements of $\mathcal{B}$ are of the form $\big([\theta]_\mathcal{X}, [\tau]_{\mathcal{Y}_{[\theta]_\mathcal{X}}}\big)$ for partial computable θ and τ with $\forall n\,(\theta(n)\downarrow \,\rightarrow\, \theta(n) \in |\mathcal{K}_n|)$, $C \subseteq^* \mathrm{dom}(\tau) \subseteq \mathrm{dom}(\theta)$, and $\forall n\,(\tau(n)\downarrow \,\rightarrow\, \tau(n) \in |\mathcal{M}_{n,\theta(n)}|)$.

Define a function $F\colon |\mathcal{A}| \to |\mathcal{B}|$ as follows. For $[\varphi]_\mathcal{A} \in |\mathcal{A}|$, we have that $\varphi(n) \in |\mathcal{L}_n|$ and therefore that $\varphi(n) = \langle k, m\rangle$ for some $k \in |\mathcal{K}_n|$ and $m \in |\mathcal{M}_{n,k}|$ whenever $\varphi(n)\downarrow$. Let $\theta = \pi_0 \circ \varphi$, and let $\tau = \pi_1 \circ \varphi$. Then $[\theta]_\mathcal{X} \in |\mathcal{X}|$ and $[\tau]_{\mathcal{Y}_{[\theta]_\mathcal{X}}} \in |\mathcal{Y}_{[\theta]_\mathcal{X}}|$. Set $F([\varphi]_\mathcal{A}) = \big([\theta]_\mathcal{X}, [\tau]_{\mathcal{Y}_{[\theta]_\mathcal{X}}}\big)$. To see that F is well-defined, observe that if $\varphi =_C \psi$, then also $\pi_0 \circ \varphi =_C \pi_0 \circ \psi$ and $\pi_1 \circ \varphi =_C \pi_1 \circ \psi$.

To show that F is an isomorphism between the linear orders $\mathcal{A}$ and $\mathcal{B}$, it suffices to show that F is surjective and order-preserving.

For surjectivity, consider an element $\big([\theta]_\mathcal{X}, [\tau]_{\mathcal{Y}_{[\theta]_\mathcal{X}}}\big)$ of $\mathcal{B}$. Define a partial computable φ by $\varphi(n) \simeq \langle \theta(n), \tau(n)\rangle$. Then $\varphi(n) \in |\mathcal{L}_n|$ whenever $\varphi(n)\downarrow$, and $C \subseteq^* \mathrm{dom}(\varphi)$ because $C \subseteq^* \mathrm{dom}(\tau) \subseteq \mathrm{dom}(\theta)$. Therefore $[\varphi]_\mathcal{A} \in |\mathcal{A}|$ and $F([\varphi]_\mathcal{A}) = \big([\theta]_\mathcal{X}, [\tau]_{\mathcal{Y}_{[\theta]_\mathcal{X}}}\big)$.

For order-preserving, suppose that $[\varphi]_\mathcal{A}$ and $[\psi]_\mathcal{A}$ are members of $\mathcal{A}$ with $[\varphi]_\mathcal{A} \prec_\mathcal{A} [\psi]_\mathcal{A}$. Then $(\forall^\infty n \in C)(\varphi(n) \prec_{\mathcal{L}_n} \psi(n))$. Write $\theta = \pi_0 \circ \varphi$, $\tau = \pi_1 \circ \varphi$, $\alpha = \pi_0 \circ \psi$, and $\beta = \pi_1 \circ \psi$. Then

$$(\forall^\infty n \in C)\Big(\big(\theta(n) \prec_{\mathcal{K}_n} \alpha(n)\big) \;\vee\; \big(\theta(n) = \alpha(n) \;\wedge\; \tau(n) \prec_{\mathcal{M}_{n,\theta(n)}} \beta(n)\big)\Big)$$

By cohesiveness, either

- $(\forall^\infty n \in C)\big(\theta(n) \prec_{\mathcal{K}_n} \alpha(n)\big)$ or
- $(\forall^\infty n \in C)\big(\theta(n) = \alpha(n) \;\wedge\; \tau(n) \prec_{\mathcal{M}_{n,\theta(n)}} \beta(n)\big)$.

In the first case, $[\theta]_\mathcal{X} \prec_\mathcal{X} [\alpha]_\mathcal{X}$. In the second case, $[\theta]_\mathcal{X} = [\alpha]_\mathcal{X}$ and $[\tau]_{\mathcal{Y}_{[\theta]_\mathcal{X}}} \prec_{\mathcal{Y}_{[\theta]_\mathcal{X}}} [\beta]_{\mathcal{Y}_{[\theta]_\mathcal{X}}}$. Thus in either case,

$$F([\varphi]_\mathcal{A}) = \big([\theta]_\mathcal{X}, [\tau]_{\mathcal{Y}_{[\theta]_\mathcal{X}}}\big) \prec_\mathcal{B} \big([\alpha]_\mathcal{X}, [\beta]_{\mathcal{Y}_{[\alpha]_\mathcal{X}}}\big) = F([\psi]_\mathcal{A}),$$

as desired. □

3. Encoding information into cohesive powers

We first show that there are computable structures whose cohesive powers are capable of encoding arbitrary information via the cohesive set.

Proposition 3.1. *There is a fixed computable structure $\mathcal{A}$ such that for every set $X \subseteq \mathbb{N}$, there is a cohesive set C such that every presentation of $\prod_C \mathcal{A}$ computes X.*

Proof. The language consists of a computable sequence of unary predicates $(U_\sigma : \sigma \in 2^{<\mathbb{N}})$, which we think of as names for sets. Compute a structure $\mathcal{A} = \big(\mathbb{N}; (U_\sigma : \sigma \in 2^{<\mathbb{N}})\big)$ as follows. Let $U_\emptyset = \mathbb{N}$. Given U_σ enumerated in order as $U_\sigma = \{p_0 < p_1 < p_2 < \cdots\}$, let $U_{\sigma^\frown 0} = \{p_{2n} : n \in \mathbb{N}\}$ and $U_{\sigma^\frown 1} = \{p_{2n+1} : n \in \mathbb{N}\}$. If $\sigma \sqsubseteq \tau$, then $U_\sigma \supseteq U_\tau$; and if σ and τ are incomparable, then $U_\sigma \cap U_\tau = \emptyset$.

Let $X \subseteq \mathbb{N}$ be any set. Choose an increasing sequence $c_0 < c_1 < c_2 < \cdots$ with $c_n \in U_{X \restriction n}$ for each n. Let C be any cohesive subset of $\{c_n : n \in \mathbb{N}\}$. Then for any $\sigma \in 2^{<\mathbb{N}}$, we have that $C \subseteq^* U_\sigma$ if and only if $\sigma \sqsubseteq X$. Let $f\colon \mathbb{N} \to \mathbb{N}$ be the computable function $f(n) = n$. For any $\sigma \in 2^{<\mathbb{N}}$,

$$U_\sigma^{\prod_C \mathcal{A}}([f]) \;\Leftrightarrow\; C \subseteq^* \{n : U_\sigma(f(n))\} \;\Leftrightarrow\; C \subseteq^* \{n : U_\sigma(n)\} \;\Leftrightarrow\; C \subseteq^* U_\sigma \;\Leftrightarrow\; \sigma \sqsubseteq X.$$

Every presentation $\mathcal{P}$ of $\prod_C \mathcal{A}$ computes X because if $a \in |\mathcal{P}|$ is the element corresponding to $[f]$, then the $\mathcal{P}$-computable set $\{\sigma \in 2^{<\mathbb{N}} : U_\sigma^{\mathcal{P}}(a)\}$ consists of exactly the initial segments of X. □

The use of an infinite language in Proposition 3.1 is just for expedience. With a little more coding, the same effect may be achieved with a finite language.

In light of Proposition 3.1, if cohesive products are to be thought of as effectivized ultraproducts, then some effectivity assumption should be imposed on the cohesive sets over which the products are taken. The next proposition shows that if we restrict to Δ_k ($k \geq 2$) cohesive sets, then the cohesive products over these cohesive sets always have Δ_{k+1} presentations.

Proposition 3.2. *Let $\mathfrak{L}$ be a computable language, and let $(\mathcal{A}_n : n \in \mathbb{N})$ be a uniformly computable sequence of $\mathfrak{L}$-structures. Let C be a Δ_k cohesive set for some $k \geq 2$. Then $\prod_C \mathcal{A}_n$ has a Δ_{k+1} presentation.*

Proof. We describe a Δ_{k+1} presentation $\mathcal{P}$ of $\prod_C \mathcal{A}_n$. The elements of $\prod_C \mathcal{A}_n$ are equivalence classes $[\varphi]$ of partial computable functions $\varphi\colon \mathbb{N} \to \mathbb{N}$ where $\forall n\,(\varphi(n){\downarrow} \to \varphi(n) \in |\mathcal{A}_n|)$ and $C \subseteq^* \mathrm{dom}(\varphi)$. Let

$$D = \{\langle e, N\rangle : \forall n\,(\varphi_e(n){\downarrow} \to \varphi_e(n) \in |\mathcal{A}_n|) \wedge (\forall n > N)(n \in C \to \varphi_e(n){\downarrow})\}.$$

The set D is Π_k because C is Δ_k (and $k \geq 2$). The elements of D are intended to represent the elements of $\prod_C \mathcal{A}_n$. Thus we must identify when different elements of D represent the same element of $\prod_C \mathcal{A}_n$. Define

$$\langle e, N\rangle \sim \langle i, M\rangle \quad\equiv\quad \langle e, N\rangle \in D \wedge \langle i, M\rangle \in D \wedge (\exists K)(\forall n > K)(n \in C \to \varphi_e(n) = \varphi_i(n))$$

As written, $\langle e, N\rangle \sim \langle i, M\rangle$ is a Σ_{k+1} relation because C is Δ_k and D is Π_k. However, if $\langle e, N\rangle \in D$ and $\langle i, M\rangle \in D$ but $\neg(\exists K)(\forall n > K)(n \in C \to \varphi_e(n) = \varphi_i(n))$, then by cohesiveness $(\exists K)(\forall n > K)(n \in C \to \varphi_e(n) \neq \varphi_i(n))$. Therefore

$$\langle e, N\rangle \nsim \langle i, M\rangle \quad\equiv\quad \langle e, N\rangle \notin D \vee \langle i, M\rangle \notin D \vee (\exists K)(\forall n > K)(n \in C \to \varphi_e(n) \neq \varphi_i(n)),$$

which is also Σ_{k+1}. Thus $\langle e, N\rangle \sim \langle i, M\rangle$ is a Δ_{k+1} relation.

Let X be the set of least representatives:

$$X = \Big\{\langle e, N\rangle : \langle e, N\rangle \in D \wedge \big(\forall \langle i, M\rangle < \langle e, N\rangle\big)\big(\langle e, N\rangle \nsim \langle i, M\rangle\big)\Big\}.$$

Then X is Δ_{k+1}, and we take it to be the domain of our presentation $\mathcal{P}$.

For each m-ary relation symbol R, define

$$R^{\mathcal{P}}\big(\langle e_0, N_0\rangle, \dots, \langle e_{m-1}, N_{m-1}\rangle\big) \equiv \\ (\forall i < m)\big(\langle e_i, N_i\rangle \in X\big) \wedge (\exists K)(\forall n > K)\big(n \in C \to R^{\mathcal{A}_n}(\varphi_{e_0}(n), \dots, \varphi_{e_{m-1}}(n))\big).$$

By cohesiveness and the same analysis as for the $\langle e, N\rangle \sim \langle i, M\rangle$ relation, $R^{\mathcal{P}}$ is a Δ_{k+1} relation on X^m uniformly in R.

For each m-ary function symbol F, allowing $m = 0$ to include the constant symbols, define

$$F^{\mathcal{P}}\big(\langle e_0, N_0\rangle, \dots, \langle e_{m-1}, N_{m-1}\rangle\big) = \langle e_m, N_m\rangle \equiv \\ (\forall i \leq m)\big(\langle e_i, N_i\rangle \in X\big) \wedge (\exists K)(\forall n > K)\big(n \in C \to \varphi_{e_m}(n) = F^{\mathcal{A}_n}(\varphi_{e_0}(n), \dots, \varphi_{e_{m-1}}(n))\big)$$

Again by cohesiveness, $F^{\mathcal{P}}$ is a Δ_{k+1} function $X^m \to X$ uniformly in F.

We have defined a Δ_{k+1} $\mathfrak{L}$-structure $\mathcal{P}$ with domain X. An element $\langle e, N\rangle \in X$ corresponds to the element $[\varphi_e]$ of $\prod_C \mathcal{A}_n$. Conversely, an element $[\varphi]$ of $\prod_C \mathcal{A}_n$ corresponds to the element $\langle e, N\rangle$ of X, where $\langle e, N\rangle$ is the least pair in D such that $(\exists K)(\forall n > K)(n \in C \to \varphi_e(n) = \varphi(n))$. It is straightforward to check that this correspondence is bijective and gives an isomorphism between $\mathcal{P}$ and $\prod_C \mathcal{A}_n$, so $\mathcal{P}$ is a Δ_{k+1} presentation of $\prod_C \mathcal{A}_n$. □

Now we compute a graph $\mathcal{G}$ for which every presentation of every cohesive power computes $0''$. Thus if we restrict to Δ_2 cohesive sets, the cohesive powers of $\mathcal{G}$ have maximum complexity. If C is a Δ_2 cohesive set, then $\prod_C \mathcal{G}$ has a $0''$-computable presentation, and every presentation of $\prod_C \mathcal{G}$ computes $0''$.

Theorem 3.3. *There is a computable graph $\mathcal{G}$ such that for every cohesive set C, every presentation of the cohesive power $\prod_C \mathcal{G}$ computes $0''$.*

Proof. Given a Σ_3 set X, the plan is to compute a graph $\mathcal{G}$ such that for every cohesive set C, the set X is c.e. in every presentation of $\prod_C \mathcal{G}$. If we take X to be the particular Σ_3 set $0'' \oplus (\mathbb{N} \setminus 0'')$, then both $0''$ and its complement are c.e. and hence computable in every presentation of $\prod_C \mathcal{G}$ for every cohesive set C.

Let $X = \{k : \Phi(k)\}$ be a Σ_3 set, where $\Phi(k) \equiv \exists \ell\, \forall i\, \exists j\, R(k, \ell, i, j)$ for the computable relation R. We compute a graph $\mathcal{G}$ on domain $|\mathcal{G}| = A \cup B$, where $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$. For specificity, say $a_n = 2n$ and $b_n = 2n + 1$. The graph $\mathcal{G}$ consists of cycles of various lengths having exactly one vertex from A and the rest from B (plus some vertices in B may be isolated). Each vertex in A may be in many cycles, but each vertex in B is in at most one cycle. This will be done in such a way so that if $f \colon \mathbb{N} \to \mathbb{N}$ is the computable function $f(n) = a_n$, then, for any cohesive set C, $\Phi(k)$ holds if and only if, in $\prod_C \mathcal{G}$, there is an ℓ and a cycle of length $\langle k, \ell\rangle + 3$ containing $[f]$. This makes X c.e. in every presentation of $\prod_C \mathcal{G}$ by enumerating the k for which there is a cycle of length $\langle k, \ell\rangle + 3$ for some ℓ that contains the element corresponding to $[f]$.

Compute $\mathcal{G}$ in stages. At stage s, consider the triples $\langle n, k, \ell\rangle < s$. For each such triple, if $(\forall i < n)(\exists j < s)\, R(k, \ell, i, j)$ and a_n does not already belong to a cycle of length $\langle k, \ell\rangle + 3$, then add a cycle of length $\langle k, \ell\rangle + 3$ to $\mathcal{G}$ consisting of a_n and $\langle k, \ell\rangle + 2$ fresh elements $> s$ from B. This completes the computation of $\mathcal{G}$.

Let C be a cohesive set, and let $f \colon \mathbb{N} \to \mathbb{N}$ be the computable function $f(n) = a_n$. Given k, we show that $\Phi(k)$ holds if and only if there is an ℓ and a cycle of length $\langle k, \ell\rangle + 3$ in $\prod_C \mathcal{G}$ containing $[f]$.

Suppose that $\Phi(k)$ holds, and let ℓ witness $\forall i\, \exists j\, R(k, \ell, i, j)$. Then $(\forall i < n)(\exists j < s)\, R(k, \ell, i, j)$ holds for every n and every sufficiently large s. So for every n, $\mathcal{G}$ has a cycle of length $\langle k, \ell\rangle + 3$ containing a_n. The formula $\Psi_{k,\ell}(v) \equiv$ "there is a cycle of length $\langle k, \ell\rangle + 3$ containing v" is Σ_1, and $\mathcal{G} \models \Psi_{k,\ell}(f(n))$ for every n. Therefore $\prod_C \mathcal{G} \models \Psi_{k,\ell}([f])$ by Theorem 2.4. That is, $\prod_C \mathcal{G}$ has a cycle of length $\langle k, \ell\rangle + 3$ containing $[f]$.

Suppose instead that $\Phi(k)$ fails, and consider an ℓ. This ℓ does not witness $\forall i\, \exists j\, R(k, \ell, i, j)$, so there is an n_ℓ such that $\forall j\, \neg R(k, \ell, n_\ell, j)$. Thus $(\forall i < n)(\exists j < s)\, R(k, \ell, i, j)$ fails for every $n > n_\ell$ and every s. Therefore, if $n > n_\ell$, then $\mathcal{G}$ does not have a cycle of length $\langle k, \ell\rangle + 3$ containing a_n. The formula $\neg\Psi_{k,\ell}(v)$ is Π_1, and $\mathcal{G} \models \neg\Psi_{k,\ell}(f(n))$ for all $n > n_\ell$. Therefore $\prod_C \mathcal{G} \models \neg\Psi_{k,\ell}([f])$ by Theorem 2.4, so $\prod_C \mathcal{G}$ does not have a cycle of length $\langle k, \ell\rangle + 3$ containing $[f]$. This analysis was for an arbitrary ℓ, thus for no ℓ does $\prod_C \mathcal{G}$ have a cycle of length $\langle k, \ell\rangle + 3$ containing $[f]$. □

Corollary 3.4. *There is a computable graph $\mathcal{G}$ such that every cohesive power $\prod_C \mathcal{G}$ of $\mathcal{G}$ by a Δ_2 cohesive set C has degree $0''$.*

Proof. Let $\mathcal{G}$ be the computable graph from Theorem 3.3, and let C be a Δ_2 cohesive set. Then $\prod_C \mathcal{G}$ has a Δ_3 presentation (i.e., a $0''$-computable presentation) by Proposition 3.2, and every presentation of $\prod_C \mathcal{G}$ computes $0''$ by Theorem 3.3. □

Note that this construction is not possible with a linear order by Theorem 2.1. The next section is dedicated to seeing what is possible with a linear order.

4. A linear order exhibiting a Tennenbaum-like phenomenon

We compute a linear order $\mathcal{L}$ with the following property. If C is a cohesive set and $\mathcal{P}$ is a presentation of $\prod_C \mathcal{L}$, then $\mathcal{P}''$ has PA-degree relative to $0''$. Such a $\mathcal{P}$ cannot be computable because $0''$ does not have PA-degree relative to itself. Therefore $\mathcal{L}$ exhibits a Tennenbaum-like phenomenon, as it is a computable linear order whose cohesive powers do not have computable presentations. This answers our Question (3) from the introduction.

To compute $\mathcal{L}$, we first compute a sequence $(\mathcal{L}_n : n \in \mathbb{N})$ of linear orders where $\prod_C \mathcal{L}_n$ has the desired property for every cohesive set C: the double-jump of every presentation of $\prod_C \mathcal{L}_n$ has PA-degree relative to $0''$. Then we show that we can essentially take the desired $\mathcal{L}$ to be the sum of the linear orders $(\mathcal{L}_n : n \in \mathbb{N})$.

Theorem 4.1. *There is a uniformly computable sequence $(\mathcal{L}_n : n \in \mathbb{N})$ of linear orders such that for every cohesive set C and every presentation $\mathcal{P}$ of $\prod_C \mathcal{L}_n$, the double-jump $\mathcal{P}''$ has* PA*-degree relative to $0''$. Thus $\prod_C \mathcal{L}_n$ has no computable presentation.*

Proof. Let A and B be two disjoint Σ_3 sets. We compute $(\mathcal{L}_n : n \in \mathbb{N})$ so that whenever C is a cohesive set and $\mathcal{P}$ is a presentation of $\prod_C \mathcal{L}_n$, there is an AB-separator computable from $\mathcal{P}''$. The desired result follows because there are disjoint Σ_3 sets A and B whose separators are exactly the functions that are DNC_2 relative to $0''$.

Given any Σ_3 set A, we can compute a sequence of sets $(A_k)_{k\in\mathbb{N}}$ such that, for all k, $k \in A$ if and only if A_k has an infinite column: $(\exists i)(\forall s_0)(\exists s > s_0)(A_k(\langle i, s\rangle) = 1)$. For our disjoint Σ_3 sets A and B, we therefore have two uniformly computable sequences of sets $(A_k)_{k\in\mathbb{N}}$ and $(B_k)_{k\in\mathbb{N}}$ where, for every k, at most one of A_k and B_k has an infinite column.

We uniformly compute linear orders $(\mathcal{L}_n : n \in \mathbb{N})$ so that for every cohesive set C, the cohesive product $\prod_C \mathcal{L}_n$ has the form

$$\prod_C \mathcal{L}_n \quad = \quad \sum_{k\in\mathbb{N}}(\mathcal{S}_k + \mathcal{R}_k) + \mathcal{J} \quad = \quad (\mathcal{S}_0 + \mathcal{R}_0 + \mathcal{S}_1 + \mathcal{R}_1 + \cdots) + \mathcal{J}$$

with the following properties.

- For each k, $\mathcal{S}_k \;\cong\; (\boldsymbol{k+2}) + \eta + (\boldsymbol{k+2}) + \eta + (\boldsymbol{k+2})$. The orders $\mathcal{S}_k$ and $\mathcal{S}_{k+1}$ serve as delimiters around the order $\mathcal{R}_k$.
- For each k, $\mathcal{R}_k$ has no non-trivial dense interval. In fact, every element of $\mathcal{R}_k$ other than the maximum element (should it exist) has a successor.
- $\mathcal{J}$ does not have finite blocks of size 2 or more.

It follows that for each k, $\mathcal{S}_k$ is the unique interval of $\prod_C \mathcal{L}_n$ with order-type $(\boldsymbol{k+2}) + \eta + (\boldsymbol{k+2}) + \eta + (\boldsymbol{k+2})$. Furthermore, if $\mathcal{P}$ is a presentation of $\prod_C \mathcal{L}_n$, then $\mathcal{P}''$ can identify $\mathcal{S}_k$ by searching for elements $u_0, \dots, u_{k+1}, v_0, \dots, v_{k+1}, w_0, \dots, w_{k+1}$ such that the following two-quantifier properties hold:

- $u_0 \prec_\mathcal{P} \cdots \prec_\mathcal{P} u_{k+1} \prec_\mathcal{P} v_0 \prec_\mathcal{P} \cdots \prec_\mathcal{P} v_{k+1} \prec_\mathcal{P} w_0 \prec_\mathcal{P} \cdots \prec_\mathcal{P} w_{k+1}$,
- for each $i \leq k$, the intervals $(u_i, u_{i+1})_\mathcal{P}$, $(v_i, v_{i+1})_\mathcal{P}$, and $(w_i, w_{i+1})_\mathcal{P}$ are empty, and
- the intervals $[u_{k+1}, v_0]_\mathcal{P}$ and $[v_{k+1}, w_0]_\mathcal{P}$ are dense.

The encoding plan is to arrange for $\mathcal{R}_k$ to have a maximum element if A_k has an infinite column (i.e., if $k \in A$) and for $\mathcal{R}_k$ to have no maximum element if B_k has an infinite column (i.e., if $k \in B$). As we are only trying to compute a separator for A and B, we will not need any control over whether $\mathcal{R}_k$ will have a maximum element in the case that neither A_k nor B_k has an infinite column. Again, let $\mathcal{P}$ be a presentation of $\prod_C \mathcal{L}_n$. To decode, use $\mathcal{P}''$ to compute the set $X = \{k : \mathcal{R}_k \text{ has a maximum element}\}$, which is an AB-separator. To do this, given k, use $\mathcal{P}''$ to identify the intervals $\mathcal{S}_k$ and $\mathcal{S}_{k+1}$ as explained above. Specifically, identify the right endpoint w of $\mathcal{S}_k$ and the left endpoint u of S_{k+1} so that $\mathcal{R}_k$ is the interval (w, u) of $\mathcal{P}$. Then we use $\mathcal{P}''$ again to determine whether $\mathcal{R}_k$ has a maximum element and thus whether $k \in X$. The idea of using orders like the $\mathcal{S}_k$'s to delimit diagonalization or, in this case, encoding regions like the $\mathcal{R}_k$'s comes from [11]. The '$\mathcal{J}$' at the end of $\prod_C \mathcal{L}_n$ stands for 'junk.' We do not use this region for encoding or decoding, but we ensure that it does not contain finite blocks of size ≥ 2 to ensure that $\prod_C \mathcal{L}_n$ does not contain unintended intervals of the form $\mathcal{S}_k$.

To implement the encoding, we uniformly compute linear orders $(\mathcal{M}_{n,k} : n, k \in \mathbb{N})$ and take $\mathcal{L}_n = \sum_{k\in\mathbb{N}} \mathcal{M}_{n,k}$ for each n. In fact, we compute $(\mathcal{M}_{n,k} : n, k \in \mathbb{N})$ in such a way that the successor relation on $\mathcal{M}_{n,k}$ is c.e. and hence computable uniformly in n and k. Throughout we use '$\mathbb{N}$' to also denote the usual presentation of $(\mathbb{N}; <)$.

For any cohesive set C, we have that

$$\prod_C \mathcal{L}_n \quad = \quad \prod_C \sum_{k\in\mathbb{N}} \mathcal{M}_{n,k} \quad \cong \quad \sum_{[\theta]\in|\prod_C \mathbb{N}|} \prod_C \mathcal{M}_{n,\theta(n)}$$

by Theorem 2.7. We also have that $\prod_C \mathbb{N} \cong \omega + \zeta \cdot \eta$ by [4, Theorem 4.5]. Using the terminology of [4], we call the initial ω of $\prod_C \mathbb{N}$ the *standard part* and denote it $|\prod_C \mathbb{N}|_{\mathrm{std}}$, and we call the remaining $\zeta \cdot \eta$ within $\prod_C \mathbb{N}$ the *non-standard part* and denote it $|\prod_C \mathbb{N}|_{\mathrm{nonstd}}$. An element $[\theta]$ of $\prod_C \mathbb{N}$ is standard if and only if it is eventually constant on C, and $[\theta]$ is non-standard if and only if $\lim_{n\in C} \theta(n) = \infty$ ([4, Lemma 4.2]). In particular, the elements of $|\prod_C \mathbb{N}|_{\mathrm{std}}$ are exactly those represented by constant functions. Therefore

$$\sum_{[\theta]\in|\prod_C \mathbb{N}|} \prod\nolimits_C \mathcal{M}_{n,\theta(n)} = \sum_{[\theta]\in|\prod_C \mathbb{N}|_{\mathrm{std}}} \prod\nolimits_C \mathcal{M}_{n,\theta(n)} + \sum_{[\theta]\in|\prod_C \mathbb{N}|_{\mathrm{nonstd}}} \prod\nolimits_C \mathcal{M}_{n,\theta(n)}$$

$$= \sum_{k\in\mathbb{N}} \prod_{n\in C} \mathcal{M}_{n,k} + \sum_{[\theta]\in|\prod_C \mathbb{N}|_{\mathrm{nonstd}}} \prod\nolimits_C \mathcal{M}_{n,\theta(n)}.$$

In the last expression, we use the notation $\prod_{n\in C}\mathcal{M}_{n,k}$ to emphasize that k is fixed and that the cohesive product is taken over the sequence $\mathcal{M}_{0,k}, \mathcal{M}_{1,k}, \dots$. Put

$$\begin{aligned} \mathcal{S}_k &= \prod_{n\in C}\mathcal{M}_{n,6k} + \prod_{n\in C}\mathcal{M}_{n,6k+1} + \prod_{n\in C}\mathcal{M}_{n,6k+2} + \prod_{n\in C}\mathcal{M}_{n,6k+3} + \prod_{n\in C}\mathcal{M}_{n,6k+4} && \text{for each } k, \\ \mathcal{R}_k &= \prod_{n\in C}\mathcal{M}_{n,6k+5} && \text{for each } k, \\ \mathcal{J} &= \sum_{[\theta]\in|\prod_C \mathbb{N}|_{\mathrm{nonstd}}} \prod\nolimits_C \mathcal{M}_{n,\theta(n)} \end{aligned}$$

to write $\prod_C \mathcal{L}_n$ in the desired form

$$\prod\nolimits_C \mathcal{L}_n = (\mathcal{S}_0 + \mathcal{R}_0 + \mathcal{S}_1 + \mathcal{R}_1 + \cdots) + \mathcal{J}.$$

We now describe the $\mathcal{M}_{n,k}$. The orders that are used to build up the $\mathcal{S}_k$ are straightforward to describe and directly reflect the desired structure of $\mathcal{S}_k$. For each n and k, set

$$\begin{aligned} \mathcal{M}_{n,6k} = \mathcal{M}_{n,6k+2} &= \mathcal{M}_{n,6k+4} = k+2 \\ \mathcal{M}_{n,6k+1} &= \mathcal{M}_{n,6k+3} = \mathbb{Q}, \end{aligned}$$

where $k+2$ is the usual presentation of the linear order $\boldsymbol{k+2}$ and $\mathbb{Q}$ is the usual presentation of $(\mathbb{Q};<)$. Note that the successor relation is uniformly computable for these orders. Then

$$\prod_{n\in C}\mathcal{M}_{n,6k} = \prod_{n\in C}\mathcal{M}_{n,6k+2} = \prod_{n\in C}\mathcal{M}_{n,6k+4} \cong \boldsymbol{k+2}$$

for each n and k because $k+2$ is finite (see [4, Section 2]) and

$$\prod_{n\in C}\mathcal{M}_{n,6k+1} = \prod_{n\in C}\mathcal{M}_{n,6k+3} \cong \eta$$

for each n and k by [4, Proposition 3.4]. Therefore we indeed have $\mathcal{S}_k \cong (\boldsymbol{k+2})+\eta+(\boldsymbol{k+2})+\eta+(\boldsymbol{k+2})$ for each k.

We now describe how to compute $\mathcal{M}_{n,6k+5}$ for all n and k. Each $\mathcal{M}_{n,6k+5}$ will either have order-type $\omega\cdot\boldsymbol{\ell}$ for some $\ell>0$ or order-type $\omega\cdot\boldsymbol{\ell}+\boldsymbol{q}$ for some $\ell>0$ and $q>k$. Moreover, the successor relation on $\mathcal{M}_{n,6k+5}$ will be c.e. uniformly in n and k.

Think of k as being fixed and of computing the sequence $(\mathcal{M}_{n,6k+5} : n\in\mathbb{N})$ of linear orders. The goal is to ensure that $\mathcal{R}_k = \prod_{n\in C}\mathcal{M}_{n,6k+5}$ has a maximum element if A_k has an infinite column and that $\mathcal{R}_k$ does not have a maximum element if B_k has an infinite column. To do this, the columns of A_k play finite sequences of length $k+1$ at the top of $\mathcal{M}_{n,6k+5}$ for larger and larger n, and the columns of B_k play ω-sequences at the top of $\mathcal{M}_{n,6k+5}$ for larger and larger n. We impose priority on the plays, meaning that a lower priority column is not allowed to play in $\mathcal{M}_{n,6k+5}$ if a higher priority column has already played there. In detail, the computation proceeds as follows.

At stage 0, $\mathcal{M}_{n,6k+5}$ starts as an ω-sequence for every n. Declare $\langle 0,x\rangle\in|\mathcal{M}_{n,6k+5}|$ for every x, with $\langle 0,x\rangle \prec_{\mathcal{M}_{n,6k+5}} \langle 0,y\rangle$ if and only if $x<y$. Furthermore, enumerate $\langle\langle 0,x\rangle,\langle 0,x+1\rangle\rangle$ as a successor pair in $\mathcal{M}_{n,6k+5}$ for every x.

At stage $s>0$, let $i<s$ be least such that either $A_k(\langle i,s\rangle)=1$ or $B_k(\langle i,s\rangle)=1$. If there is no such i, then declare $\langle s,x\rangle\notin|\mathcal{M}_{n,6k+5}|$ for all x and all n. Go on to stage $s+1$.

Suppose there is such an i. If $A_k(\langle i,s\rangle)=1$, then we have noticed that column i of A_k has a new element s. In this case, A_k plays a copy of $\boldsymbol{k+1}$ at the top of $\mathcal{M}_{n,6k+5}$ for the least $n>i$ such that neither A_k nor B_k has yet played in $\mathcal{M}_{n,6k+5}$ on account of a column $j\leq i$. Specifically, declare $\langle s,x\rangle\in|\mathcal{M}_{n,6k+5}|$ for every $x\leq k$, with $\langle s,x\rangle\prec_{\mathcal{M}_{n,6k+5}}\langle s,y\rangle$ if and only if $x<y$. Declare $\langle s,x\rangle\notin|\mathcal{M}_{n,6k+5}|$ for every $x>k$. Declare $p\prec_{\mathcal{M}_{n,6k+5}}\langle s,x\rangle$ for all $x\leq k$ and all p already in $|\mathcal{M}_{n,6k+5}|$ at the start of stage s. Enumerate $\langle\langle s,x\rangle,\langle s,x+1\rangle\rangle$ as a successor pair in $\mathcal{M}_{n,6k+5}$ for every $x<k$. Finally, if $\mathcal{M}_{n,6k+5}$ had a maximum element p at the start of stage s (which is if A_k was last to play in $\mathcal{M}_{n,6k+5}$), then also enumerate $\langle p,\langle s,0\rangle\rangle$ as a successor pair in $\mathcal{M}_{n,6k+5}$. Declare $\langle s,x\rangle\notin|\mathcal{M}_{m,6k+5}|$ for all x and all $m\neq n$. We say that A_k *has played in* $\mathcal{M}_{n,6k+5}$ *on account of column* i or that *column* i *of* A_k *played in* $\mathcal{M}_{n,6k+5}$. Go on to stage $s+1$.

If $A_k(\langle i,s\rangle)=0$ but $B_k(\langle i,s\rangle)=1$, then we have noticed that column i of B_k has a new element s. In this case, B_k plays a copy of ω at the top of $\mathcal{M}_{n,6k+5}$ for the least $n>i$ such that neither A_k nor B_k has yet played in $\mathcal{M}_{n,6k+5}$ on account of a column $j\leq i$. Specifically, declare $\langle s,x\rangle\in|\mathcal{M}_{n,6k+5}|$ for every x, with $\langle s,x\rangle\prec_{\mathcal{M}_{n,6k+5}}\langle s,y\rangle$ if and only if $x<y$. Declare also $p\prec_{\mathcal{M}_{n,6k+5}}\langle s,x\rangle$ for all x and all p already in $|\mathcal{M}_{n,6k+5}|$ at the start of stage s. Enumerate $\langle\langle s,x\rangle,\langle s,x+1\rangle\rangle$ as a successor pair in $\mathcal{M}_{n,6k+5}$ for every x. Finally, if $\mathcal{M}_{n,6k+5}$ had a maximum element p at the start of stage s (which is if A_k was last to play in $\mathcal{M}_{n,6k+5}$), then also enumerate $\langle p,\langle s,0\rangle\rangle$ as a successor pair in $\mathcal{M}_{n,6k+5}$. Declare $\langle s,x\rangle\notin|\mathcal{M}_{m,6k+5}|$ for all x and all $m\neq n$. We say that B_k *has played in* $\mathcal{M}_{n,6k+5}$ *on account of column* i or that *column* i *of* B_k *played in* $\mathcal{M}_{n,6k+5}$. Go on to stage $s+1$.

For a given n, $\mathcal{M}_{n,6k+5}$ starts as an order of type ω, and then only the columns $i<n$ of A_k and B_k can play further elements into $\mathcal{M}_{n,6k+5}$. Moreover, for each $i<n$, once either of A_k or B_k plays in $\mathcal{M}_{n,6k+5}$ on account of column i, neither does so on account of column i again. Thus $\mathcal{M}_{n,6k+5}$ is a finite sum of orders of type ω and orders of type $\boldsymbol{k+1}$, starting with an order of type ω. In particular, $\mathcal{M}_{n,6k+5}$ either has order-type $\omega\cdot\boldsymbol{\ell}$ for some $\ell>0$ or order-type $\omega\cdot\boldsymbol{\ell}+\boldsymbol{q}$ for some $\ell>0$ and $q>k$.

We now show that $\prod_{n\in C}\mathcal{M}_{n,6k+5}$ has a maximum element if A_k has an infinite column and does not have a maximum element if B_k has an infinite column.

Suppose first that A_k has an infinite column and that i is the least infinite column of A_k. In this case, by disjointness, B_k does not have an infinite column. Thus the columns $j<i$ of A_k are finite, and the columns $j\leq i$ of B_k are finite. So A_k plays in $\mathcal{M}_{n,6k+5}$ for finitely many n on account of the columns $j<i$, and B_k plays in $\mathcal{M}_{n,6k+5}$ for finitely many n on account of the columns $j\leq i$. Let n_0 be such that for all $n>n_0$, the columns $j<i$ of A_k and the columns $j\leq i$ of B_k never play in $\mathcal{M}_{n,6k+5}$. Then for each $n>n_0$, column i of A_k eventually plays a copy of $\boldsymbol{k+1}$ at the top of $\mathcal{M}_{n,6k+5}$, after which no further elements are played in $\mathcal{M}_{n,6k+5}$. To see this, suppose that column i of A_k has played in $\mathcal{M}_{m,6k+5}$ for all m with $n_0<m<n$ by some stage s_0. By increasing s_0 if necessary, assume it is greater than all the elements of $\{s:\exists j<i\ A_k(\langle j,s\rangle)=1\}\cup\{s:\exists j\leq i\ B_k(\langle j,s\rangle)=1\}$. Let $s>s_0$ be least with $A_k(\langle i,s\rangle)=1$. Then at stage s, i is least such that $A_k(\langle i,s\rangle)=1$ or $B_k(\langle i,s\rangle)=1$. Thus at stage s, column i of A_k plays a copy of $\boldsymbol{k+1}$ at the top of $\mathcal{M}_{n,6k+5}$ if it has not done so already. Once column i of A_k plays in $\mathcal{M}_{n,6k+5}$ for an $n>n_0$, no further elements are ever added to $\mathcal{M}_{n,6k+5}$. No column $j<i$ of A_k or column $j\leq i$ of B_k ever plays in $\mathcal{M}_{n,6k+5}$ by choice of n_0. After column i of A_k plays in $\mathcal{M}_{n,6k+5}$, no column $j\geq i$ of A_k or of B_k is allowed to play in $\mathcal{M}_{n,6k+5}$ by the priority restriction. Therefore the function

$$\varphi(n)=\begin{cases}\text{the top element of the sequence of length } k+1 & \text{if } n>n_0\\ \text{played by column } i \text{ of } A_k \text{ in } \mathcal{M}_{n,6k+5} & \\ & \\ \uparrow & \text{if } n\leq n_0\end{cases}$$

is partial computable, and $\varphi(n)$ is the maximum element of $\mathcal{M}_{n,6k+5}$ when $n>n_0$. It follows that $[\varphi]$ is the maximum element of $\prod_{n\in C}\mathcal{M}_{n,6k+5}$ by Theorem 2.4.

Suppose instead that B_k has an infinite column and that i is the least infinite column of B_k. In this case, A_k does not have an infinite column. Arguing as in the previous case, there is an n_0 such that for each $n>n_0$, column i of B_k eventually plays a copy of ω at the top of $\mathcal{M}_{n,6k+5}$, after which no further elements are played in $\mathcal{M}_{n,6k+5}$. Therefore the linear order $\mathcal{M}_{n,6k+5}$ has no maximum element for all $n>n_0$. It follows that $\prod_{n\in C}\mathcal{M}_{n,6k+5}$ has no maximum element by Theorem 2.4.

We have now computed the linear orders $(\mathcal{M}_{n,k} : n, k \in \mathbb{N})$. Taking $\mathcal{L}_n = \sum_{k\in\mathbb{N}} \mathcal{M}_{n,k}$ for each n, we have that

$$\prod_C \mathcal{L}_n \quad = \quad \sum_{k\in\mathbb{N}} (\mathcal{S}_k + \mathcal{R}_k) + \mathcal{J},$$

Where $\mathcal{S}_k$ and $\mathcal{R}_k$ for each k and $\mathcal{J}$ are defined as above. We have shown that $\mathcal{S}_k \cong (\boldsymbol{k+2}) + \eta + (\boldsymbol{k+2}) + \eta + (\boldsymbol{k+2})$ for each k, that $\mathcal{R}_k = \prod_{n\in C} \mathcal{M}_{n,6k+5}$ has a maximum element if A_k has an infinite column, and that $\mathcal{R}_k$ has no maximum element if B_k has an infinite column. We need to show that, for every k, every non-maximum element of $\mathcal{R}_k$ has a successor, and that $\mathcal{J}$ does not have a finite block of size ≥ 2.

Fix k and consider a non-maximum element $[\varphi]$ of $\mathcal{R}_k$. Then $\varphi(n)$ is not the maximum element of $\mathcal{M}_{n,6k+5}$ for almost every $n \in C$ by Theorem 2.4. The successor relation on the orders $\mathcal{M}_{n,6k+5}$ is uniformly computable. Therefore, we may partially compute the function

$$\psi(n) = \begin{cases} \text{the successor of } \varphi(n) \text{ in } \mathcal{M}_{n,6k+5} & \text{if } \varphi(n)\downarrow \text{ is not maximum in } \mathcal{M}_{n,6k+5} \\ \uparrow & \text{otherwise.} \end{cases}$$

By assumption, $\psi(n)$ is defined for almost every $n \in C$, therefore $[\psi]$ is the successor of $[\varphi]$ in $\mathcal{R}_k$ by Theorem 2.4.

Now we show that $\mathcal{J}$ does not have a finite block of size ≥ 2. Consider an element $[\varphi]$ of a summand $\prod_C \mathcal{M}_{n,\theta(n)}$ of $\mathcal{J}$, where $[\theta] \in |\prod_C \mathbb{N}|_{\text{nonstd}}$. Recall that $\lim_{n\in C} \theta(n) = \infty$ in this case. By cohesiveness, there is a $d < 6$ such that $\theta(n) \equiv d \mod 6$ for almost every $n \in C$. If d is 1 or 3, then $\mathcal{M}_{n,\theta(n)} = \mathbb{Q}$ for almost every $n \in C$. It follows from Theorem 2.4 that $\prod_C \mathcal{M}_{n,\theta(n)}$ is a (countable) dense linear order without endpoints, so $\prod_C \mathcal{M}_{n,\theta(n)} \cong \eta$. Thus $[\varphi]$ is a block of size 1 in this case.

Say that an element x of a linear order *has p successors* if there are $x_0, \dots, x_p$ with $x = x_0 \prec x_1 \prec \cdots \prec x_p$, where x_{i+1} is the successor of x_i for each $i < p$. In this situation, say also that x_i is the *i^{th} successor of x*. Define x to *have p predecessors* and the *i^{th} predecessor of x* in the analogous way. In the remaining cases $d \in \{0, 2, 4, 5\}$, we show that for every p, $[\varphi]$ has either p successors or p predecessors (or both) in $\prod_C \mathcal{M}_{n,\theta(n)}$. Therefore $[\varphi]$ is in an infinite block. Fix p. If $d \in \{0, 2, 4\}$, then $\mathcal{M}_{n,\theta(n)}$ is a finite linear order of size $> 2p$ for all sufficiently large $n \in C$ because $\lim_{n\in C} \theta(n) = \infty$. Similarly, if $d = 5$ then for all sufficiently large $n \in C$, either $\mathcal{M}_{n,\theta(n)} \cong \omega \cdot \boldsymbol{\ell}$ for some $\ell > 0$ or $\mathcal{M}_{n,\theta(n)} \cong \omega \cdot \boldsymbol{\ell} + \boldsymbol{q}$ for some $\ell > 0$ and $q > 2p$. The point is that in all cases, if $n \in C$ is sufficiently large, then every element of $\mathcal{M}_{n,\theta(n)}$ has either p successors or p predecessors. Furthermore, the successor relation on $\mathcal{M}_{n,\theta(n)}$ is uniformly partial computable in n. (The 'partial' here is on account of the partiality of θ.) Therefore "$\varphi(n)$ has p successors in $\mathcal{M}_{n,\theta(n)}$" and "$\varphi(n)$ has p predecessors in $\mathcal{M}_{n,\theta(n)}$" are Σ_1 predicates. Thus by cohesiveness, it is either the case that $\varphi(n)$ has p successors in $\mathcal{M}_{n,\theta(n)}$ for almost every $n \in C$ or that $\varphi(n)$ has p predecessors in $\mathcal{M}_{n,\theta(n)}$ for almost every $n \in C$ (or both). Consider the 'successors' case. The 'predecessors' case is similar. Again, the successor relation on $\mathcal{M}_{n,\theta(n)}$ is uniformly partial computable. Therefore we may partially compute functions $\psi_1, \dots, \psi_p$, where

$$\psi_i(n) = \begin{cases} \text{the } i^{\text{th}} \text{ successor of } \varphi(n) \text{ in } \mathcal{M}_{n,\theta(n)} & \text{if } \varphi(n) \text{ has } i \text{ successors in } \mathcal{M}_{n,\theta(n)} \\ \uparrow & \text{otherwise} \end{cases}$$

for each $1 \leq i \leq p$. Then each $\psi_i(n)$ is defined for almost every $n \in C$, and

$$[\varphi] \prec_{\prod_C \mathcal{M}_{n,\theta(n)}} [\psi_1] \prec_{\prod_C \mathcal{M}_{n,\theta(n)}} \cdots \prec_{\prod_C \mathcal{M}_{n,\theta(n)}} [\psi_p]$$

are p successors of $[\varphi]$ in $\prod_C \mathcal{M}_{n,\theta(n)}$. This completes the proof. □

We now explain how to turn the sequence of linear orders defined in the above theorem into a single linear order exhibiting a Tennenbaum-like phenomenon. The single linear order is essentially the sum of the orders constructed above with additional separators. We now analyze orders of this form.

Lemma 4.2. *Let $(\mathcal{L}_n : n \in \mathbb{N})$ be a uniformly computable sequence of linear orders. Let*

$$\mathcal{S} \quad = \quad \sum_{n\in\mathbb{N}} (\{0\} + \mathcal{L}_n) \quad = \quad \{0\} + \mathcal{L}_0 + \{0\} + \mathcal{L}_1 + \cdots.$$

Let $f, g \colon \mathbb{N} \to \mathbb{N}$ be the computable functions given by $f(n) = \langle 2n, 0\rangle$ (i.e., the 0 to the left of $\mathcal{L}_n$) and $g(n) = \langle 2n+2, 0\rangle$ (i.e., the 0 to the right of $\mathcal{L}_n$). Let C be a cohesive set. Then the interval $([f], [g])$ in $\prod_C \mathcal{S}$ is isomorphic to $\prod_C \mathcal{L}_n$.

Proof. The linear order $\mathcal{S}$ has domain $\{\langle 2n, 0\rangle : n \in \mathbb{N}\} \cup \{\langle 2n+1, \ell\rangle : n \in \mathbb{N} \wedge \ell \in |\mathcal{L}_n|\}$ and is ordered in the obvious way, so it is easy to identify each $\mathcal{L}_n$ with its corresponding copy $\{2n+1\} \times |\mathcal{L}_n|$ in $\mathcal{S}$.

Let $[\varphi]_{\prod_C \mathcal{S}}$ be an element of $\prod_C \mathcal{S}$ with $[f] \prec_{\prod_C \mathcal{S}} [\varphi] \prec_{\prod_C \mathcal{S}} [g]$. Then $(\forall^\infty n \in C)(f(n) \prec_{\mathcal{S}} \varphi(n) \prec_{\mathcal{S}} g(n))$, so $(\forall^\infty n \in C)(\langle 2n, 0\rangle \prec_{\mathcal{S}} \varphi(n) \prec_{\mathcal{S}} \langle 2n+2, 0\rangle)$, so $(\forall^\infty n \in C)(\varphi(n) \in \{2n+1\} \times |\mathcal{L}_n|)$. Let

$$\psi(n) = \begin{cases} \pi_1(\varphi(n)) & \text{if } \pi_0(\varphi(n)) = 2n+1 \\ \uparrow & \text{otherwise.} \end{cases}$$

Then $[\varphi]_{\prod_C \mathcal{S}}$ corresponds to the element $[\psi]_{\prod_C \mathcal{L}_n}$ of $\prod_C \mathcal{L}_n$. Conversely, let $[\psi]_{\prod_C \mathcal{L}_n}$ be an element of $\prod_C \mathcal{L}_n$. Thus $C \subseteq^* \mathrm{dom}(\psi)$ and $\forall n\, (\psi(n)\downarrow \ \to\ \psi(n) \in |\mathcal{L}_n|)$. Let $\varphi(n) \simeq \langle 2n+1, \psi(n)\rangle$ so that $[\varphi]_{\prod_C \mathcal{S}}$ is an element of $\prod_C \mathcal{S}$. Then $(\forall^\infty n \in C)(\langle 2n, 0\rangle \prec_{\mathcal{S}} \varphi(n) \prec_{\mathcal{S}} \langle 2n+2, 0\rangle)$, so in fact $[\varphi]_{\prod_C \mathcal{S}}$ is an element of the interval $([f], [g])$ in $\prod_C \mathcal{S}$ corresponding to $[\psi]_{\prod_C \mathcal{L}_n}$. These correspondences are inverses of each other and preserve order, so the interval $([f], [g])$ in $\prod_C \mathcal{S}$ is isomorphic to $\prod_C \mathcal{L}_n$. □

In the statement of Lemma 4.2, it would be more pleasing to write $\mathcal{S} = \sum_{n \in \mathbb{N}}(\mathbf{1} + \mathcal{L}_n)$. The reason for the somewhat awkward formulation is to give names to the singleton breakpoints to describe the functions f and g. Regardless, this construction yields our desired result.

Theorem 4.3. *There is a computable linear order $\mathcal{L}$ such that for every cohesive set C and every presentation $\mathcal{P}$ of $\prod_C \mathcal{L}$, the double-jump $\mathcal{P}''$ has* PA*-degree relative to $0''$. Thus $\prod_C \mathcal{L}$ has no computable presentation.*

Proof. Let $(\mathcal{L}_n : n \in \mathbb{N})$ be the uniformly computable sequence of linear orders from Theorem 4.1. So for every cohesive set C, the double-jump $\mathcal{Q}''$ of every presentation $\mathcal{Q}$ of the cohesive product $\prod_C \mathcal{L}_n$ has PA-degree relative to $0''$. Let $\mathcal{L} = \sum_{n \in \mathbb{N}}(\{0\} + \mathcal{L}_n)$, and let $f, g \colon \mathbb{N} \to \mathbb{N}$ be the computable functions $f(n) = \langle 2n, 0\rangle$ and $g(n) = \langle 2n+2, 0\rangle$ as in the statement of Lemma 4.2. Let C be a cohesive set, and let $\mathcal{P}$ be a presentation of $\prod_C \mathcal{L}$. Let $a, b \in |\mathcal{P}|$ be the elements corresponding to $[f]$ and $[g]$ respectively. Then the interval (a, b) in $\mathcal{P}$ is isomorphic to $\prod_C \mathcal{L}_n$ by Lemma 4.2. Thus $\mathcal{P}$ computes a presentation $\mathcal{Q}$ of $\prod_C \mathcal{L}_n$, and therefore $\mathcal{P}'' \geq_{\mathrm{T}} \mathcal{Q}''$, which has PA-degree relative to $0''$. So $\mathcal{P}''$ has PA-degree relative to $0''$. □

Corollary 4.4. *There is a computable structure $\mathcal{L}$ such that for every cohesive set C, $\prod_C \mathcal{L}$ has no degree. In other words,* $\mathrm{DgSp}(\prod_C \mathcal{L})$ *has no minimum element.*

Proof. The $\mathcal{L}$ described above has the property that, for every cohesive set C, $\mathbf{0} \notin \mathrm{DgSp}(\prod_C \mathcal{L})$. It follows from Theorem 2.1 that for every cohesive set C, $\mathrm{DgSp}(\prod_C \mathcal{L})$ has no minimum element. □

5. Further questions

In addition to Questions (1) and (2) from the introduction, a number of further questions concerning the degree spectra of cohesive powers come to mind. We relay a few of them here.

Further Questions.

(5) What other Turing degrees can be realized as degrees of cohesive powers in the sense of Corollary 3.4?

(6) What can be encoded into cohesive powers of linear orders? Which degree spectra of linear orders are possible to achieve in this way?

(7) Is Proposition 3.2 best possible when $k > 2$? Is there a structure $\mathcal{G}$ such that for every strictly Δ_k cohesive set C, $\prod_C \mathcal{G}$ has no Δ_k copies?

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References

[1] Christopher J. Ash, Carl G. Jockusch Jr., and Julia F. Knight, *Jumps of orderings*, Transactions of the American Mathematical Society **319** (1990), no. 2, 573–599.

[2] Christopher J. Ash and Julia F. Knight, *Computable Structures and the Hyperarithmetical Hierarchy*, Studies in Logic and the Foundations of Mathematics, vol. 144, North-Holland Publishing Co., Amsterdam, 2000.

[3] Rumen Dimitrov, *Cohesive powers of computable structures*, Godishnik na Sofiĭskiya Universitet "Sv. Kliment Ohridski". Fakultet po Matematika i Informatika. Annuaire de l'Université de Sofia "St. Kliment Ohridski". Faculté de Mathématiques et Informatique **99** (2009), 193–201.

[4] Rumen Dimitrov, Valentina Harizanov, Andrey Morozov, Paul Shafer, Alexandra A. Soskova, and Stefan V. Vatev, *On cohesive powers of linear orders*, The Journal of Symbolic Logic **88** (2023), no. 3, 947–1004.

[5] Solomon Feferman, Dana S. Scott, and Stanley Tennenbaum, *Models of arithmetic through function rings*, Notices of the American Mathematical Society **6** (1959), no. 2, 173–174. Abstract 556-31.

[6] Harvey Friedman and Lee Stanley, *A Borel reducibility theory for classes of countable structures*, The Journal of Symbolic Logic **54** (1989), no. 3, 894–914.

[7] Su Gao, *Some dichotomy theorems for isomorphism relations of countable models*, The Journal of Symbolic Logic **66** (2001), no. 2, 902–922.

[8] David Gonzalez and Matthew Harrison-Trainor, *Scott spectral gaps are bounded for linear orderings*, 2025. ArXiv preprint arXiv:2411.12084.

[9] Yoram Hirschfeld, *Models of arithmetic and recursive functions*, Israel Journal of Mathematics **20** (1975), no. 2, 111–126.

[10] Yoram Hirschfeld and William H. Wheeler, *Forcing, Arithmetic, Division Rings*, Lecture Notes in Mathematics, vol. 454, Springer-Verlag, Berlin–New York, 1975.

[11] Carl G. Jockusch Jr. and Robert I. Soare, *Degrees of orderings not isomorphic to recursive linear orderings*, 1991, pp. 39–64. International Symposium on Mathematical Logic and its Applications (Nagoya, 1988).

[12] Richard Kaye, *Models of Peano Arithmetic*, Oxford Logic Guides, vol. 15, The Clarendon Press, Oxford University Press, New York, 1991.

[13] Julia F. Knight, *Degrees coded in jumps of orderings*, The Journal of Symbolic Logic **51** (1986), no. 4, 1034–1042.

[14] Manuel Lerman, *Recursive functions modulo co-r-maximal sets*, Transactions of the American Mathematical Society **148** (1970), 429–444.

[15] ———, *Degrees of Unsolvability: Local and Global Theory*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1983.

[16] Thomas G. McLaughlin, *Embeddings of and into Nerode semirings*, Israel Journal of Mathematics **60** (1987), no. 1, 65–88.

[17] ———, *Some extension and rearrangement theorems for Nerode semirings*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik **35** (1989), no. 3, 197–209.

[18] ———, *Sub-arithmetical ultrapowers: a survey*, Annals of Pure and Applied Logic **49** (1990), no. 2, 143–191.

[19] ———, *Recursive ultrapowers, simple models, and cofinal extensions*, Archive for Mathematical Logic **31** (1992), no. 4, 287–296.

[20] ———, *A note on effective ultrapowers: uniform failure of bounded collection*, Mathematical Logic Quarterly **39** (1993), no. 4, 431–435.

[21] ———, *Δ_1 ultrapowers are totally rigid*, Archive for Mathematical Logic **46** (2007), no. 5-6, 379–384.

[22] Russell Miller, *The Δ^0_2-spectrum of a linear order*, The Journal of Symbolic Logic **66** (2001), no. 2, 470–486.

[23] Antonio Montalbán, *Computable Structure Theory: Within the Arithmetic*, Perspectives in Logic, Cambridge University Press, 2021.

[24] ———, *Computable Structure Theory: Beyond the Arithmetic*, Perspectives in Logic, Cambridge University Press, 2026.

[25] Linda Jean Richter, *Degrees of structures*, The Journal of Symbolic Logic **46** (1981), no. 4, 723–731.

[26] Joseph G. Rosenstein, *Linear Orderings*, Pure and Applied Mathematics, vol. 98, Academic Press, New York–London, 1982.

[27] Paul Shafer, *Effective powers of ω over Δ_2 cohesive sets and infinite Π_1 sets without Δ_2 cohesive subsets*, 2023. To appear in The Journal of Symbolic Logic.

[28] V. Yu. Shavrukov, *R.e. prime powers and total rigidity*, Advances in Mathematics **360** (2020), 106884, 50.

[29] Thoralf Skolem, *Über die nicht-charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler aussagen mit ausschliesslich Zahlenvariablen*, Fundamenta Mathematicae **23** (1934), no. 1, 150–161.

[30] Robert I. Soare, *Recursively Enumerable Sets and Degrees*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1987.

[31] Stanley Tennenbaum, *Non-Archimedean models for arithmetic*, Notices of the American Mathematical Society **6** (1959), no. 270.

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