Leeds Computability Days 2024
University of Leeds School of Mathematics
Leeds, UK
02 – 05 July 2024

Tuesday, 02 July 2024
Lecture Theatre D, Chemistry Building, University of Leeds

Before 10:00am:	Coffee available!
10:00am – 10:50am:	Keita Yokoyama (Tohoku University) *Proof interpretations based on low basis type theorems and forcing*
10:50am – 11:10am:	Morning break!
11:10am – 12:00pm:	Katarzyna Kowalik (University of Würzburg) *Computable ultrapowers, forcing and proof size*
12:00pm – 2:00pm:	Lunch!
2:00pm – 2:50pm:	Mathieu Hoyrup (University of Lorraine / LORIA) *Computable presentations of topological spaces*
2:50pm – 3:20pm:	Afternoon break!
3:20pm – 4:10pm:	Sam Sanders (Ruhr University Bochum) *The Big, Bigger, and Biggest Five of reverse mathematics*
4:10pm – 5:00pm:	Rupert Hölzl (Bundeswehr University Munich) *Benign approximations, superspeedability, and randomness*

Wednesday, 03 July 2024
MALL (Level 8), School of Mathematics, University of Leeds

Before 10:00am:	Coffee available!
10:00am – 10:50am:	Dino Rossegger (TU Wien) *Learning equivalence relations on Polish spaces*
10:50am – 11:10am:	Morning break!
11:10am – 12:00pm:	David Fernández-Duque (University of Barcelona) *Provable well-orders and hyperarithmetical soundness*
12:00pm – 2:00pm:	Lunch!
2:00pm – 2:50pm:	Johanna Franklin (Hofstra University) *Highness in the reticent sense*
2:50pm – 3:20pm:	Afternoon break!
3:20pm – 4:10pm:	David Gonzalez (University of California, Berkeley) *Hybrid maximal filter spaces*
4:30pm – 6:00pm	Reception!

Thursday, 04 July 2024
MALL (Level 8), School of Mathematics, University of Leeds

Before 10:00am:	Coffee available!
10:00am – 10:50am:	Patrick Uftring (University of Würzburg) *Weihrauch degrees without roots*
10:50am – 11:10am:	Morning break!
11:10am – 12:00pm:	Emanuele Frittaion (University of Leeds) *Games for Peano arithmetic and elementary descent recursion*
12:00pm – 2:00pm:	Lunch!
2:00pm – 2:50pm:	Shuwei Wang (University of Leeds) Σ^1_1-*computability and realisability of a global well-ordering*
2:50pm – 3:20pm:	Afternoon break!
3:20pm – 4:10pm:	Alice Vidrine (University of Wisconsin–Madison) *Some results on enumeration Weihrauch reduction*
4:10pm – 5:00pm:	Ellen Hammatt (Victoria University of Wellington) *Structures computable without delay*

Friday, 05 July 2024
MALL (Level 8), School of Mathematics, University of Leeds

Before 10:00am:	Coffee available!
10:00am – 10:25am:	Heidi Benham (University of Connecticut) (*online talk*) *The Ginsburg–Sands theorem and computability theory*
10:25am – 10:50am:	Damir Dzhafarov (University of Connecticut) (*online talk*) *The strength of the Ginsburg–Sands theorem for T_1 spaces*
10:50am – 11:10am:	Morning break!
11:10am – 12:00pm:	Dan Turetsky (Victoria University of Wellington) *The descriptive complexity of* high_α
12:00pm – 2:00pm:	Lunch!
2:00pm – 2:50pm:	Arno Pauly (Swansea University) *More on the indivisibility of* $\mathbb{Q}$
2:50pm – 3:20pm:	Afternoon break!
3:20pm – 4:10pm:	Yudai Suzuki (NIT, Okinawa College) *On some subtheories of* $\Pi^1_1\text{-}\mathsf{CA}_0$

Abstracts (alphabetical by author)

Heidi Benham (University of Connecticut)

The Ginsburg–Sands theorem and computability theory

At present, topological theorems have a relatively small presence in the reverse mathematical zoo. One topological result that has turned out to be a fertile starting point for reverse mathematical analysis is a theorem due to Ginsburg and Sands. The theorem states that every infinite topological space has an infinite subspace that is homeomorphic to exactly one of the following five topologies on the natural numbers: indiscrete, discrete, initial segment, final segment, or cofinite. The original proof, which is nonconstructive and uses an application of Ramsey's theorem for pairs, was given in the context of topology. This left open the question of which axioms are necessary to prove this theorem. Using Dorais's formalization of CSC spaces, we analyze the location of this theorem in the reverse math zoo, as well as the location of several related theorems. The Ginsburg–Sands Theorem for CSC spaces turns out to be equivalent to ACA_0. We also look at the theorem restricted to certain types of topological spaces. On the weaker end of the spectrum, we have that the Ginsburg–Sands theorem when restricted to Hausdorff spaces is equivalent to RCA_0. Interestingly, when restricting the theorem to T_1 CSC spaces, the strength is equivalent to none of the big five subsystems of second order arithmetic, but rather lies strictly between the strength of ACA_0 and RT^2_2, which is an unusual place for a natural mathematical theorem. This talk is based on work done jointly with Andrew DeLapo, Damir Dzhafarov, Reed Solomon, and Java Darleen Villano.

Damir Dzhafarov (University of Connecticut)

The strength of the Ginsburg–Sands theorem for T_1 spaces

The Ginsburg–Sands theorem asserts that every infinite T_1 topological space either has an infinite discrete subspace or an infinite subspace homeomorphic to the cofinite topology on the naturals. In the context of reverse mathematics and computability theory, where the theorem is studied for CSC spaces in the sense of Dorais, this theorem (which we denote GST_1) has a very interesting and somewhat surprising strength. In particular, GST_1 lies strictly between ACA_0 and Ramsey's theorem for pairs and two colors (RT^2_2). In this talk, I will focus on the connection of GST_1 with RT^2_2, outlining the implication and the main ideas used in the separation.

David Fernández-Duque (University of Barcelona)

Provable well-orders and hyperarithmetical soundness

Ordinal analysis traditionally measures the strength of formal theories using the supremum of the order types of their provable well-orders. In this talk, we will instead propose the use of proof-theoretic ordinals to measure the degree of soundness of a theory: to be precise, we define an ordinal-theoretic measure of the degree of hyperarithmetical correctness of a mathematical theory. We then characterize the ordinals that are assigned in this way to theories of various degrees of hyperarithmetical soundness.

Joint work with Juan Aguilera.

Johanna Franklin (Hofstra University)

Highness in the reticent sense

We consider the distinction between positive information and complete information in the context of highness for computable structure theory. For instance, we say that a degree that is uniformly high for isomorphism "in the reticent sense" will not only compute an isomorphism when one exists but also fail to compute an apparent isomorphism where one does not. We will consider reticence for this and other structural highness notions.

This work is joint with Wesley Calvert and Dan Turetsky.

Emanuele Frittaion (University of Leeds)

Games for Peano arithmetic and elementary descent recursion

I will discuss a "constructive" game semantics for classical first-order arithmetic due to Coquand. In such "semantics," winning strategies correspond to cut-free proofs in a certain infinitary propositional logic. Cut elimination corresponds to debates between winning strategies. The proof of cut elimination, i.e., the proof that these debates eventually terminate, is by transfinite induction on certain "interaction" sequences of ordinals. In this talk, I will present an effective implementation of Coquand's cut elimination, one that allows us to describe winning strategies by elementary descent recursive functions. A byproduct of this analysis is, of course, a characterization of the provably recursive functions and functionals of Peano arithmetic.

David Gonzalez (University of California, Berkeley)

Hybrid maximal filter spaces

There are many ways to represent topological spaces for the purposes of reverse math. Because a topology is usually defined as a third-order object, these second-order representations tend to lose a lot of generality or be quite cumbersome to work with. We introduce a new second-order coding for general topological spaces called a hybrid maximal filter space. I will discuss the benefits of this encoding when compared to the previously proposed methods and argue for the naturalness of the hybrid maximal filter formalism.

We will conclude with a discussion of metrization in the context of hybrid maximal filter spaces. In contrast to the encoding used by Mummert and Simpson (in [1]) where the metrization theorem sits at Π^1_2-CA_0, the metrization theorem for hybrid maximal filter spaces can be proven all the way down at ACA_0.

[1] Carl Mummert and Stephen G. Simpson. Reverse mathematics and Π^1_2 comprehension. *Bull. Symbolic Logic*, 11(4):526–533, 2005.

Ellen Hammatt (Victoria University of Wellington)

Structures computable without delay

In this talk we investigate what happens when we take concepts from computable structure theory and forbid the use of unbounded search. In other words, we discuss the primitive recursive content of structure theory. The central definition is that of punctual structures, introduced by Kalimullin, Melnikov and Ng in 2017. We will investigate the primitive recursive analogues of various concepts from computable structure theory, including categoricity, computable dimension and 1-decidability. A common theme is that new techniques are required in the primitive recursive case. We also discuss a degree structure within punctual presentations which is induced by primitive recursive isomorphisms. This degree structure is a new concept that does not arise in computable structure theory.

Rupert Hölzl (Bundeswehr University Munich)

Benign approximations, superspeedability, and randomness

Speedable numbers are real numbers which are algorithmically approximable from below and whose approximations can be accelerated nonuniformly in some sense. We begin by reviewing recent results on these numbers, including a negative answer to the question of Merkle and Titov whether, among the left-computable numbers, being Martin-Löf random is equivalent to being non-speedable.

To achieve this answer it was necessary to study strict subsets of the left-computable numbers whose elements possess left-approximations that are in some sense better behaved than general left-computation; we collectively refer to these as benign approximations and continue with further results on them: First we answer a question of Barmpalias by separating a strict subclass that we will refer to as superspeedable from the speedable numbers; for elements of this subclass, acceleration is possible uniformly and to an even higher degree. This new notion then integrates itself into the hierarchy of benign approximation notions. We add a new perspective to the study of this hierarchy by juxtaposing it with the well-studied hierarchy of algorithmic randomness notions.

Based on joint works with Peter Hertling, Philip Janicki, Wolfgang Merkle, and Frank Stephan.

Mathieu Hoyrup (University of Lorraine / LORIA)

Computable presentations of topological spaces

A computable presentation of a countably-based space is an enumeration of a basis of non-empty open sets whose finite intersections can be computed. It is a rather weak notion of presentation: Melnikov and Ng have shown the existence of arbitrarily complex Polish spaces having computable presentations; Bazhenov, Melnikov and Ng have proved that every Δ^0_2 Polish space has a computable topological presentation. In a joint work with Melnikov and Ng, we show how the latter result can be considerably extended.

Katarzyna Kowalik (University of Würzburg)

Computable ultrapowers, forcing and proof size

Definable ultrapowers are a classical tool used to construct nonstandard models of arithmetic. In contrast to ultrapowers obtained from the whole $\mathbb{N}^{\mathbb{N}}$, one can reason about definable ultrapowers inside arithmetical theories using satisfaction predicates. Thus, one can syntactically simulate a model-theoretic construction of an ultrapower and, by means of a method called a forcing interpretation, obtain some information about proof size.

We will illustrate this technique using computable ultrapowers and prove a non-speedup result for the chain/antichain principle (CAC) over the weak base theory RCA_0^*: there exists a polynomial-time algorithm such that, given a proof of a Π^0_3 sentence in the theory $\mathsf{RCA}_0^* + \mathsf{CAC}$, outputs a proof of this sentence in RCA_0^*. This stands in sharp contrast with non-elementary speedup of RT^2_2 over RCA_0^*, a result proved earlier by Kołodziejczyk, Wong, and Yokoyama.

Arno Pauly (Swansea University)

More on the indivisibility of $\mathbb{Q}$

If we colour the rationals (seen here as the dense countable linear order) with finitely many colours, we are guaranteed the existence of a monochromatic isomorphic copy. I will discuss the Weihrauch degree of the problem of finding such an isomorphic copy, given the colouring as input. Recently, this problem was independently studied by Gill, and by Dzhafarov, Solomon and Valenti. I will answer some of the open questions they posed, and paint a more complete picture of where in the Weihrauch lattice this problem is situated.

Dino Rossegger (TU Wien)

Learning equivalence relations on Polish spaces

In joint work with Ted Slaman and Tomasz Steifer, we introduced frameworks that give a formal notion of algorithmic learnability for equivalence relations on Polish spaces. Our main results characterize learnability in these frameworks via the descriptive complexity of the equivalence relations, and, using techniques from higher recursion theory and effective descriptive set theory, we calculate the complexity of the class of learnable equivalence relations. Computability theory gives rise to many important examples of equivalence relations, such as Turing equivalence and recursive isomorphism of sets. In this talk, I will introduce our framework, summarize the main results, and discuss the complexity of equivalence relations arising in computability theory.

Sam Sanders (Ruhr University Bochum)

The Big, Bigger, and Biggest Five of reverse mathematics

I provide an overview of my recent joint work with Dag Normann (University of Oslo) on the Reverse Mathematics (RM for short) of the uncountable ([3–9]).

The well-known *Big Five phenomenon* of RM (see [2, 10]) is the observation that a large number of theorems from ordinary mathematics are either provable in the base theory of RM or equivalent to one of only four systems; these five systems together are called the 'Big Five' of RM. The aim of this paper is to **greatly** extend the Big Five phenomenon, working in Kohlenbach's *higher-order* RM ([1]).

In particular, we have established numerous equivalences involving the **second-order** Big Five systems on one hand, and well-known **third-order** theorems from analysis about (possibly) discontinuous functions on the other hand. We study both relatively tame notions, like cadlag or Baire 1, and potentially wild ones, like quasi-continuity.

We also show that *slight* generalisations and variations (involving e.g. the notions Baire 2 and cliquishness) of the aforementioned third-order theorems fall *far* outside of the Big Five. These observation give rise to four new 'Big' third-order systems that boast many equivalences, namely the *uncountability of* $\mathbb{R}$ ([5, 7, 9]), the *Jordan decomposition theorem* ([4, 9]), the *Baire category theorem* ([3, 8]), and Tao's *pigeon hole principle for measure* ([8, 9]).

Finally, we indicate connections to Kleene's higher-order computability theory when relevant.

[1] Ulrich Kohlenbach, *Higher order reverse mathematics*, Reverse mathematics 2001, Lect. Notes Log., vol. 21, ASL, 2005, pp. 281–295.

[2] Antonio Montalbán, *Open questions in reverse mathematics*, Bull. Sym. Logic **17** (2011), no. 3, 431–454.

[3] Dag Normann and Sam Sanders, *Open sets in computability theory and reverse mathematics*, Journal of Logic and Computation **30** (2020), no. 8, pp. 40.

[4] ______, *On robust theorems due to Bolzano, Jordan, Weierstrass, and Cantor in Reverse Mathematics*, Journal of Symbolic Logic, DOI: `doi.org/10.1017/jsl.2022.71` (2022), pp. 51.

[5] ______, *On the uncountability of* $\mathbb{R}$, Journal of Symbolic Logic, DOI: `doi.org/10.1017/jsl.2022.27` (2022), pp. 43.

[6] ______, *The Biggest Five of Reverse Mathematics*, Journal of Mathematical Logic, doi: `https://doi.org/10.1142/S0219061324500077` (2023), pp. 56.

[7] Sam Sanders, *Big in Reverse Mathematics: the uncountability of the real numbers*, Journal of Symbolic Logic, doi:`https://doi.org/10.1017/jsl.2023.42` (2023), pp. 26.

[8] ______, *Big in Reverse Mathematics: measure and category*, Journal of Symbolic Logic, doi: `https://doi.org/10.1017/jsl.2023.65` (2023), pp. 44.

[9] ______, *Bernstein Polynomials throughout Reverse Mathematics*, To appear in the Journal of Symbolic Logic, arxiv: `https://arxiv.org/abs/2311.11036` (2023), pp. 28.

[10] Stephen G. Simpson, *Subsystems of second order arithmetic*, 2nd ed., Perspectives in Logic, CUP, 2009.

Yudai Suzuki (NIT, Okinawa College)

On some subtheories of Π^1_1-CA_0

It is well-known that there is no Π^1_2 sentence equivalent to Π^1_1-CA_0. Despite this fact, there are interesting mathematical theorems expressed by a Π^1_2 sentence provable from Π^1_1-CA_0 that may not be provable from ATR_0 such as Kruskal's theorem on embeddings of trees. To study the complexity of those theorems, Towsner introduced a hierarchy of Π^1_2 theorems provable from Π^1_1-CA_0 [1]. In this talk, we generalize his idea and introduce a new hierarchy which covers the set $\{\sigma \in \Pi^1_2 : \Pi^1_1\text{-}\mathsf{CA}_0 \vdash \sigma\}$. We then introduce a new descriptive set-theoretic principle based on Galvin-Prikry's theorem and give another characterization of this set. This is joint work with Keita Yokoyama [2].

In addition, I would like to talk about some related new results.

[1] Towsner, Henry. *Partial impredicativity in reverse mathematics.* The Journal of Symbolic Logic 78.2 (2013): 459-488.

[2] Suzuki, Yudai, and Yokoyama, Keita. *On the* Π^1_2 *consequences of* Π^1_1-CA_0. arXiv preprint arXiv:2402.07136 (2024).

Dan Turetsky (Victoria University of Wellington)

The descriptive complexity of high_α

What is the complexity of the set of high_α reals? You can get a complexity by writing down a natural formula, but how do you show that's tight? In this talk, I will focus on the proof techniques involved. This includes some interesting forcing techniques and the method of true stages. This is joint work with Noam Greenberg, Joe Miller and Mariya Soskova.

Patrick Uftring (University of Würzburg)

Weihrauch degrees without roots

We answer the following question by Arno Pauly: "Is there a square root operator on the Weihrauch degrees?" In fact, we show that there are uncountably many pairwise incomparable Weihrauch degrees without any roots. We also prove that the omniscience principles of LPO and LLPO do not have roots.

Alice Vidrine (University of Wisconsin–Madison)

Some results on enumeration Weihrauch reduction

Enumeration Weihrauch (eW) reduction is a variant of Weihrauch reduction that replaces Turing functionals by enumeration operators, using a notion of "positive information" computation. In this setting, several of the benchmark choice problems in the Weihrauch setting become separated. We discuss some of the more striking examples and their proofs, and show a first order difference between the Weihrauch and eW degrees.

Shuwei Wang (University of Leeds)

Σ^1_1-computability and realisability of a global well-ordering

Using Kleene's normal form theorem for Σ^1_1-formulae, we can organise the collection of Σ^1_1-definable partial functions into a partial combinatory algebra. This can then be used as a realisability structure to interpret intuitionistic theories of arithmetic that allow comprehension over all arithmetic formulae.

In this talk, I will use CM, a novel formal system of intuitionistic third-order arithmetic with a classical first-order fragment in Weaver [1], as an example to discuss the increase in power of this realisability model as we move to stronger fragments of second-order arithmetic in the meta-theory. The goal is to interpret an axiom of global well-ordering on all second-order objects, which functions as a choice-like extension of CM and provides an analogue of Zorn's lemma for the theory.

We will demonstrate that this approach gives an ordinal analysis of Weaver's theory. CM itself will have the proof-theoretic ordinal $\varphi_{\varepsilon_0}0$, while the addition of the global well-ordering will increase its strength to the Bachmann–Howard ordinal.

[1] Nik Weaver *Axiomatizing mathematical conceptualism in third order arithmetic* (2009). Available at `arXiv:0905.1675[math.HO]`.

Keita Yokoyama (Tohoku University)

Proof interpretations based on low basis type theorems and forcing

In reverse mathematics, numerous conservation theorems are established using low basis theorems and their variations. Specifically, given two Π^1_2 theories T and T', if for any model (M, S) of T, there exists an extension $\bar{S} \supseteq S$ (obtained via a low basis theorem or forcing) such that $(M, \bar{S})$ is a model of T', then T' is Π^1_1-conservative over T. As this conservation is proven model-theoretically, extracting a proof of $T \vdash \varphi$ from a proof of $T' \vdash \varphi$ for a Π^1_1-sentence φ is not straightforward.

In this talk, we will present a framework for converting these model-theoretic arguments into syntactic interpretations. We will then reprove several conservation theorems together with polynomial-size proof transformations. This work is partly a collaboration with Hiroyuki Ikari and Leszek Kołodziejczyk.