CODING TRUE ARITHMETIC IN THE MEDVEDEV AND MUCHNIK DEGREES

PAUL SHAFER

Abstract. We prove that the first-order theory of the Medvedev degrees, the first-order theory of the Muchnik degrees, and the third-order theory of true arithmetic are pairwise recursively isomorphic (obtained independently by Lewis, Nies, and Sorbi [7]). We then restrict our attention to the degrees of closed sets and prove that the following theories are pairwise recursively isomorphic: the first-order theory of the closed Medvedev degrees, the first-order theory of the compact Medvedev degrees, the first-order theory of the closed Muchnik degrees, the first-order theory of the compact Muchnik degrees, and the second-order theory of true arithmetic. Our coding methods also prove that neither the closed Medvedev degrees nor the compact Medvedev degrees are elementarily equivalent to either the closed Muchnik degrees or the compact Muchnik degrees.

1. Introduction

The complexities of the first-order theories of degree structures are a central topic in computability theory. The results typically show that these theories are computationally as complicated as possible. Major results include (in chronological order):

- The first-order theory of the Turing degrees is recursively isomorphic to the second-order theory of true arithmetic (Simpson [15]).
- The first-order theory of the Turing degrees below $0'$ is recursively isomorphic to the first-order theory of true arithmetic (Shore [14]).
- The first-order theory of the Turing degrees of r.e. sets is recursively isomorphic to the first-order theory of true arithmetic (Harrington and Slaman, unpublished; see also Nies, Shore, and Slaman [12]).

We continue in this vein by proving two main theorems:

- Theorem 3.13: The first-order theory of the Medvedev degrees, the first-order theory of the Muchnik degrees, and the third-order theory of true arithmetic are pairwise recursively isomorphic (obtained independently by Lewis, Nies, and Sorbi [7]).
- Theorem 5.12: The following theories are pairwise recursively isomorphic: the first-order theory of the closed Medvedev degrees, the first-order theory of the compact Medvedev degrees, the first-order theory of the closed Muchnik degrees, the first-order theory of the compact Muchnik degrees, and the second-order theory of true arithmetic.

In addition we prove:

- Theorem 6.3: Neither the closed Medvedev degrees nor the compact Medvedev degrees are elementarily equivalent to either the closed Muchnik degrees or the compact Muchnik degrees.

Our codings of arithmetic into the Medvedev and Muchnik degree structures are direct. We define parameters coding ω, $\leq$, $+$, and $\times$, and then we explain how to simulate quantification. In the third-order case, we show that any Medvedev degree or Muchnik degree codes both a subset of ω and a subset of 2^ω. Hence quantification over the Medvedev degrees or over the Muchnik degrees simulates both quantification over 2^ω and quantification over 2^{2^ω}. In the second-order case, we use

This research was partially supported by NSF grants DMS-0554855 and DMS-0852811.

a different coding and again show that quantification over the closed degrees or over the compact degrees simulates quantification over 2^ω. In contrast, Lewis, Nies, and Sorbi's proof of Theorem 3.13 relies on the following facts: The third-order theory of arithmetic is recursively isomorphic to the second-order theory of the reals, and the reals can be coded as a symmetric graph.

This paper is organized as follows: The rest of the introduction establishes notation and defines the objects considered. Section 2 interprets the various degree structures in third-order arithmetic or in second-order arithmetic. Section 3 interprets third-order arithmetic in the Medvedev degrees and in the Muchnik degrees. Section 4 interprets second-order arithmetic in the closed Muchnik degrees and in the compact Muchnik degrees. Section 5 interprets second-order arithmetic in the closed Medvedev degrees and in the compact Medvedev degrees. Section 6 distinguishes the first-order theories of the closed Medvedev degrees and the compact Medvedev degrees from the first-order theories of the closed Muchnik degrees and the compact Muchnik degrees

1.1. **Basic notation.** Φ_e denotes the e^{th} Turing functional. The function $\langle \cdot, \cdot \rangle \colon \omega \times \omega \to \omega$ is a fixed recursive bijection. For $f, g \in \omega^\omega$, $f \oplus g \in \omega^\omega$ is the function where $(f \oplus g)(2n) = f(n)$ and $(f \oplus g)(2n+1) = g(n)$. For finite sequences $\sigma, \tau \in \omega^{<\omega}$, $\sigma \subset \tau$ means that σ is an initial segment of τ. Similarly, $\sigma \subset f$ means that σ is an initial segment of f. The sequence $\sigma^\frown\tau$ is the concatenation of sequences σ and τ. Similarly $\sigma^\frown f$ is the concatenation of σ and f. For $\sigma \in \omega^{<\omega}$ and $\mathcal{A} \subseteq \omega^\omega$, $\sigma^\frown\mathcal{A}$ denotes $\{\sigma^\frown f \mid f \in \mathcal{A}\}$. The sequence $f \upharpoonright n$ is the initial segment of f of length n. The length of a sequence σ is denoted by $|\sigma|$. A tree is a set $T \subseteq \omega^{<\omega}$ closed under initial segments. A function f is a path through T if $(f \upharpoonright n) \in T$ for all $n \in \omega$. For $A, B \subseteq \omega$, we write $A \leq_1 B$ if there is a one-to-one recursive function f such that $\forall n(n \in A \leftrightarrow f(n) \in B)$. A and B are *recursively isomorphic* if there is such an f that is a bijection. The Myhill isomorphism theorem states that A and B are recursively isomorphic if and only if $A \leq_1 B$ and $B \leq_1 A$ (see [18] Section I.5).

Our coding will make use of the following familiar definitions from recursion theory:

Definition 1.1. $\mathcal{A} \subseteq \omega^\omega$ is a *Turing antichain* if $f \mid_T g$ for any distinct $f, g \in \mathcal{A}$.

Definition 1.2. $\mathcal{A} \subseteq \omega^\omega$ is *independent* if $g \not\leq_{\mathrm{T}} f_1 \oplus \cdots \oplus f_n$ for any distinct $g, f_1, \ldots, f_n \in \mathcal{A}$.

Infinite independent sets exist. See [6] section II.3 for an example. An independent set is a Turing antichain.

1.2. **Standard relational models of arithmetic.** We describe what we mean by "true arithmetic" by defining the standard relational models of first-order, second-order, and third-order arithmetic. In what follows, equality is always part of the language and is always interpreted as true equality on ω. Equality on 2^ω and 2^{2^ω} is defined in terms of membership via extensionality.

The standard model of arithmetic is the structure $\mathfrak{N} = \langle \omega, \leq, +, \times \rangle$. The relations $\leq \subseteq \omega^2$, $+ \subseteq \omega^3$, and $\times \subseteq \omega^3$ are interpreted as the usual less-than-or-equal-to, plus, and times. Variables x range over ω.

The standard model of second-order arithmetic is the structure $\mathfrak{N}_2 = \langle \omega, 2^\omega, \leq, +, \times, \in \rangle$. The relations $\leq$, $+$, and $\times$ are interpreted as usual. The relation $\in \subseteq \omega \times 2^\omega$ is interpreted as membership. Variables x range over ω and variables X range over 2^ω. $\mathrm{Th}(\mathfrak{N}_2)$ denotes the theory of $\mathfrak{N}_2$, the set of all sentences in this language true in $\mathfrak{N}_2$.

The standard model of third-order arithmetic is the structure $\mathfrak{N}_3 = \langle \omega, 2^\omega, 2^{2^\omega}, \leq, +, \times, \in_2, \in_3 \rangle$. The relations $\leq$, $+$, and $\times$ are interpreted as usual. The relation $\in_2 \subseteq \omega \times 2^\omega$ is interpreted as second-order membership and the relation $\in_3 \subseteq 2^\omega \times 2^{2^\omega}$ is interpreted as third-order membership. Variables x range over ω, variables X range over 2^ω, and variables $\mathcal{X}$ range over 2^{2^ω}. $\mathrm{Th}(\mathfrak{N}_3)$ denotes the theory of $\mathfrak{N}_3$, the set of all sentences in this language true in $\mathfrak{N}_3$.

We consider arithmetic with $+$ and $\times$ as relations $\subseteq \omega^3$ instead of as the usual functions $\omega^2 \to \omega$ because our coding methods most naturally code relations. Any formula in which $+$ and $\times$ are relation symbols can be trivially translated into an equivalent formula in which $+$ and $\times$ are

function symbols. Translations in the other direction require *unnesting*. In general, a formula is said to be *unnested* if all its atomic subformulas are of the form $x = y$, $c = y$, $f(x_1, \dots, x_n) = y$, or $R(x_1, \dots, x_n)$, where x, y, and the x_i are variables, c is a constant symbol, f is a function symbol, and R is a relation symbol. Every formula can be recursively translated into an equivalent unnested formula. See for example [4] section 2.6. When unnesting is applied to a first-order formula in the functional language of arithmetic, we get an equivalent formula whose atomic subformulas are of the form

$$x = y \qquad x \leq y \qquad x + y = z \qquad x \times y = z,$$

which can easily be translated into a formula in the relational language of arithmetic. Unnesting second-order or third-order formulas is the same but allows additional atomic formulas $x \in X$ (second-order case) or $x \in_2 X$ and $X \in_3 \mathcal{X}$ (third-order case). Thus the functional and relational theories of second-order arithmetic are recursively isomorphic, as are the functional and relational theories of third-order arithmetic.

1.3. **Mass problems and reducibilities.** A *mass problem* is a set of functions $\mathcal{A} \subseteq \omega^\omega$. We say mass problem $\mathcal{A}$ *Medvedev reduces* to mass problem $\mathcal{B}$ (written $\mathcal{A} \leq_{\mathrm{M}} \mathcal{B}$) if there is a Turing functional Φ such that for every $f \in \mathcal{B}$, Φ^f computes a total function that is in $\mathcal{A}$ (written $\Phi(\mathcal{B}) \subseteq \mathcal{A}$). We say $\mathcal{A}$ and $\mathcal{B}$ are *Medvedev equivalent* (written $\mathcal{A} \equiv_{\mathrm{M}} \mathcal{B}$) if $\mathcal{A} \leq_{\mathrm{M}} \mathcal{B}$ and $\mathcal{B} \leq_{\mathrm{M}} \mathcal{A}$. The relation $\equiv_{\mathrm{M}}$ is an equivalence relation on 2^{ω^ω}, and the equivalence class $[\mathcal{A}]$ is called the *Medvedev degree* of $\mathcal{A}$. Medvedev reducibility induces a partial order on degrees: $[\mathcal{A}] \leq_{\mathrm{M}} [\mathcal{B}]$ if and only if $\mathcal{A} \leq_{\mathrm{M}} \mathcal{B}$. The structure $\mathfrak{M} = (2^{\omega^\omega} / \equiv_{\mathrm{M}}, \leq_{\mathrm{M}})$ introduced by Medvedev in [9] is called the *Medvedev degrees*. $\mathfrak{M}$ is a lattice. For mass problems $\mathcal{A}$ and $\mathcal{B}$, let

$$\mathcal{A} + \mathcal{B} = \{f \oplus g \mid f \in \mathcal{A} \wedge g \in \mathcal{B}\}$$
$$\mathcal{A} \times \mathcal{B} = 0^\frown \mathcal{A} \cup 1^\frown \mathcal{B}.$$

Then join is given by $[\mathcal{A}] + [\mathcal{B}] = [\mathcal{A} + \mathcal{B}]$ and meet is given by $[\mathcal{A}] \times [\mathcal{B}] = [\mathcal{A} \times \mathcal{B}]$. $\mathrm{Th}(\mathfrak{M})$ denotes the first-order theory of the Medvedev degrees.

We say mass problem $\mathcal{A}$ *Muchnik reduces* (or *weakly reduces*) to mass problem $\mathcal{B}$ (written $\mathcal{A} \leq_{\mathrm{w}} \mathcal{B}$) if for every $f \in \mathcal{B}$ there is a $g \in \mathcal{A}$ with $g \leq_{\mathrm{T}} f$. Muchnik reducibility is the non-uniform version of Medvedev reducibility. We say $\mathcal{A}$ and $\mathcal{B}$ are *Muchnik equivalent* (or *weakly equivalent*, written $\mathcal{A} \equiv_{\mathrm{w}} \mathcal{B}$) if $\mathcal{A} \leq_{\mathrm{w}} \mathcal{B}$ and $\mathcal{B} \leq_{\mathrm{w}} \mathcal{A}$. The equivalence class $[\mathcal{A}]_{\mathrm{w}}$ is called the *Muchnik degree* of $\mathcal{A}$. Muchnik reducibility induces a partial order on degrees $[\mathcal{A}]_{\mathrm{w}}$, and this partial order is a lattice with join and meet computed as in the Medvedev case: $[\mathcal{A}]_{\mathrm{w}} + [\mathcal{B}]_{\mathrm{w}} = [\mathcal{A} + \mathcal{B}]_{\mathrm{w}}$ and $[\mathcal{A}]_{\mathrm{w}} \times [\mathcal{B}]_{\mathrm{w}} = [\mathcal{A} \times \mathcal{B}]_{\mathrm{w}}$. Notice that in the Muchnik case $\mathcal{A} \times \mathcal{B} \equiv_{\mathrm{w}} \mathcal{A} \cup \mathcal{B}$, so one may think of $\mathcal{A} \times \mathcal{B}$ as being defined as $\mathcal{A} \cup \mathcal{B}$ in this case. The structure $\mathfrak{M}_{\mathrm{w}} = (2^{\omega^\omega} / \equiv_{\mathrm{w}}, \leq_{\mathrm{w}})$ introduced by Muchnik in [11] is called the *Muchnik degrees*. $\mathrm{Th}(\mathfrak{M}_{\mathrm{w}})$ denotes the first-order theory of the Muchnik degrees.

$\mathfrak{M}$ and $\mathfrak{M}_{\mathrm{w}}$ both have a least element and a greatest element. In both lattices, ω^ω has minimum degree. In fact, a mass problem has minimum degree if and only if it contains a recursive function. The empty mass problem has maximum degree, and it is the only such mass problem. $\mathfrak{M}$ and $\mathfrak{M}_{\mathrm{w}}$ are also both *distributive lattices*. That is, they satisfy $\forall x \forall y \forall z [x + (y \times z) = (x + y) \times (x + z)]$ and $\forall x \forall y \forall z [x \times (y + z) = (x \times y) + (x \times z)]$. Sorbi's [22] is a good introduction to $\mathfrak{M}$ and $\mathfrak{M}_{\mathrm{w}}$.

We note that Lewis, Nies, and Sorbi [7] prove that $\mathfrak{M}$ and $\mathfrak{M}_{\mathrm{w}}$ are not elementary equivalent. Thus the theorem $\mathrm{Th}(\mathfrak{M}) \equiv_1 \mathrm{Th}(\mathfrak{M}_{\mathrm{w}})$ is nontrivial.

For the sake of definiteness, the official language of $\mathfrak{M}$ (and of all lattices considered here) is that of partial orders. In any lattice, $+$ and $\times$ are first-order definable from $\leq$, so we will freely use the symbols $+$ and $\times$ with the understanding that they are abbreviations for their first-order definitions.

The notation $\mathrm{Th}(\mathfrak{M})$ is also used to denote the collection of propositional formulas valid in $\mathfrak{M}$ when studying $\mathfrak{M}$ as a Brouwer algebra. In fact, this interpretation was the main motivation

behind Medvedev's introduction of $\mathfrak{M}$ in [9], and in [10] he proves that $\mathfrak{M}$ provides semantics for intuitionistic logic plus the additional axiom $\neg p \vee \neg\neg p$ (the so-called Jankov's logic [5]). There are many interesting results and problems in this direction. See for example [17], [20], [21], [23], and [13]. However, we do not consider propositional logics here, and for us the notation $\mathrm{Th}(\mathfrak{M})$ always denotes the first-order theory of $\mathfrak{M}$.

Our notation for join and meet in lattices conflicts with the notation for plus and times in arithmetic. The lattice join and meet operations are denoted in the literature variously as $+$, $\times$, as $\vee$, $\wedge$, and confusingly as $\wedge$, $\vee$. We prefer to conflict with the arithmetic notation rather than the logical notation.

1.4. **Mass problems and topology.** We consider Baire space ω^ω and Cantor space 2^ω, both with their usual product topologies. Basic open sets in ω^ω have the form $I(\sigma) = \{f \in \omega^\omega \mid \sigma \subset f\}$ for $\sigma \in \omega^{<\omega}$, and similarly for 2^ω. If $\mathcal{A} \subseteq \omega^\omega$ is closed, then $\mathcal{A}$ is the set of paths through the tree $T \subseteq \omega^{<\omega}$ defined by $T = \{\sigma \mid (\exists f \in \mathcal{A})[\sigma \subset f]\}$. Conversely, if $T \subseteq \omega^{<\omega}$ is a tree, then the set of paths through T is a closed subset of ω^ω. A set $\mathcal{A} \subseteq \omega^\omega$ is compact if and only if it is closed and bounded if and only if it is the set of paths through a finitely branching tree (here bounded means there is a $g\colon \omega \to \omega$ such that $f(n) \leq g(n)$ for all $f \in \mathcal{A}$ and $n \in \omega$).

A Medvedev degree is said to be *closed* (*compact*) if it is of the form $[\mathcal{A}]$ where $\mathcal{A}$ is closed (compact) in ω^ω. By inspecting the definitions, one can check that if $\mathcal{A}$ and $\mathcal{B}$ are closed (compact) then so are $0^\frown\mathcal{A} \cup 1^\frown\mathcal{B}$ and $\{f \oplus g \mid f \in \mathcal{A} \wedge g \in \mathcal{B}\}$. Thus the closed Medvedev degrees form a distributive sublattice of $\mathfrak{M}$ which we denote by $\mathfrak{M}_{\mathrm{cl}}$, and the compact Medvedev degrees form a distributive sublattice of $\mathfrak{M}$ (and of $\mathfrak{M}_{\mathrm{cl}}$) which we denote by $\mathfrak{M}^{01}_{\mathrm{cl}}$, both as in [8] (the "01" notation is explained below). Both $\mathfrak{M}_{\mathrm{cl}}$ and $\mathfrak{M}^{01}_{\mathrm{cl}}$ inherit the least element and the greatest element from $\mathfrak{M}$. $\mathrm{Th}(\mathfrak{M}_{\mathrm{cl}})$ denotes the first-order theory of $\mathfrak{M}_{\mathrm{cl}}$, and $\mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{cl}})$ denotes the first-order theory of $\mathfrak{M}^{01}_{\mathrm{cl}}$.

Similarly, a Muchnik degree is said to be *closed* (*compact*) if it is of the form $[\mathcal{A}]_{\mathrm{w}}$ where $\mathcal{A}$ is closed (compact) in ω^ω. The closed (compact) Muchnik degrees form a distributive sublattice of $\mathfrak{M}_{\mathrm{w}}$ denoted by $\mathfrak{M}_{\mathrm{w,cl}}$ ($\mathfrak{M}^{01}_{\mathrm{w,cl}}$). $\mathfrak{M}^{01}_{\mathrm{w,cl}}$ is also a distributive sublattice of $\mathfrak{M}_{\mathrm{w,cl}}$. Both $\mathfrak{M}_{\mathrm{w,cl}}$ and $\mathfrak{M}^{01}_{\mathrm{w,cl}}$ inherit the least element and the greatest element from $\mathfrak{M}_{\mathrm{w}}$. $\mathrm{Th}(\mathfrak{M}_{\mathrm{w,cl}})$ denotes the first-order theory of $\mathfrak{M}_{\mathrm{w,cl}}$, and $\mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{w,cl}})$ denotes the first-order theory of $\mathfrak{M}^{01}_{\mathrm{w,cl}}$.

The closed subsets of ω^ω (and of 2^ω) are the topologically simplest classes which yield non-trivial degree structures because every nonempty open set contains a recursive function. As such, they are worthy objects of study. For example, Bianchini and Sorbi [1] studied the filter (in $\mathfrak{M}$) generated by the nonminimum closed degrees. Lewis, Shore, and Sorbi [8] have made a recent study of topologically-defined collections of Medvedev degrees.

In general, every $\mathcal{A} \subseteq \omega^\omega$ is Medvedev equivalent (and hence also Muchnik equivalent) to some $\mathcal{B} \subseteq 2^\omega$:

Lemma 1.3. *If $\mathcal{A} \subseteq \omega^\omega$ then there is a $\mathcal{B} \subseteq 2^\omega$ with $\mathcal{A} \equiv_{\mathrm{M}} \mathcal{B}$.*

Proof. For $f \in \omega^\omega$, let $\mathrm{graph}\, f \subseteq \omega$ denote $\{\langle n, m\rangle \mid f(n) = m\}$ Given $\mathcal{A}$, let $\mathcal{B} = \{\mathrm{graph}\, f \mid f \in \mathcal{A}\}$. Let Φ be the functional such that $\Phi^f(\langle n, m\rangle) = 1$ if $f(n) = m$ and $\Phi^f(\langle n, m\rangle) = 0$ otherwise. Then $\Phi^f = \mathrm{graph}\, f$ for all f. Thus $\Phi(\mathcal{A}) = \mathcal{B}$. Let Ψ be the functional such that $\Psi^g(n)$ searches for an m such that $g(\langle n, m\rangle) = 1$ and outputs such an m if it is found. If g is the characteristic function of graph f, then Ψ^g is total and equals f. Hence $\Psi(\mathcal{B}) = \mathcal{A}$. □

If we let $\mathfrak{M}^{01}$ denote the Medvedev degrees of mass problems $\mathcal{A} \subseteq 2^\omega$ and let $\mathfrak{M}^{01}_{\mathrm{w}}$ denote the Muchnik degrees of mass problems $\mathcal{A} \subseteq 2^\omega$, then Lemma 1.3 says $\mathfrak{M} = \mathfrak{M}^{01}$ and $\mathfrak{M}_{\mathrm{w}} = \mathfrak{M}^{01}_{\mathrm{w}}$. However, if $\mathcal{A} \subseteq \omega^\omega$ is closed, the $\mathcal{B} \subseteq 2^\omega$ produced by Lemma 1.3 need not be. Turing functionals are continuous, but ω^ω and 2^ω are not homeomorphic. Nevertheless, if $\mathcal{A} \subseteq \omega^\omega$ is compact, Lemma 1.3 produces a closed $\mathcal{B} \subseteq 2^\omega$. So every compact $\mathcal{A} \subseteq \omega^\omega$ is Medvedev equivalent (and hence also Muchnik equivalent) to a closed (hence compact) $\mathcal{B} \subseteq 2^\omega$. This explains the notations $\mathfrak{M}^{01}_{\mathrm{cl}}$ and $\mathfrak{M}^{01}_{\mathrm{w,cl}}$ for the collections of compact degrees.

We will prove that neither $\mathfrak{M}_{\mathrm{cl}}$ nor $\mathfrak{M}_{\mathrm{cl}}^{01}$ are elementarily equivalent to either $\mathfrak{M}_{\mathrm{w,cl}}$ or $\mathfrak{M}_{\mathrm{w,cl}}^{01}$ (Theorem 6.3 below). The relationship between $\mathfrak{M}_{\mathrm{cl}}$ and $\mathfrak{M}_{\mathrm{cl}}^{01}$ and the relationship between $\mathfrak{M}_{\mathrm{w,cl}}$ and $\mathfrak{M}_{\mathrm{w,cl}}^{01}$ require further study.

Question 1.4.

- Is every closed $\mathcal{X} \subseteq \omega^\omega$ Medvedev equivalent to some closed $\mathcal{Y} \subseteq 2^\omega$? If not, are $\mathfrak{M}_{\mathrm{cl}}$ and $\mathfrak{M}_{\mathrm{cl}}^{01}$ isomorphic? If not, are $\mathfrak{M}_{\mathrm{cl}}$ and $\mathfrak{M}_{\mathrm{cl}}^{01}$ elementarily equivalent?
- Is every closed $\mathcal{X} \subseteq \omega^\omega$ Muchnik equivalent to some closed $\mathcal{Y} \subseteq 2^\omega$? If not, are $\mathfrak{M}_{\mathrm{w,cl}}$ and $\mathfrak{M}_{\mathrm{w,cl}}^{01}$ isomorphic? If not, are $\mathfrak{M}_{\mathrm{w,cl}}$ and $\mathfrak{M}_{\mathrm{w,cl}}^{01}$ elementarily equivalent?

Our topological considerations of Medvedev reducibility are consequences of the familiar *use property* (see [6] section I.3): If $\Phi^f(m) = n$, then there is a finite $\sigma \subset f$ such that σ contains all the answers to the oracle queries made during the computation of $\Phi^f(m) = n$. This is written $\Phi^\sigma(m) = n$ and implies $\Phi^g(m) = n$ for any $g \supset \sigma$. The starting point is the following simple lemma:

Lemma 1.5. *Let $m, n \in \omega$. For any program Φ, the set $\{f \in \omega^\omega \mid \Phi^f(m) = n\}$ is open. If Φ^f is total for all $f \in \mathcal{A}$, then $\{f \in \mathcal{A} \mid \Phi^f(m) = n\}$ is clopen in $\mathcal{A}$ (i.e. it is both the intersection of $\mathcal{A}$ with a set open in ω^ω and the intersection of $\mathcal{A}$ with a set closed in ω^ω).*

Proof. If $\Phi^f(m) = n$, then by the use property there is some $\sigma \subset f$ such that $\Phi^\sigma(m) = n$. Hence $\{f \in \omega^\omega \mid \Phi^f(m) = n\} = \bigcup\{I(\sigma) \mid \Phi^\sigma(m) = n\}$.

If Φ is total on $\mathcal{A}$, then $\{f \in \mathcal{A} \mid \Phi^f(m) = n\} = \mathcal{A} \cap \{f \in \omega^\omega \mid \Phi^f(m) = n\} = \mathcal{A} \cap \big(\bigcap_{i \neq n}\{f \in \omega^\omega \mid \Phi^f(m) \neq i\}\big)$. The last equality holds because if Φ^f is total and $\Phi^f(m) \neq i$ for all $i \neq n$, then it must be that $\Phi^f(m) = n$. □

2. Interpreting the Medvedev degrees and the Muchnik degrees in arithmetic

In this section we prove that $\mathrm{Th}(\mathfrak{M})$, $\mathrm{Th}(\mathfrak{M}_{\mathrm{w}}) \leq_1 \mathrm{Th}(\mathfrak{N}_3)$ and also that $\mathrm{Th}(\mathfrak{M}_{\mathrm{cl}})$, $\mathrm{Th}(\mathfrak{M}_{\mathrm{cl}}^{01})$, $\mathrm{Th}(\mathfrak{M}_{\mathrm{w,cl}})$, $\mathrm{Th}(\mathfrak{M}_{\mathrm{w,cl}}^{01}) \leq_1 \mathrm{Th}(\mathfrak{N}_2)$.

The reductions $\mathrm{Th}(\mathfrak{M})$, $\mathrm{Th}(\mathfrak{M}_{\mathrm{w}}) \leq_1 \mathrm{Th}(\mathfrak{N}_3)$ follow from the fact that every mass problem $\mathcal{A}$ is equivalent to some $\mathcal{B} \subseteq 2^\omega$ (i.e. Lemma 1.3) and that the Medvedev and Muchnik reducibilities are definable in $\mathfrak{N}_3$.

Lemma 2.1. $\mathrm{Th}(\mathfrak{M})$, $\mathrm{Th}(\mathfrak{M}_{\mathrm{w}}) \leq_1 \mathrm{Th}(\mathfrak{N}_3)$.

Proof. The relation $R(X, e, m, n)$ expressing $\Phi_e^X(m) = n$ is definable by a formula which says "there exists a number s coding a sequence of configurations witnessing the computation $\Phi_e^X(m) = n$." The relation $S(X, Y, e)$ expressing $\Phi_e^X = Y$ is definable by the formula

$$\forall m[(m \in Y \to R(X, e, m, 1)) \wedge (m \notin Y \to R(X, e, m, 0))].$$

Thus the relation $\mathcal{A} \leq_{\mathrm{M}} \mathcal{B}$ is definable by the formula

$$\varphi(\mathcal{A}, \mathcal{B}) := \exists e \forall X[X \in \mathcal{B} \to \exists Y(Y \in \mathcal{A} \wedge S(X, Y, e))].$$

Now, given a sentence ψ in the language of partial orders, produce a sentence ψ' in the language of $\mathfrak{N}_3$ by replacing quantifications $\forall x$ and $\exists x$ with third-order quantifications $\forall \mathcal{X}$ and $\exists \mathcal{X}$, by replacing atomic formulas $x \leq y$ with $\varphi(\mathcal{X}, \mathcal{Y})$, and by replacing atomic formulas $x = y$ with $\varphi(\mathcal{X}, \mathcal{Y}) \wedge \varphi(\mathcal{Y}, \mathcal{X})$. Then $\mathfrak{N}_3 \models \psi'$ if and only if $\mathfrak{M} \models \psi$.

The reduction $\mathrm{Th}(\mathfrak{M}_{\mathrm{w}}) \leq_1 \mathrm{Th}(\mathfrak{N}_3)$ is obtained by switching the quantifiers $\exists e$ and $\forall X$ in the definition of the formula φ above. □

The interpretations of $\mathfrak{M}_{\mathrm{cl}}$ and $\mathfrak{M}_{\mathrm{w,cl}}$ ($\mathfrak{M}_{\mathrm{cl}}^{01}$ and $\mathfrak{M}_{\mathrm{w,cl}}^{01}$) in $\mathfrak{N}_2$ rely on the fact that $\mathcal{A} \subseteq \omega^\omega$ ($\mathcal{A} \subseteq 2^\omega$) is closed if and only if it is the set of paths through some tree $T \subseteq \omega^{<\omega}$ ($T \subseteq 2^{<\omega}$). Thus we quantify over all closed mass problems by quantifying over all trees. So fix some definable coding of sequences, trees, and functions in $\mathfrak{N}_2$. See [16] section II.2 for a particularly careful method.

Lemma 2.2. $\mathrm{Th}(\mathfrak{M}_{\mathrm{cl}})$, $\mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{cl}})$, $\mathrm{Th}(\mathfrak{M}_{\mathrm{w,cl}})$, $\mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{w,cl}}) \leq_1 \mathrm{Th}(\mathfrak{N}_2)$.

Proof. The relation $P(f,T)$ expressing "function f is a path through tree T" is definable by the formula $\forall n \exists \sigma(\sigma \in T \wedge |\sigma| = n \wedge (\forall i < |\sigma|)[\sigma(i) = f(i)])$. Relations $R(f,e,m,n)$ expressing $\Phi^f_e(m) = n$ and $S(f,g,e)$ expressing $\Phi^f_e = g$ are definable as in Lemma 2.1. Thus the relation $T \leq_{\mathrm{M}} S$ (expressing that the set of paths through T Medvedev reduces to the set of paths through S) is definable by the formula $\varphi(T,S) := \exists e \forall f[P(f,S) \to \exists g(P(g,T) \wedge S(f,g,e))]$.

Now, given a sentence ψ in the language of partial orders, produce a sentence ψ' in the language of $\mathfrak{N}_2$ by replacing quantifications $\forall x$ and $\exists x$ with second-order quantifications $\forall T_x$ and $\exists T_x$ quantifying over trees $T_x \subseteq \omega^{<\omega}$, by replacing atomic formulas $x \leq y$ with $\varphi(T_x, T_y)$, and by replacing atomic formulas $x = y$ with $\varphi(T_x,T_y) \wedge \varphi(T_y,T_x)$. Then $\mathfrak{N}_2 \models \psi'$ if and only if $\mathfrak{M}_{\mathrm{cl}} \models \psi$. The reduction $\mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{cl}}) \leq_1 \mathrm{Th}(\mathfrak{N}_2)$ is exactly the same, except we quantify over trees $T \subseteq 2^{<\omega}$.

The reductions $\mathrm{Th}(\mathfrak{M}_{\mathrm{w,cl}}), \mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{w,cl}}) \leq_1 \mathrm{Th}(\mathfrak{N}_2)$ are obtained by switching the quantifiers $\exists e$ and $\forall f$ in the definition of the formula φ above. □

3. Interpreting arithmetic in the Medvedev degrees and in the Muchnik degrees

In this section we prove $\mathrm{Th}(\mathfrak{N}_3) \leq_1 \mathrm{Th}(\mathfrak{M}_{\mathrm{w}}) \leq_1 \mathrm{Th}(\mathfrak{M})$, thereby completing the proof that all three theories are pairwise recursively isomorphic. The proof of $\mathrm{Th}(\mathfrak{N}_3) \leq_1 \mathrm{Th}(\mathfrak{M}_{\mathrm{w}})$ is also valid with $\mathfrak{M}$ in place of $\mathfrak{M}_{\mathrm{w}}$. This makes the $\mathrm{Th}(\mathfrak{M}_{\mathrm{w}}) \leq_1 \mathrm{Th}(\mathfrak{M})$ step unnecessary, but the definability of $\mathfrak{M}_{\mathrm{w}}$ in $\mathfrak{M}$ is still worthwhile to notice.

3.1. Defining $\mathfrak{M}_{\mathrm{w}}$ in $\mathfrak{M}$.

The Muchnik degrees are definable in the Medvedev degrees [2], thereby giving $\mathrm{Th}(\mathfrak{M}_{\mathrm{w}}) \leq_1 \mathrm{Th}(\mathfrak{M})$.

Definition 3.1. For a mass problem $\mathcal{A}$, let $C(\mathcal{A})$ denote the Turing upward-closure of $\mathcal{A}$: $C(\mathcal{A}) = \{f \mid (\exists g \in \mathcal{A})[g \leq_{\mathrm{T}} f]\}$.

Definition 3.2. A Medvedev degree $\mathbf{s}$ is called a *degree of solvability* if $\mathbf{s} = [\{f\}]$ for some $f \in \omega^\omega$.

Definition 3.3. A Medvedev degree $\mathbf{m}$ is called a *Muchnik degree* if $\mathbf{m} = [C(\mathcal{A})]$ for some mass problem $\mathcal{A}$.

Notice that $C(\mathcal{A}) \leq_{\mathrm{M}} \mathcal{B}$ if and only if $\mathcal{B} \subseteq C(\mathcal{A})$. Medvedev degrees of the form $[C(\mathcal{A})]$ are called Muchnik degrees because $\mathcal{A} \leq_{\mathrm{w}} \mathcal{B}$ if and only if $C(\mathcal{B}) \subseteq C(\mathcal{A})$ if and only if $C(\mathcal{A}) \leq_{\mathrm{M}} C(\mathcal{B})$. The mapping $[\mathcal{A}]_{\mathrm{w}} \mapsto [C(\mathcal{A})]$ embeds $\mathfrak{M}_{\mathrm{w}}$ into $\mathfrak{M}$ as an upper-semilattice but not as a lattice [19].

Lemma 3.4 (Medvedev [9], Dyment [2])**.** *The degrees of solvability and the Muchnik degrees are definable in* $\mathfrak{M}$.

The formula defining the degrees of solvability is $\theta(x) := \exists y[x < y \wedge \forall z(x < z \to y \leq z)]$. For a degree of solvability $\mathbf{x} = [\{f\}]$, the witnessing $\mathbf{y}$ is the degree $[\{e^\frown g \mid \Phi^g_e = f \wedge g \not\leq_{\mathrm{T}} f\}]$. Complete proofs that θ defines the degrees of solvability are found in [2] and [22]. We reproduce the definability of the Muchnik degrees here. The result essentially appears in [2], but is not phrased in terms of definability.

Proof that the Muchnik degrees are definable in $\mathfrak{M}$.
The defining formula is $\chi(x) := \forall y[\forall z[(\theta(z) \wedge y \leq z) \to x \leq z] \to x \leq y]$, where θ is the formula defining the degrees of solvability as above. Let $[C(\mathcal{A})]$ be a Muchnik degree. If $\mathcal{B}$ satisfies $(\forall f \in \omega^\omega)[\mathcal{B} \leq_{\mathrm{M}} \{f\} \to C(\mathcal{A}) \leq_{\mathrm{M}} \{f\}]$, then in particular we must have $C(\mathcal{A}) \leq_{\mathrm{M}} \{f\}$ for all $f \in \mathcal{B}$. Hence $\mathcal{B} \subseteq C(\mathcal{A})$ and so $\chi([C(\mathcal{A})])$ holds. Conversely, suppose $\chi([\mathcal{A}])$. As $(\forall f \in \omega^\omega)[C(\mathcal{A}) \leq_{\mathrm{M}} \{f\} \to \mathcal{A} \leq_{\mathrm{M}} \{f\}]$, we have $\mathcal{A} \leq_{\mathrm{M}} C(\mathcal{A})$. Thus $\mathcal{A} \equiv_{\mathrm{M}} C(\mathcal{A})$, so $[\mathcal{A}]$ is a Muchnik degree. □

Corollary 3.5. $\mathrm{Th}(\mathfrak{M}_{\mathrm{w}}) \leq_1 \mathrm{Th}(\mathfrak{M})$.

Proof. Interpret $\mathrm{Th}(\mathfrak{M}_{\mathrm{w}})$ inside $\mathrm{Th}(\mathfrak{M})$ by restricting quantification in $\mathfrak{M}$ to quantify only over degrees of the form $[C(\mathcal{A})]$. That is, given a sentence ψ in the language of partial orders, generate a sentence ψ' by inductively replacing subformulas $\exists x\varphi$ and $\forall x\varphi$ by formulas $\exists x(\chi(x)\wedge\varphi)$ and $\forall x(\chi(x)\to\varphi)$. Then $\mathfrak{M}_{\mathrm{w}}\models\psi$ if and only if $\mathfrak{M}\models\psi'$. □

In $\mathfrak{M}_{\mathrm{w}}$, a degree $\mathbf{s}$ is also called a degree of solvability if $\mathbf{s}=[\{f\}]_{\mathrm{w}}$ for some $f\in\omega^\omega$. The formula $\theta(x)$ as above defines the degrees of solvability in $\mathfrak{M}_{\mathrm{w}}$, and the proof is similar to that for $\mathfrak{M}$.

3.2. **Defining a code for $\mathfrak{N}_3$.** We code $\mathfrak{N}_3$ into $\mathfrak{M}_{\mathrm{w}}$ to prove that $\mathrm{Th}(\mathfrak{N}_3)\leq_1\mathrm{Th}(\mathfrak{M}_{\mathrm{w}})$. Although we phrase what follows in terms of $\mathfrak{M}_{\mathrm{w}}$, the same coding can be used to code $\mathfrak{N}_3$ into $\mathfrak{M}$ and thus to prove $\mathrm{Th}(\mathfrak{N}_3)\leq_1\mathrm{Th}(\mathfrak{M})$ without appealing to the definability of $\mathfrak{M}_{\mathrm{w}}$ in $\mathfrak{M}$.

We view each degree $\mathbf{w}$ as coding the set of minimal degrees of solvability above it. Degree $\mathbf{s}$ is a minimal degree of solvability above $\mathbf{w}$ if and only if $\eta(\mathbf{s},\mathbf{w})$ where

$$\eta(s,w):=\theta(s)\wedge w\leq s\wedge\forall z[(\theta(z)\wedge w\leq z)\to z\not\leq s]$$

and $\theta(x)$ is the formula defining the degrees of solvability from Lemma 3.4.

Definition 3.6. For $\mathbf{w}\in\mathfrak{M}_{\mathrm{w}}$, $E(\mathbf{w})=\{\mathbf{s}\in\mathfrak{M}_{\mathrm{w}}\mid\eta(\mathbf{s},\mathbf{w})\}$ denotes the set of minimal degrees of solvability above $\mathbf{w}$.

Our coding makes use of the following obvious lemma:

Lemma 3.7. *If $\mathcal{W}$ is a Turing antichain, then $E([\mathcal{W}]_{\mathrm{w}})=\{[\{f\}]_{\mathrm{w}}\mid f\in\mathcal{W}\}$.*

Proof. Obvious □

Definition 3.8. A *code for $\mathfrak{N}_3$ in $\mathfrak{M}_{\mathrm{w}}$* is a collection of degrees $\mathbf{w}_0,\mathbf{w}_1,\mathbf{w}_2,\mathbf{m},\mathbf{l},\mathbf{p},\mathbf{t},\mathbf{r}\in\mathfrak{M}_{\mathrm{w}}$ such that:

(i) For every degree $\mathbf{a}$ there is an $\mathbf{s}\in E(\mathbf{r})$ such that for all $\mathbf{u}\in E(\mathbf{w_0})$, $\mathbf{u}\in E(\mathbf{a})$ if and only if $\mathbf{u}\leq_{\mathrm{w}}\mathbf{s}$.

(ii) If $R_\leq$ is the following 2-ary relation defined on $E(\mathbf{w}_0)^2$ and R_+ and $R_\times$ are the following 3-ary relations defined on $E(\mathbf{w}_0)^3$:
- $R_\leq(\mathbf{s}_0,\mathbf{u}_0)$ if and only if there is a $\mathbf{u}_1\in E(\mathbf{w}_1)$ with $\mathbf{u}_0+\mathbf{u}_1\in E(\mathbf{m})$ and $\mathbf{s}_0+\mathbf{u}_1\in E(\mathbf{l})$,
- $R_+(\mathbf{s}_0,\mathbf{u}_0,\mathbf{v}_0)$ if and only if there is a $\mathbf{u}_1\in E(\mathbf{w}_1)$ and a $\mathbf{v}_2\in E(\mathbf{w}_2)$ with $\mathbf{u}_0+\mathbf{u}_1\in E(\mathbf{m})$, $\mathbf{v}_0+\mathbf{v}_2\in E(\mathbf{m})$, and $\mathbf{s}_0+\mathbf{u}_1+\mathbf{v}_2\in E(\mathbf{p})$,
- $R_\times(\mathbf{s}_0,\mathbf{u}_0,\mathbf{v}_0)$ if and only if there is a $\mathbf{u}_1\in E(\mathbf{w}_1)$ and a $\mathbf{v}_2\in E(\mathbf{w}_2)$ with $\mathbf{u}_0+\mathbf{u}_1\in E(\mathbf{m})$, $\mathbf{v}_0+\mathbf{v}_2\in E(\mathbf{m})$, and $\mathbf{s}_0+\mathbf{u}_1+\mathbf{v}_2\in E(\mathbf{t})$,

then $\mathfrak{M}_{\mathrm{w}}$ satisfies the formula that says $E(\mathbf{w}_0)$ is a discretely ordered commutative semiring with unity and for every $\mathbf{a}\in\mathfrak{M}_{\mathrm{w}}$, if there is an $\mathbf{s}\in E(\mathbf{a})\cap E(\mathbf{w}_0)$, then there is a least such $\mathbf{s}$, where $\leq$, $+$, and $\times$ are interpreted as $R_\leq$, R_+, and $R_\times$ respectively.

The property "$\mathbf{w}_0,\mathbf{w}_1,\mathbf{w}_2,\mathbf{m},\mathbf{l},\mathbf{p},\mathbf{t},\mathbf{r}$ is a code for $\mathfrak{N}_3$ in $\mathfrak{M}_{\mathrm{w}}$" is first-order definable. The relation $\mathbf{s}\in E(\mathbf{w})$ is defined by the first-order formula $\eta(s,w)$. By inspecting Definition 3.8, we see that the property in item (i) is first-order and that the relations $R_\leq$, R_+, and $R_\times$ in item (ii) are first-order. The axioms of a discretely ordered commutative semiring with unity are first-order, so if we change these axioms to make quantification be over $E(\mathbf{w}_0)$ and to make $\leq$, $+$, and $\times$ be interpreted as the relations $R_\leq$, R_+, and $R_\times$ respectively, we have a first-order formula in the language of partial orders expressing that $E(\mathbf{w}_0)$ is a discretely ordered commutative semiring with unity. Therefore the property in item (ii) is also first-order.

In Definition 3.8, think $\mathbf{w}$ for "ω," $\mathbf{m}$ for "match," $\mathbf{l}$ for "less," $\mathbf{p}$ for "plus," $\mathbf{t}$ for "times," and $\mathbf{r}$ for "reals." Our intention is that $\mathbf{w}_0$ codes ω as $E(\mathbf{w}_0)$ and that the auxiliary degrees $\mathbf{w}_1,\mathbf{w}_2,\mathbf{m},\mathbf{l},\mathbf{p},\mathbf{t}$ code $\leq,+,\times$ on $E(\mathbf{w}_0)$. The idea is that, for $\mathbf{s}$ and $\mathbf{u}$ in some $E(\mathbf{w})$, we would like to code the tuple $(\mathbf{s},\mathbf{u})$ as the degree $\mathbf{s}+\mathbf{u}$. However, with this coding we would not be able to tell $(\mathbf{s},\mathbf{u})$ from $(\mathbf{u},\mathbf{s})$ because $\mathbf{s}+\mathbf{u}=\mathbf{u}+\mathbf{s}$. To fix this problem, we let $\mathbf{w}_0$ code both the "true" ω and

the first-coordinate version of ω, and we introduce $\mathbf{w}_1$ and $\mathbf{w}_2$ to code second- and third-coordinate versions of ω. The degree $\mathbf{m}$ matches first-coordinate "numbers" with their corresponding second- and third-coordinate "numbers." We think of $\mathbf{u}_0 \in E(\mathbf{w}_0)$ and $\mathbf{u}_1 \in E(\mathbf{w}_1)$ as coding the same number if $\mathbf{u}_0 + \mathbf{u}_1 \in E(\mathbf{m})$. Similarly, $\mathbf{v}_0 \in E(\mathbf{w}_0)$ and $\mathbf{v}_2 \in E(\mathbf{w}_2)$ code the same number if $\mathbf{v}_0 + \mathbf{v}_2 \in E(\mathbf{m})$. Now, for $\mathbf{s}_0, \mathbf{u}_0 \in E(\mathbf{w}_0)$, if there is a $\mathbf{u}_1 \in E(\mathbf{w}_1)$ with $\mathbf{u}_0 + \mathbf{u}_1 \in E(\mathbf{m})$, then we can code the tuple $(\mathbf{s}_0, \mathbf{u}_0)$ as $\mathbf{s}_0 + \mathbf{u}_1$. For example, in item (ii) of Definition 3.8, $R_{\leq}(\mathbf{s_0}, \mathbf{u_0})$ holds if and only if there is a second-coordinate version of $\mathbf{u}_0$ (called $\mathbf{u}_1$) such that $\mathbf{s}_0 + \mathbf{u}_1 \in E(\mathbf{l})$. Similarly, $R_+(\mathbf{s_0}, \mathbf{u_0}, \mathbf{v}_0)$ holds if and only if there are a second-coordinate version of $\mathbf{u}_0$ (called $\mathbf{u}_1$) and a third-coordinate version of $\mathbf{v}_0$ (called $\mathbf{v}_2$) such that $\mathbf{s}_0 + \mathbf{u}_1 + \mathbf{v}_2 \in E(\mathbf{p})$. The degree $\mathbf{m}$ need not code bijections (or even functions) between $E(\mathbf{w}_0)$ and $E(\mathbf{w}_1)$ and between $E(\mathbf{w}_0)$ and $E(\mathbf{w}_2)$. However, this is irrelevant because the definitions of the relations $R_{\leq}$, R_+, and $R_\times$ make sense for any degree $\mathbf{m}$.

A degree $\mathbf{a}$ codes the set $E(\mathbf{a}) \cap E(\mathbf{w}_0) \subseteq E(\mathbf{w}_0)$. Every subset of $E(\mathbf{w}_0)$ has a code: If $X \subseteq E(\mathbf{w}_0)$, then for each $\mathbf{s} \in X$ fix an $f_{\mathbf{s}} \in \omega^\omega$ such that $\mathbf{s} = [\{f_{\mathbf{s}}\}]_{\mathrm{w}}$. Let $\mathcal{A} = \{f_{\mathbf{s}} \mid \mathbf{s} \in X\}$ and let $\mathbf{a} = [\mathcal{A}]_{\mathrm{w}}$. $\mathcal{A}$ is a Turing antichain, so $E(\mathbf{a}) = X$ by Lemma 3.7. Thus $\mathbf{a}$ is a code for X. We then quantify over all subsets of $E(\mathbf{w}_0)$ by quantifying over all degrees $\mathbf{a}$ and interpreting each as a subset of $E(\mathbf{w}_0)$. Therefore item (ii) above ensures that, for a code for $\mathfrak{N}_3$ in $\mathfrak{M}_{\mathrm{w}}$, the structure $\langle E(\mathbf{w}_0), R_{\leq}, R_+, R_\times \rangle$ is a well-founded model of arithmetic and as such is isomorphic to $\mathfrak{N}$.

A degree $\mathbf{b}$ can also be interpreted as coding a subset $S(\mathbf{b})$ of $2^{E(\mathbf{w}_0)}$ as follows.

Definition 3.9. For $\mathbf{w}_0$ as in a code for $\mathfrak{N}_3$ in $\mathfrak{M}_{\mathrm{w}}$ and $\mathbf{b} \in \mathfrak{M}_{\mathrm{w}}$, $S(\mathbf{b}) = \{X \subseteq E(\mathbf{w}_0) \mid (\exists \mathbf{s} \in E(\mathbf{b}))(\forall \mathbf{u} \in E(\mathbf{w}_0))[\mathbf{u} \in X \leftrightarrow \mathbf{u} \leq_{\mathrm{w}} \mathbf{s}]\}$.

Let $\pi(a, b, w)$ be the formula

$$\pi(a, b, w) := (\exists s \in E(b))(\forall u \in E(w))[u \in E(a) \leftrightarrow u \leq s].$$

We write $\mathbf{a} \in S(\mathbf{b})$ for $\pi(\mathbf{a}, \mathbf{b}, \mathbf{w}_0)$, which expresses that the subset of $E(\mathbf{w}_0)$ coded by $\mathbf{a}$ is an element of the subset of $2^{E(\mathbf{w}_0)}$ coded by $\mathbf{b}$. Every subset of $2^{E(\mathbf{w}_0)}$ has a code: If $\mathcal{X} \subseteq 2^{E(\mathbf{w}_0)}$ then for each $X \in \mathcal{X}$ fix a degree $\mathbf{a}_X$ with $E(\mathbf{a}_X) \cap E(\mathbf{w}_0) = X$. Then by item (i), for each $\mathbf{a}_X$ find a degree $\mathbf{s}_X \in E(\mathbf{r})$ such that $(\forall \mathbf{u} \in E(\mathbf{w}_0))[\mathbf{u} \in E(\mathbf{a}_X) \leftrightarrow \mathbf{u} \leq_{\mathrm{w}} \mathbf{s}_X]$. For each $\mathbf{s}_X$, fix $f_X \in \omega^\omega$ such that $\mathbf{s}_X = [\{f_X\}]_{\mathrm{w}}$. Let $\mathbf{b} = [\{f_X \mid X \in \mathcal{X}\}]_{\mathrm{w}}$. Then $S(\mathbf{b}) = \mathcal{X}$.

We have seen that, for a code for $\mathfrak{N}_3$ in $\mathfrak{M}_{\mathrm{w}}$, every degree can be interpreted as a subset of $E(\mathbf{w}_0)$ and as a subset of $2^{E(\mathbf{w}_0)}$. Moreover, quantifying over all degrees quantifies over all subsets of $E(\mathbf{w}_0)$ and quantifies over all subsets of $2^{E(\mathbf{w}_0)}$. Thus for a code for $\mathfrak{N}_3$ in $\mathfrak{M}_{\mathrm{w}}$, the coded structure is exactly $\langle E(\mathbf{w}_0), 2^{E(\mathbf{w}_0)}, 2^{2^{E(\mathbf{w}_0)}}, R_{\leq}, R_+, R_\times \rangle$, and this structure is isomorphic to $\mathfrak{N}_3$. As discussed above, there is a sentence in the language of partial orders expressing the existence of a code for $\mathfrak{N}_3$. Given a sentence ψ in the language of $\mathfrak{N}_3$, we translate it into a sentence in the language of partial orders that says "there is a code for $\mathfrak{N}_3$ in $\mathfrak{M}_{\mathrm{w}}$ and ψ is true in the coded structure." It remains to prove the existence of such a code.

3.3. **Finding a code for $\mathfrak{N}_3$ in $\mathfrak{M}_{\mathrm{w}}$.** The crucial point is the existence of the degree $\mathbf{r}$ coding 2^{2^ω}. The following lemma is proved using standard recursion theoretic techniques:

Lemma 3.10. *If $\mathcal{A} = \{f_i \mid i \in \omega\}$ is a countable independent set, then there exists a Turing antichain $\mathcal{R} = \{g_X \mid X \in 2^\omega\}$ such that $\{f_i \mid i \in X\} = \{f \in \mathcal{A} \mid f \leq_{\mathrm{T}} g_X\}$ for each $X \in 2^\omega$.*

Proof. We construct partial functions $g_\sigma \colon \omega \to \omega$ for $\sigma \in 2^{<\omega}$ and put $g_X = \cup_{n \in \omega} g_{X \restriction n}$. The g_σ will have the following properties:

(i) If $\sigma \subset \tau$ then $\operatorname{dom} g_\sigma \subseteq \operatorname{dom} g_\tau$ and the two functions agree on their common domain.
(ii) If $s < |\sigma|$ and $\sigma(s) = 0$ then $g_\sigma(\langle s, j \rangle)$ is defined for all j and equals 0 for all but finitely many j.
(iii) If $s < |\sigma|$ and $\sigma(s) = 1$ then $g_\sigma(\langle s, j \rangle)$ is defined for all j and equals $f_s(j)$ for all but finitely many j.

(iv) $g_\sigma(\langle s, j \rangle)$ is defined for only finitely many $\langle s, j \rangle$ with $s \geq |\sigma|$.

Items (i) – (iii) ensure that each g_X is a total function, and item (iii) ensures $f_s \leq_{\mathrm{T}} g_X$ for all $s \in X$. In addition we satisfy the following requirements for all $e, i \in \omega$ and all $X, Y \subseteq \omega$:

- $R^X_{e,i}$: $i \notin X \to \Phi^{g_X}_e \neq f_i$
- $Q^{X,Y}_e$: $X \neq Y \to \Phi^{g_X}_e \neq g_Y$

Let $g_\emptyset = \emptyset$. At stage s we have g_σ for all σ of length s.

At stage $s = 2\langle e, i \rangle$ we handle requirement $R^X_{e,i}$. For each σ of length s do the following: If $\sigma(i) = 0$, if there is a finite partial function h_σ with domain disjoint from g_σ, and if there is a number n such that $\Phi^{g_\sigma \cup h_\sigma}_e(n) \downarrow \neq f_i(n)$, then redefine g_σ to be $g_\sigma \cup h_\sigma$. Then for each σ of length s put $g_{\sigma^\frown 0} = g_\sigma \cup \{\langle \langle s, j \rangle, 0 \rangle \mid \langle s, j \rangle \notin \operatorname{dom} g_\sigma\}$ and put $g_{\sigma^\frown 1} = g_\sigma \cup \{\langle \langle s, j \rangle, f_s(j) \rangle \mid \langle s, j \rangle \notin \operatorname{dom} g_\sigma\}$.

At stage $s = 2e{+}1$ we handle requirement $Q^{X,Y}_e$. List the pairs (σ, τ) with $|\sigma| = |\tau| = s$ and $\sigma \neq \tau$. For each such (σ, τ) do the following: Let n be least such that $n \notin \operatorname{dom} g_\tau$. If there is a finite partial function h_σ with domain disjoint from g_σ and if there is a number m such that $\Phi^{g_\sigma \cup h_\sigma}_e(n) \downarrow = m$, then redefine g_σ to be $g_\sigma \cup h_\sigma$ and redefine g_τ to be $g_\tau \cup \{\langle n, m+1 \rangle\}$. After these extensions are made for each pair (σ, τ), then for each σ of length s put $g_{\sigma^\frown 0} = g_\sigma \cup \{\langle \langle s, j \rangle, 0 \rangle \mid \langle s, j \rangle \notin \operatorname{dom} g_\sigma\}$ and put $g_{\sigma^\frown 1} = g_\sigma \cup \{\langle \langle s, j \rangle, f_s(j) \rangle \mid \langle s, j \rangle \notin \operatorname{dom} g_\sigma\}$.

We verify $i \notin X \to f_i \not\leq_{\mathrm{T}} g_X$. Suppose that $i \notin X$ and $\Phi^{g_X}_e = f_i$. Consider stage $s = 2\langle e, i \rangle$ of the construction. Let $\sigma = X \upharpoonright s$ and let $f = \bigoplus\{f_t \mid t < s \wedge \sigma(t) = 1\}$. The function f computes the graph of the partial function g_σ. Thus we can use f to simulate the computation $\Phi^{g_\sigma \cup h}_e(n)$ for any finite partial function h with domain disjoint from g_σ. We now have the contradiction $f_i \leq_{\mathrm{T}} f$ as follows: Given input n, use f to search for a finite partial function h with domain disjoint from g_σ such that $\Phi^{g_\sigma \cup h}_e(n) \downarrow = m$ for some m. There must be such an h because g_X extends g_σ and $\Phi^{g_X}_e(n) \downarrow$. Moreover, we must have $m = f_i(n)$. Otherwise at stage s we would have been able to find an h_σ such that $\Phi^{g_\sigma \cup h_\sigma}_e(n) \downarrow \neq f_i(n)$, and this would imply $\Phi^{g_X}_e \neq f_i$.

We verify $X \neq Y \to g_Y \not\leq_{\mathrm{T}} g_X$. Suppose for a contradiction that $\Phi^{g_X} = g_Y$. Choose an index e for Φ greater than the least e such that $X(e) \neq Y(e)$, put $s = 2e + 1$, and let $\sigma = X \upharpoonright s$, $\tau = Y \upharpoonright s$. Consider the g_σ and g_τ we have right before we process the pair (σ, τ) in stage s. Let n be least such that $n \notin \operatorname{dom} g_\tau$. Since g_X extends g_σ and $\Phi^{g_X}_e$ is total, we must have found a finite h_σ and number m such that $\Phi^{g_\sigma \cup h_\sigma}_e(n) \downarrow = m$. But then we extended g_τ so that $g_\tau(n) = m + 1$. Thus $\Phi^{g_X}_e(n) = m \neq g_Y(n)$, a contradiction. □

Lemma 3.11. *There is a code for $\mathfrak{N}_3$ in $\mathfrak{M}_{\mathrm{w}}$.*

Proof. Let

- $\mathcal{W}_0 = \{f_{0,i} \mid i \in \omega\}$, $\mathcal{W}_1 = \{f_{1,i} \mid i \in \omega\}$, and $\mathcal{W}_2 = \{f_{2,i} \mid i \in \omega\}$ be such that $\mathcal{W}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2$ is independent,
- $\mathcal{M} = \{f_{0,i} \oplus f_{1,i} \mid i \in \omega\} \cup \{f_{0,i} \oplus f_{2,i} \mid i \in \omega\}$,
- $\mathcal{L} = \{f_{0,i} \oplus f_{1,j} \mid i \leq j\}$,
- $\mathcal{P} = \{f_{0,i} \oplus f_{1,j} \oplus f_{2,k} \mid i + j = k\}$,
- $\mathcal{T} = \{f_{0,i} \oplus f_{1,j} \oplus f_{2,k} \mid i \times j = k\}$,
- by Lemma 3.10, let $\mathcal{R} = \{g_X \mid X \in 2^\omega\}$ be a Turing antichain such that $\{f_{0,i} \in \mathcal{W}_0 \mid i \in X\} = \{f_{0,i} \in \mathcal{W}_0 \mid f_{0,i} \leq_{\mathrm{T}} g_X\}$ for each $X \in 2^\omega$.

Put $\mathbf{w}_0 = [\mathcal{W}_0]_{\mathrm{w}}$, $\mathbf{w}_1 = [\mathcal{W}_1]_{\mathrm{w}}$, $\mathbf{w}_2 = [\mathcal{W}_2]_{\mathrm{w}}$, $\mathbf{m} = [\mathcal{M}]_{\mathrm{w}}$, $\mathbf{l} = [\mathcal{L}]_{\mathrm{w}}$, $\mathbf{p} = [\mathcal{P}]_{\mathrm{w}}$, $\mathbf{t} = [\mathcal{T}]_{\mathrm{w}}$, $\mathbf{r} = [\mathcal{R}]_{\mathrm{w}}$. We check the two cases of Definition 3.8. Notice that the above mass problems are all Turing antichains.

(i) Given a degree $\mathbf{a}$, let $X = \{i \mid [\{f_{0,i}\}]_{\mathrm{w}} \in E(\mathbf{a}) \cap E(\mathbf{w}_0)\}$ and let $\mathbf{s} = [\{g_X\}]_{\mathrm{w}}$. Then $\mathbf{s} \in E(\mathbf{r})$ and $(\forall \mathbf{u} \in E(\mathbf{w}_0))[\mathbf{u} \in E(\mathbf{a}) \leftrightarrow \mathbf{u} \leq_{\mathrm{w}} \mathbf{s}]$.

(ii) For $[\{f_{0,i}\}]_{\mathrm{w}}, [\{f_{0,j}\}]_{\mathrm{w}} \in E(\mathbf{w}_0)$ we have $R_{\leq}([\{f_{0,i}\}]_{\mathrm{w}}, [\{f_{0,j}\}]_{\mathrm{w}})$ if and only if there is a $\mathbf{u}_1 \in E(\mathbf{w}_1)$ with $[\{f_{0,j}\}]_{\mathrm{w}} + \mathbf{u}_1 \in E(\mathbf{m})$ and $[\{f_{0,i}\}]_{\mathrm{w}} + \mathbf{u}_1 \in E(\mathbf{l})$. By the independence

of $\mathcal{W}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2$ and Lemma 3.7, this happens if and only if $\mathbf{u}_1 = [\{f_{1,j}\}]_{\mathrm{w}}$ and $i \leq j$. Thus $R_{\leq}([\{f_{0,i}\}]_{\mathrm{w}}, [\{f_{0,j}\}]_{\mathrm{w}})$ if and only if $i \leq j$. Similarly $R_{+}([\{f_{0,i}\}]_{\mathrm{w}}, [\{f_{0,j}\}]_{\mathrm{w}}, [\{f_{0,k}\}]_{\mathrm{w}})$ if and only if $i + j = k$ and $R_{\times}([\{f_{0,i}\}]_{\mathrm{w}}, [\{f_{0,j}\}]_{\mathrm{w}}, [\{f_{0,k}\}]_{\mathrm{w}})$ if and only if $i \times j = k$. Hence $E(\mathbf{w}_0)$ is a discretely ordered commutative semiring with unity. Moreover, if $E(\mathbf{a}) \cap E(\mathbf{w}_0)$ is nonempty, then there is a least i for which $\mathbf{s} = [\{f_{0,i}\}]_{\mathrm{w}}$ is in $E(\mathbf{a}) \cap E(\mathbf{w}_0)$. This $\mathbf{s}$ is the $R_{\leq}$-least element of $E(\mathbf{a}) \cap E(\mathbf{w}_0)$.

□

We are ready to interpret $\mathfrak{N}_3$ in $\mathfrak{M}_{\mathrm{w}}$.

Lemma 3.12. $\mathrm{Th}(\mathfrak{N}_3) \leq_1 \mathrm{Th}(\mathfrak{M}_{\mathrm{w}})$.

Proof. Let φ be a sentence in the language of $\mathfrak{N}_3$. Each atomic subformula of φ has one of the following forms:

$$x = y \qquad x \leq y \qquad x + y = z \qquad x \times y = z \qquad x \in_2 X \qquad X \in_3 \mathcal{X}.$$

Now let $\varphi'(w_0, w_1, w_2, m, \ell, p, t, r)$ be the formula (with the displayed variables free) in the language of partial orders obtained from φ by making the replacements below. The second-order variable X in φ corresponds to the variable v_X in φ' and the third-order variable $\mathcal{X}$ in φ corresponds to the variable $v_{\mathcal{X}}$ in φ'.

- Replace $x \leq y$ by the formula defining $R_{\leq}(x, y)$.
- Replace $x + y = z$ by the formula defining $R_{+}(x, y, z)$.
- Replace $x \times y = z$ by the formula defining $R_{\times}(x, y, z)$.
- Replace $x \in_2 X$ by the formula expressing $x \in E(v_X)$.
- Replace $X \in_3 \mathcal{X}$ by the formula expressing $v_X \in S(v_{\mathcal{X}})$.
- Replace quantifiers $\exists x$ and $\forall x$ by $\exists x \in E(w_0)$ and $\forall x \in E(w_0)$.
- Replace quantifiers $\exists X$ and $\forall X$ by $\exists v_X$ and $\forall v_X$.
- Replace quantifiers $\exists \mathcal{X}$ and $\forall \mathcal{X}$ by $\exists v_{\mathcal{X}}$ and $\forall v_{\mathcal{X}}$.

Let ψ be the sentence saying "there is a code w_0, w_1, w_2, m, ℓ, p, t, r for $\mathfrak{N}_3$ in $\mathfrak{M}_{\mathrm{w}}$ and $\varphi'(w_0, w_1, w_2, m, \ell, p, t, r)$." A code for $\mathfrak{N}_3$ in $\mathfrak{M}_{\mathrm{w}}$ codes a structure isomorphic to $\mathfrak{N}_3$, and so $\mathfrak{M}_{\mathrm{w}} \models \psi$ if and only if $\mathfrak{N}_3 \models \varphi$. □

Theorem 3.13 (Independently by Lewis, Nies, and Sorbi [7])**.** $\mathrm{Th}(\mathfrak{M}_{\mathrm{w}}) \equiv_1 \mathrm{Th}(\mathfrak{M}) \equiv_1 \mathrm{Th}(\mathfrak{N}_3)$.

Proof. We have $\mathrm{Th}(\mathfrak{M}) \leq_1 \mathrm{Th}(\mathfrak{N}_3)$ by Lemma 2.1, $\mathrm{Th}(\mathfrak{M}_{\mathrm{w}}) \leq_1 \mathrm{Th}(\mathfrak{M})$ by Corollary 3.5, and we have $\mathrm{Th}(\mathfrak{N}_3) \leq_1 \mathrm{Th}(\mathfrak{M}_{\mathrm{w}})$ by Lemma 3.12. □

4. Interpreting arithmetic in the closed and compact Muchnik degrees

Our coding of third-order arithmetic in $\mathfrak{M}_{\mathrm{w}}$ relied on the definability of the degrees of solvability in $\mathfrak{M}_{\mathrm{w}}$. The definability of degrees of solvability in $\mathfrak{M}_{\mathrm{cl}}$, $\mathfrak{M}^{01}_{\mathrm{cl}}$, $\mathfrak{M}_{\mathrm{w,cl}}$, and $\mathfrak{M}^{01}_{\mathrm{w,cl}}$ would give an immediate proof of $\mathrm{Th}(\mathfrak{N}_2) \leq_1 \mathrm{Th}(\mathfrak{M}_{\mathrm{cl}})$, $\mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{cl}})$, $\mathrm{Th}(\mathfrak{M}_{\mathrm{w,cl}})$, $\mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{w,cl}})$. This is because the Turing degrees are isomorphic to the degrees of solvability and because the first-order theory of the Turing degrees is recursively isomorphic to $\mathrm{Th}(\mathfrak{N}_2)$ [15]. Singleton mass problems $\{f\}$ are compact, so the degrees of solvability are in $\mathfrak{M}_{\mathrm{cl}}$, $\mathfrak{M}^{01}_{\mathrm{cl}}$, $\mathfrak{M}_{\mathrm{w,cl}}$, and $\mathfrak{M}^{01}_{\mathrm{w,cl}}$. However, we do not know if the degrees of solvability are definable in any of these structures.

Question 4.1. Are the degrees of solvability definable in $\mathfrak{M}_{\mathrm{cl}}$, $\mathfrak{M}^{01}_{\mathrm{cl}}$, $\mathfrak{M}_{\mathrm{w,cl}}$, or $\mathfrak{M}^{01}_{\mathrm{w,cl}}$?

In this section we prove that $\mathrm{Th}(\mathfrak{N}_2) \leq_1 \mathrm{Th}(\mathfrak{M}_{\mathrm{w,cl}})$, $\mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{w,cl}})$, and in Section 5 we prove that $\mathrm{Th}(\mathfrak{N}_2) \leq_1 \mathrm{Th}(\mathfrak{M}_{\mathrm{cl}})$, $\mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{cl}})$. We will use the same definition of a code for $\mathfrak{N}_2$ (Definition 4.6 below) in all four cases. The difference between the Muchnik cases and the Medvedev cases is in how we prove that each subset of ω has a code.

4.1. **Defining a code for $\mathfrak{N}_2$.** In Section 3, a degree $\mathbf{w}$ coded the set of minimal degrees of solvability above it. Now a degree $\mathbf{w}$ codes the set of minimal meet-irreducible degrees that meet to it.

Definition 4.2. An element of a lattice is called *meet-reducible* if it satisfies the formula $\widetilde{\theta}(x) := \exists y \exists z(x < y \wedge x < z \wedge x = y \times z)$. An element of a lattice is called *meet-irreducible* if it is not meet-reducible.

We will take advantage of the following easily checkable fact: In a distributive lattice, if $\mathbf{x}$ is meet-irreducible and $\mathbf{x} \geq \mathbf{y} \times \mathbf{z}$, then $\mathbf{x} \geq \mathbf{y}$ or $\mathbf{x} \geq \mathbf{z}$.

Definition 4.3. For elements $\mathbf{s}$ and $\mathbf{w}$ of a lattice, we say $\mathbf{s}$ *meets to* $\mathbf{w}$ if $\mathbf{s}$ and $\mathbf{w}$ satisfy the formula $\widetilde{\chi}(s, w) := \exists y(y > w \wedge s \times y = w)$.

Hence for $\mathbf{s}$ and $\mathbf{w}$ in a lattice, $\mathbf{s}$ is a minimal meet-irreducible that meets to $\mathbf{w}$ if and only if $\widetilde{\eta}(\mathbf{s}, \mathbf{w})$ where

$$\widetilde{\eta}(s, w) := \neg\widetilde{\theta}(s) \wedge \widetilde{\chi}(s, w) \wedge \forall y[y < s \rightarrow (w \not\leq y \vee \widetilde{\theta}(y))].$$

Definition 4.4. For a lattice $\mathfrak{L}$ and an element $\mathbf{w} \in \mathfrak{L}$, $F(\mathbf{w}) = \{\mathbf{s} \in \mathfrak{L} \mid \mathbf{s} \geq \mathbf{w}\}$ denotes the set of elements above $\mathbf{w}$ and $\widetilde{E}(\mathbf{w}) = \{\mathbf{s} \in \mathfrak{L} \mid \widetilde{\eta}(\mathbf{s}, \mathbf{w})\}$ denotes the set of minimal meet-irreducibles that meet to $\mathbf{w}$.

Notice that $\widetilde{E}(\mathbf{w})$ is an antichain by the minimality of its elements.

Keep in mind that the lattices we now consider are $\mathfrak{M}_{\mathrm{w,cl}}$ and $\mathfrak{M}^{01}_{\mathrm{w,cl}}$ and that meet-reducible means meet-reducible in these lattices. If a closed (compact) $\mathcal{W}$ has meet-reducible degree in $\mathfrak{M}_{\mathrm{w,cl}}$ ($\mathfrak{M}^{01}_{\mathrm{w,cl}}$), then it has meet-reducible degree in $\mathfrak{M}_{\mathrm{w}}$. However, we do not know the converse.

Question 4.5. If $\mathcal{W}$ is closed (compact) and $\mathcal{W} \equiv_{\mathrm{w}} \mathcal{X} \times \mathcal{Y}$ for $\mathcal{X}, \mathcal{Y} >_{\mathrm{w}} \mathcal{W}$, then are there closed (compact) such $\mathcal{X}$ and $\mathcal{Y}$?

The converse does hold in the Medvedev cases: A closed (compact) $\mathcal{W}$ has meet-reducible degree in $\mathfrak{M}_{\mathrm{cl}}$ ($\mathfrak{M}^{01}_{\mathrm{cl}}$) if and only if it has meet-reducible degree in $\mathfrak{M}$. See Lemma 5.1 below.

Definition 4.6. Let $\mathfrak{L}$ be one of $\mathfrak{M}_{\mathrm{cl}}, \mathfrak{M}^{01}_{\mathrm{cl}}, \mathfrak{M}_{\mathrm{w,cl}}, \mathfrak{M}^{01}_{\mathrm{w,cl}}$. A *code for* $\mathfrak{N}_2$ *in* $\mathfrak{L}$ is a collection of degrees $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \mathbf{m}, \mathbf{l}, \mathbf{p}, \mathbf{t} \in \mathfrak{L}$ such that if $R_{\leq}$ is the following 2-ary relation defined on $\widetilde{E}(\mathbf{w}_0)^2$ and R_+ and $R_\times$ are the following 3-ary relations defined on $\widetilde{E}(\mathbf{w}_0)^3$:

- $R_{\leq}(\mathbf{s}_0, \mathbf{u}_0)$ if and only if there is a $\mathbf{u}_1 \in \widetilde{E}(\mathbf{w}_1)$ with $\mathbf{u}_0 + \mathbf{u}_1 \in \widetilde{E}(\mathbf{m})$ and $\mathbf{s}_0 + \mathbf{u}_1 \in \widetilde{E}(\mathbf{l})$,
- $R_+(\mathbf{s}_0, \mathbf{u}_0, \mathbf{v}_0)$ if and only if there is a $\mathbf{u}_1 \in \widetilde{E}(\mathbf{w}_1)$ and a $\mathbf{v}_2 \in \widetilde{E}(\mathbf{w}_2)$ with $\mathbf{u}_0 + \mathbf{u}_1 \in \widetilde{E}(\mathbf{m})$, $\mathbf{v}_0 + \mathbf{v}_2 \in \widetilde{E}(\mathbf{m})$, and $\mathbf{s}_0 + \mathbf{u}_1 + \mathbf{v}_2 \in \widetilde{E}(\mathbf{p})$,
- $R_\times(\mathbf{s}_0, \mathbf{u}_0, \mathbf{v}_0)$ if and only if there is a $\mathbf{u}_1 \in \widetilde{E}(\mathbf{w}_1)$ and a $\mathbf{v}_2 \in \widetilde{E}(\mathbf{w}_2)$ with $\mathbf{u}_0 + \mathbf{u}_1 \in \widetilde{E}(\mathbf{m})$, $\mathbf{v}_0 + \mathbf{v}_2 \in \widetilde{E}(\mathbf{m})$, and $\mathbf{s}_0 + \mathbf{u}_1 + \mathbf{v}_2 \in \widetilde{E}(\mathbf{t})$,

then $\mathfrak{L}$ satisfies the formula that says $\widetilde{E}(\mathbf{w}_0)$ is a discretely ordered commutative semiring with unity and for every $\mathbf{a} \in \mathfrak{L}$, if there is an $\mathbf{s} \in F(\mathbf{a}) \cap \widetilde{E}(\mathbf{w}_0)$, then there is a least such $\mathbf{s}$, where $\leq$, $+$, and $\times$ are interpreted as $R_{\leq}$, R_+, and $R_\times$ respectively.

We think of $\mathbf{w}_0$ as coding ω as $\widetilde{E}(\mathbf{w}_0)$ and any degree $\mathbf{a}$ as coding $F(\mathbf{a}) \cap \widetilde{E}(\mathbf{w}_0) \subseteq \widetilde{E}(\mathbf{w}_0)$. If we can show that every subset of $\widetilde{E}(\mathbf{w}_0)$ has a code, then we will know that the coded structure is exactly $\langle\, \widetilde{E}(\mathbf{w}_0), 2^{\widetilde{E}(\mathbf{w}_0)}, R_{\leq}, R_+, R_\times \,\rangle$ and is isomorphic to $\mathfrak{N}_2$. In fact, it suffices to show that every countable subset of $\widetilde{E}(\mathbf{w}_0)$ has a code. This is because if there is a nonempty $S \subseteq \widetilde{E}(\mathbf{w}_0)$ with no $R_{\leq}$-least element, then there is a countable such S. So if every countable subset of $\widetilde{E}(\mathbf{w}_0)$ has a code and every nonempty coded subset of $\widetilde{E}(\mathbf{w}_0)$ has an $R_{\leq}$-least element, then $\langle\, \widetilde{E}(\mathbf{w}_0), R_{\leq}, R_+, R_\times \,\rangle$ is a well-founded model of arithmetic and, as such, is isomorphic to $\mathfrak{N}$. In particular, $\widetilde{E}(\mathbf{w}_0)$ is

countable, so every subset is countable and hence has a code. Our attention now turns to finding these codes.

4.2. **Coding subsets of ω in $\mathfrak{M}_{\mathrm{w,cl}}$ and $\mathfrak{M}^{01}_{\mathrm{w,cl}}$.** It is well-known that $\mathfrak{M}_{\mathrm{w}}$ is a *complete* lattice. That is, every arbitrary collection of degrees $S \subseteq \mathfrak{M}_{\mathrm{w}}$ has a least upper bound and a greatest lower bound. Let $\langle\, \mathcal{X}_i \mid i \in I \,\rangle$ be a selection of one representative for each degree in S. Then the least upper bound of S is $\big[\bigcap_{i\in I} C(\mathcal{X}_i)\big]_{\mathrm{w}}$ and the greatest lower bound of S is $\big[\bigcup_{i\in I} C(\mathcal{X}_i)\big]_{\mathrm{w}}$ (which equals $\big[\bigcup_{i\in I} X_i\big]_{\mathrm{w}}$). In $\mathfrak{M}_{\mathrm{w,cl}}$ and $\mathfrak{M}^{01}_{\mathrm{w,cl}}$, arbitrary countable collections of degrees have greatest lower bounds. This fact allows us to code all countable subsets of an $\widetilde{E}(\mathbf{w})$.

Lemma 4.7. *Both $\mathfrak{M}_{\mathrm{w,cl}}$ and $\mathfrak{M}^{01}_{\mathrm{w,cl}}$ are countably meet-complete.*

Proof. For $\mathfrak{M}_{\mathrm{w,cl}}$, let $\{\mathbf{x}_i \mid i \in \omega\} \subseteq \mathfrak{M}_{\mathrm{w,cl}}$ be a countable set of degrees and let $\mathcal{X}_i \subseteq \omega^\omega$ be a closed representative of $\mathbf{x}_i$ for each i. The degree $\mathbf{a} = \big[\bigcup_{i\in\omega} i^\frown \mathcal{X}_i\big]_{\mathrm{w}}$ is in $\mathfrak{M}_{\mathrm{w,cl}}$ and is a lower bound for the degrees $\mathbf{x}_i$. Suppose $\mathbf{b}$ is any other lower bound for the $\mathbf{x}_i$ and let $\mathcal{B}$ be a representative for $\mathbf{b}$. Then $\mathcal{B} \leq_{\mathrm{w}} \mathcal{X}_i$ for each i which means $(\forall i \in \omega)(\forall f \in \mathcal{X}_i)(\exists g \in \mathcal{B})[g \leq_{\mathrm{T}} f]$. So $\big(\forall f \in \bigcup_{i\in\omega} i^\frown \mathcal{X}_i\big)\big(\exists g \in \mathcal{B}\big)\big[g \leq_{\mathrm{T}} f\big]$. Hence $\mathbf{b} \leq_{\mathrm{w}} \mathbf{a}$.

The above proof does not work for $\mathfrak{M}^{01}_{\mathrm{w,cl}}$ because $\bigcup_{i\in\omega} i^\frown \mathcal{X}_i$ is not compact. We provide a modified proof for $\mathfrak{M}^{01}_{\mathrm{w,cl}}$. Let $\{\mathbf{x}_i \mid i \in \omega\} \subseteq \mathfrak{M}^{01}_{\mathrm{w,cl}}$ be a countable set of degrees and let $\mathcal{X}_i \subseteq 2^\omega$ be a closed representative of $\mathbf{x}_i$ for each i. Choose any g in any non-empty $\mathcal{X}_i$ (if all the $\mathcal{X}_i$ are empty, then $[\emptyset]_{\mathrm{w}}$ is the greatest-lower-bound). Let $\sigma_i = (g \upharpoonright i)^\frown(1 - g(i))$ for each $i \in \omega$. The set $\mathcal{A} = \{g\} \cup \big(\bigcup_{i\in\omega} \sigma_i^\frown \mathcal{X}_i\big)$ is closed in 2^ω, so let $\mathbf{a} = [\mathcal{A}]_{\mathrm{w}}$. Then $\mathbf{a} \in \mathfrak{M}^{01}_{\mathrm{w,cl}}$ and the rest of the proof proceeds as in the $\mathfrak{M}_{\mathrm{w,cl}}$ case. □

In contrast, $\mathfrak{M}$, $\mathfrak{M}_{\mathrm{cl}}$, and $\mathfrak{M}^{01}_{\mathrm{cl}}$ are not countably complete, as shown by Dyment's Lemma 6.2 below.

Lemma 4.8. *Let $\mathfrak{L}$ be $\mathfrak{M}_{\mathrm{w,cl}}$ or $\mathfrak{M}^{01}_{\mathrm{w,cl}}$. Then for any $\mathbf{w} \in \mathfrak{L}$ and any at-most-countable $S \subseteq \widetilde{E}(\mathbf{w})$ there is an $\mathbf{a} \in \mathfrak{L}$ such that $F(\mathbf{a}) \cap \widetilde{E}(\mathbf{w}) = S$.*

Proof. In either case take $\mathbf{a}$ to be the greatest-lower-bound of S by Lemma 4.7. This ensures $S \subseteq F(\mathbf{a}) \cap \widetilde{E}(\mathbf{w})$. To see equality, let $\mathbf{x} \in \widetilde{E}(\mathbf{w}) - S$ and let $\mathbf{y}$ be such that $\mathbf{y} >_{\mathrm{w}} \mathbf{w}$ and $\mathbf{x} \times \mathbf{y} = \mathbf{w}$. If $\mathbf{s} \in S$, then $\mathbf{s} \not\geq_{\mathrm{w}} \mathbf{x}$ because $\widetilde{E}(\mathbf{w})$ is an antichain. Thus $\mathbf{s} \geq_{\mathrm{w}} \mathbf{y}$ for all $\mathbf{s} \in S$ because $\mathbf{s}$ is meet-irreducible and $\mathbf{s} \geq_{\mathrm{w}} \mathbf{w} = \mathbf{x} \times \mathbf{y}$ for all $\mathbf{s} \in S$. Therefore $\mathbf{a} \geq_{\mathrm{w}} \mathbf{y}$ which implies $\mathbf{x} \not\geq_{\mathrm{w}} \mathbf{a}$. □

It is possible for $\widetilde{E}(\mathbf{w})$ to be uncountable for $\mathbf{w} \in \mathfrak{M}_{\mathrm{w,cl}}$ or $\mathbf{w} \in \mathfrak{M}^{01}_{\mathrm{w,cl}}$. This is in contrast to the Medvedev cases, in which $\widetilde{E}(\mathbf{w})$ is always at most countable (see Corollary 5.3 below).

Lemma 4.9. *If $\mathcal{W} \subseteq \omega^\omega$ ($\mathcal{W} \subseteq 2^\omega$) is a closed Turing antichain, then, in $\mathfrak{M}_{\mathrm{w,cl}}$ ($\mathfrak{M}^{01}_{\mathrm{w,cl}}$), $\widetilde{E}([\mathcal{W}]_{\mathrm{w}}) = \{[\{f\}]_{\mathrm{w}} \mid f \in \mathcal{W}\}$.*

Proof. Assume in both cases that $|\mathcal{W}| > 1$, for otherwise the lemma is trivial.

For $\mathfrak{M}_{\mathrm{w,cl}}$, let $\mathcal{W} \subseteq \omega^\omega$ be closed and a Turing antichain, and let $f \in \mathcal{W}$. Let T be a tree whose set of paths is $\mathcal{W}$. Let $\langle\, \tau_i \mid i \in \omega \,\rangle$ list the sequences in T that are not initial segments of f (so that, for $g \in \mathcal{W}$, $g \neq f \leftrightarrow \exists i[\tau_i \subset g]$). Let T_i denote the full subtree of T rooted at τ_i: $T_i = \{\sigma \in \omega^{<\omega} \mid \tau_i{}^\frown\sigma \in T\}$. Let R be the tree $\bigcup_{i\in\omega} i^\frown T_i$ where $i^\frown T_i = \{i^\frown\sigma \mid \sigma \in T_i\}$ for each i. Let $\mathcal{Y}$ be the set of paths through R. If, for a mass problem $\mathcal{A}$, we let $\deg_{\mathrm{T}} \mathcal{A} = \{\deg_{\mathrm{T}} f \mid f \in \mathcal{A}\}$ denote the set of Turing degrees of the members of $\mathcal{A}$, we see that $\deg_{\mathrm{T}} \mathcal{Y} = \deg_{\mathrm{T}} \mathcal{W} - \{\deg_{\mathrm{T}} f\}$. From this and the fact that $\mathcal{W}$ is a Turing antichain, it follows that $\mathcal{Y} >_{\mathrm{w}} \mathcal{W}$ and $\mathcal{W} \equiv_{\mathrm{w}} \{f\} \times \mathcal{Y}$. Hence $[\{f\}]_{\mathrm{w}}$ is meet-irreducible and meets to $[\mathcal{W}]_{\mathrm{w}}$. We need to show that $[\{f\}]_{\mathrm{w}}$ is minimal. First suppose that $\mathcal{B} \geq_{\mathrm{w}} \mathcal{W}$ is closed and has meet-irreducible degree in $\mathfrak{M}_{\mathrm{w,cl}}$. We claim $\mathcal{B} \not\geq_{\mathrm{w}} \{f\}$ implies $\{f\} \not\geq_{\mathrm{w}} \mathcal{B}$. We have $\mathcal{B} \geq_{\mathrm{w}} \{f\}$ or $\mathcal{B} \geq_{\mathrm{w}} \mathcal{Y}$ because $\mathcal{B} \geq_{\mathrm{w}} \mathcal{W} \equiv_{\mathrm{w}} \{f\} \times \mathcal{Y}$ and $\mathcal{B}$ has meet-irreducible

degree. But $\mathcal{B} \not\geq_{\mathrm{w}} \{f\}$, so we must have $\mathcal{B} \geq_{\mathrm{w}} \mathcal{Y}$. Thus $\{f\} \not\geq_{\mathrm{w}} \mathcal{B}$ because $\{f\} \not\geq_{\mathrm{w}} \mathcal{Y}$. Therefore, if we have a closed $\mathcal{B}$ of meet-irreducible degree in $\mathfrak{M}_{\mathrm{w,cl}}$ with $\mathcal{W} \leq_{\mathrm{w}} \mathcal{B} \leq_{\mathrm{w}} \{f\}$, then the contrapositive of the claim tells us $\{f\} \leq_{\mathrm{w}} \mathcal{B}$. Thus $[\{f\}]_{\mathrm{w}}$ is minimal, making $[\{f\}]_{\mathrm{w}} \in \widetilde{E}([\mathcal{W}]_{\mathrm{w}})$.

Conversely, suppose for a contradiction that $\mathcal{B}$ is closed and $[\mathcal{B}]_{\mathrm{w}} \in \widetilde{E}([\mathcal{W}]_{\mathrm{w}})$, but $\mathcal{B} \not\geq_{\mathrm{w}} \{f\}$ for all $f \in \mathcal{W}$. By the claim, we also have $\{f\} \not\geq_{\mathrm{w}} \mathcal{B}$ for all $f \in \mathcal{W}$. Then if $\mathcal{C}$ is closed such that $\mathcal{W} \equiv_{\mathrm{w}} \mathcal{B} \times \mathcal{C}$, it must be that $\{f\} \geq_{\mathrm{w}} \mathcal{C}$ for all $f \in \mathcal{W}$. Hence $\mathcal{W} \geq_{\mathrm{w}} \mathcal{C}$. So for any closed $\mathcal{C}$ such that $\mathcal{W} \equiv_{\mathrm{w}} \mathcal{B} \times \mathcal{C}$ we have $\mathcal{W} \geq_{\mathrm{w}} \mathcal{C}$. This contradicts that $[\mathcal{B}]_{\mathrm{w}}$ meets to $[\mathcal{W}]_{\mathrm{w}}$. Thus if $\mathbf{b} \in \widetilde{E}([\mathcal{W}]_{\mathrm{w}})$, we must have $\mathbf{b} \geq_{\mathrm{w}} [\{f\}]_{\mathrm{w}}$ for some $f \in \mathcal{W}$. But $[\{f\}]_{\mathrm{w}} \in \widetilde{E}([\mathcal{W}]_{\mathrm{w}})$, hence $\mathbf{b} = [\{f\}]_{\mathrm{w}}$ by minimality.

For $\mathfrak{M}^{01}_{\mathrm{w,cl}}$, let $\mathcal{W} \subseteq 2^\omega$ be closed and a Turing antichain, and let $f \in \mathcal{W}$. Let T be a tree whose set of paths is $\mathcal{W}$. Let $\langle\, \tau_i \mid i \in \omega \,\rangle$ list the sequences in T that are not initial segments of f. Let T_i denote the full subtree of T rooted at τ_i. Choose any $g \in \mathcal{W} - \{f\}$. Let $\sigma_i = (g \upharpoonright i)^\frown(1 - g(i))$ for each $i \in \omega$. Let R be the tree $\bigcup_{i\in\omega} \sigma_i{}^\frown T_i$. Let $\mathcal{Y}$ be the set of paths through R. Then $\deg_{\mathrm{T}} \mathcal{Y} = \deg_{\mathrm{T}} \mathcal{W} - \{\deg_{\mathrm{T}} f\}$. The proof now proceeds exactly as in the $\mathfrak{M}_{\mathrm{w,cl}}$ case. □

Corollary 4.10. *Let $\mathfrak{L}$ be $\mathfrak{M}_{\mathrm{w,cl}}$ or $\mathfrak{M}^{01}_{\mathrm{w,cl}}$. Then there is a degree $\mathbf{w} \in \mathfrak{L}$ such that $\widetilde{E}(\mathbf{w})$ is uncountable.*

Proof. In either case, let $T \subseteq 2^{<\omega}$ be a perfect tree whose set of paths is a Turing antichain, and let $\mathbf{w}$ be the degree of this set of paths. See [18] Section VI.1 for the construction of such a tree. □

4.3. Finding a code for $\mathfrak{N}_2$ in $\mathfrak{M}_{\mathrm{w,cl}}$.

Definition 4.11 (Dyment [2])**.** $\mathcal{W} \subseteq \omega^\omega$ is called *effectively discrete* if $(\forall f \in \mathcal{W})(\forall g \in \mathcal{W})[f \neq g \to f(0) \neq g(0)]$.

An effectively discrete mass problem is closed and at most countable.

Lemma 4.12. *There is a code for $\mathfrak{N}_2$ in $\mathfrak{M}_{\mathrm{w,cl}}$.*

Proof. Let

- $\mathcal{W}_0 = \{i^\frown f_{0,i} \mid i \in \omega\}$, $\mathcal{W}_1 = \{i^\frown f_{1,i} \mid i \in \omega\}$, and $\mathcal{W}_2 = \{i^\frown f_{2,i} \mid i \in \omega\}$ be such that $\mathcal{W}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2$ is independent,
- $\mathcal{M} = \{(2i)^\frown(f_{0,i} \oplus f_{1,i}) \mid i \in \omega\} \cup \{(2i+1)^\frown(f_{0,i} \oplus f_{2,i}) \mid i \in \omega\}$,
- $\mathcal{L} = \{\langle\, i, j \,\rangle^\frown(f_{0,i} \oplus f_{1,j}) \mid i \leq j\}$,
- $\mathcal{P} = \{\langle\, \langle\, i, j \,\rangle, k \,\rangle^\frown(f_{0,i} \oplus f_{1,j} \oplus f_{2,k}) \mid i + j = k\}$,
- $\mathcal{T} = \{\langle\, \langle\, i, j \,\rangle, k \,\rangle^\frown(f_{0,i} \oplus f_{1,j} \oplus f_{2,k}) \mid i \times j = k\}$.

The above mass problems are effectively discrete Turing antichains. Put $\mathbf{w}_0 = [\mathcal{W}_0]_{\mathrm{w}}$, $\mathbf{w}_1 = [\mathcal{W}_1]_{\mathrm{w}}$, $\mathbf{w}_2 = [\mathcal{W}_2]_{\mathrm{w}}$, $\mathbf{m} = [\mathcal{M}]_{\mathrm{w}}$, $\mathbf{l} = [\mathcal{L}]_{\mathrm{w}}$, $\mathbf{p} = [\mathcal{P}]_{\mathrm{w}}$, $\mathbf{t} = [\mathcal{T}]_{\mathrm{w}}$. The verification that these degrees satisfy Definition 4.6 is the same as the verification that the corresponding degrees defined in Lemma 3.11 satisfy case (ii) of Definition 3.8. Use Lemma 4.9 in place of Lemma 3.7. □

We are ready to interpret $\mathfrak{N}_2$ in $\mathfrak{M}_{\mathrm{w,cl}}$.

Lemma 4.13. $\mathrm{Th}(\mathfrak{N}_2) \leq_1 \mathrm{Th}(\mathfrak{M}_{\mathrm{w,cl}})$.

Proof. Let φ be a sentence in the language of $\mathfrak{N}_2$. Each atomic subformula of φ has one of the following forms:

$$x = y \qquad x \leq y \qquad x + y = z \qquad x \times y = z \qquad x \in X.$$

Now let $\varphi'(w_0, w_1, w_2, m, \ell, p, t)$ be the formula (with the displayed variables free) in the language of partial orders obtained from φ by making the replacements below. The second-order variable X in φ corresponds to the variable v_X in φ'.

- Replace $x \leq y$ by the formula defining $R_{\leq}(x, y)$.
- Replace $x + y = z$ by the formula defining $R_+(x, y, z)$.

- Replace $x \times y = z$ by the formula defining $R_\times(x, y, z)$.
- Replace $x \in X$ by the formula expressing $x \in F(v_x)$.
- Replace quantifiers $\exists x$ and $\forall x$ by $\exists x \in \widetilde{E}(w_0)$ and $\forall x \in \widetilde{E}(w_0)$.
- Replace quantifiers $\exists X$ and $\forall X$ by $\exists v_X$ and $\forall v_X$.

Let ψ be the sentence saying "there is a code w_0, w_1, w_2, m, ℓ, p, t, for $\mathfrak{N}_2$ in $\mathfrak{M}_{\mathrm{w,cl}}$ and $\varphi'(w_0, w_1, w_2, m, \ell, p, t)$." A code for $\mathfrak{N}_2$ in $\mathfrak{M}_{\mathrm{w,cl}}$ codes a structure isomorphic to $\mathfrak{N}_2$, and so $\mathfrak{M}_{\mathrm{w,cl}} \models \psi$ if and only if $\mathfrak{N}_2 \models \varphi$. □

4.4. **Finding a code for $\mathfrak{N}_2$ in $\mathfrak{M}^{01}_{\mathrm{w,cl}}$.** An infinite effectively discrete Turing antichain is not compact, so we can no longer rely on them to provide a code. Instead we use the following definition:

Definition 4.14. Let $g \in 2^\omega$. A set $\mathcal{A} \subseteq 2^\omega$ is called a *g-spine* (or just a *spine*) if it is of the form $\{g\} \cup \{\sigma_i{}^\frown f_i \mid i \in X\}$ where $X \subseteq \omega$ is infinite, $\sigma_i = (g \upharpoonright i)^\frown(1 - g(i))$ for each $i \in X$, and $f_i \in 2^\omega$ for each $i \in X$.

Definition 4.15. Let $g \in 2^\omega$ and let $\mathcal{A} \subseteq 2^\omega$ be countable. Fix an enumeration $\langle f_i \mid i \in \omega \rangle$ of $\mathcal{A}$. We denote by $\mathrm{spine}(g, \mathcal{A})$ the g-spine $\{g\} \cup \{\sigma_i{}^\frown f_i \mid i \in \omega\}$ where $\sigma_i = (g \upharpoonright i)^\frown(1 - g(i))$ for each $i \in \omega$. We denote by $\mathrm{spine}(\mathcal{A})$ the f_0-spine $\mathrm{spine}(f_0, \mathcal{A} - \{f_0\})$.

Notice that a spine is a closed subset of 2^ω.

Lemma 4.16. *There is a code for $\mathfrak{N}_2$ in $\mathfrak{M}^{01}_{\mathrm{w,cl}}$.*

Proof. Let $\mathcal{W}'_0 = \{f_{0,i} \mid i \in \omega\}$, $\mathcal{W}'_1 = \{f_{1,i} \mid i \in \omega\}$, and $\mathcal{W}'_2 = \{f_{2,i} \mid i \in \omega\}$ be such that $\mathcal{W}'_0 \cup \mathcal{W}'_1 \cup \mathcal{W}'_2 \subseteq 2^\omega$ is independent. Then let

- $\mathcal{W}_0 = \mathrm{spine}(\mathcal{W}'_0)$, $\mathcal{W}_1 = \mathrm{spine}(\mathcal{W}'_1)$, $\mathcal{W}_2 = \mathrm{spine}(\mathcal{W}'_2)$,
- $\mathcal{M} = \mathrm{spine}(\{f_{0,i} \oplus f_{1,i} \mid i \in \omega\} \cup \{f_{0,i} \oplus f_{2,i} \mid i \in \omega\})$,
- $\mathcal{L} = \mathrm{spine}(\{f_{0,i} \oplus f_{1,j} \mid i \leq j\})$,
- $\mathcal{P} = \mathrm{spine}(\{f_{0,i} \oplus f_{1,j} \oplus f_{2,k} \mid i + j = k\})$,
- $\mathcal{T} = \mathrm{spine}(\{f_{0,i} \oplus f_{1,j} \oplus f_{2,k} \mid i \times j = k\})$.

The above mass problems are spines that are Turing antichains. Put $\mathbf{w}_0 = [\mathcal{W}_0]_{\mathrm{w}}$, $\mathbf{w}_1 = [\mathcal{W}_1]_{\mathrm{w}}$, $\mathbf{w}_2 = [\mathcal{W}_2]_{\mathrm{w}}$, $\mathbf{m} = [\mathcal{M}]_{\mathrm{w}}$, $\mathbf{l} = [\mathcal{L}]_{\mathrm{w}}$, $\mathbf{p} = [\mathcal{P}]_{\mathrm{w}}$, $\mathbf{t} = [\mathcal{T}]_{\mathrm{w}}$. The verification that these degrees satisfy Definition 4.6 is the same as the verification that the corresponding degrees defined in Lemma 3.11 satisfy case (ii) of Definition 3.8. Use Lemma 4.9 in place of Lemma 3.7. □

Lemma 4.17. $\mathrm{Th}(\mathfrak{N}_2) \leq_1 \mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{cl}})$.

Proof. As in Lemma 4.13. □

5. Interpreting arithmetic in the closed and compact Medvedev degrees

In this section we prove that $\mathrm{Th}(\mathfrak{N}_2) \leq_1 \mathrm{Th}(\mathfrak{M}_{\mathrm{cl}})$, $\mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{cl}})$. As always, the crucial point is coding any $S \subseteq \widetilde{E}(\mathbf{w})$ as some $F(\mathbf{a}) \cap \widetilde{E}(\mathbf{w})$. In the Muchnik cases, this was accomplished by assuming that S is countable, fixing a closed representative for each degree in S, and essentially taking the union of these representatives. However, the proof that this produced such an $\mathbf{a}$ relied on the non-uniformity afforded by Muchnik reducibility. Specifically, if $\mathcal{X}_i \geq_{\mathrm{w}} \mathcal{Y}$ for each $i \in \omega$, then $\bigcup_{i \in \omega} \mathcal{X}_i \geq_{\mathrm{w}} \mathcal{Y}$. In the Medvedev cases, it may be that $\mathcal{X}_i \geq_{\mathrm{M}} \mathcal{Y}$ for each $i \in \omega$ but $\bigcup_{i \in \omega} \mathcal{X}_i \not\geq_{\mathrm{M}} \mathcal{Y}$ because the reductions witnessing each $\mathcal{X}_i \geq_{\mathrm{M}} \mathcal{Y}$ cannot be combined into one uniform reduction witnessing $\bigcup_{i \in \omega} \mathcal{X}_i \geq_{\mathrm{M}} \mathcal{Y}$. We will show that it is possible to choose the representatives $\mathcal{X}_i$ in such a way that taking their union preserves uniformity.

5.1. **Coding subsets of ω.** The results of this section apply to arbitrary mass problems, not just closed and compact ones. We consider arbitrary mass problems $\mathcal{W}$ and degrees $\mathbf{w} \in \mathfrak{M}$.

The next lemma is a clarifying example. It implies that a closed (compact) $\mathcal{W}$ has meet-reducible degree in $\mathfrak{M}_{\rm cl}$ ($\mathfrak{M}^{01}_{\rm cl}$) if and only if it has meet-reducible degree in $\mathfrak{M}$.

Lemma 5.1 (Dyment [2])**.** *If $\mathcal{W} \equiv_{\rm M} \mathcal{X} \times \mathcal{Y}$, then $\mathcal{W} = \widehat{\mathcal{X}} \cup \widehat{\mathcal{Y}}$ where $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ are disjoint and clopen in $\mathcal{W}$, $\widehat{\mathcal{X}} \geq_{\rm M} \mathcal{X}$, $\widehat{\mathcal{Y}} \geq_{\rm M} \mathcal{Y}$, and $\mathcal{W} \equiv_{\rm M} \widehat{\mathcal{X}} \times \widehat{\mathcal{Y}}$.*

Proof. Let Φ be such that $\Phi(\mathcal{W}) \subseteq 0^\frown\mathcal{X} \cup 1^\frown\mathcal{Y}$. Put $\widehat{\mathcal{X}} = \{f \in \mathcal{W} \mid \Phi^f(0) = 0\}$ and put $\widehat{\mathcal{Y}} = \{f \in \mathcal{W} \mid \Phi^f(0) = 1\}$. By Lemma 1.5, $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ are clopen in $\mathcal{W}$, and it is easily checked that $\widehat{\mathcal{X}} \geq_{\rm M} \mathcal{X}$ and $\widehat{\mathcal{Y}} \geq_{\rm M} \mathcal{Y}$ (hence $\mathcal{W} \leq_{\rm M} \widehat{\mathcal{X}} \times \widehat{\mathcal{Y}}$). We have $\mathcal{W} \geq_{\rm M} 0^\frown\widehat{\mathcal{X}} \cup 1^\frown\widehat{\mathcal{Y}}$ by the reduction which sends f to $0^\frown f$ if $\Phi^f(0) = 0$ and sends f to $1^\frown f$ if $\Phi^f(0) = 1$. □

Our coding relies on the following lemma similar to Lemma 5.1.

Lemma 5.2. *If $\mathcal{W} \equiv_{\rm M} \mathcal{X} \times \mathcal{Y}$ where $\mathcal{X}$ has meet-irreducible degree and $\mathcal{Y} >_{\rm M} \mathcal{W}$, then $\mathcal{W} = \widehat{\mathcal{X}} \cup \widehat{\mathcal{Y}}$ where $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ are disjoint and clopen in $\mathcal{W}$, $\widehat{\mathcal{X}} \equiv_{\rm M} \mathcal{X}$, and $\widehat{\mathcal{X}} \not\geq_{\rm M} \widehat{\mathcal{Y}}$.*

Proof. As in Lemma 5.1, let Φ be such that $\Phi(\mathcal{W}) \subseteq 0^\frown\mathcal{X} \cup 1^\frown\mathcal{Y}$, put $\widehat{\mathcal{X}} = \{f \in \mathcal{W} \mid \Phi^f(0) = 0\}$, and put $\widehat{\mathcal{Y}} = \{f \in \mathcal{W} \mid \Phi^f(0) = 1\}$. Then $\mathcal{W} = \widehat{\mathcal{X}} \cup \widehat{\mathcal{Y}}$, $\widehat{\mathcal{X}} \cap \widehat{\mathcal{Y}} = \emptyset$, $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ are clopen in $\mathcal{W}$, $\widehat{\mathcal{X}} \geq_{\rm M} \mathcal{X}$, $\widehat{\mathcal{Y}} \geq_{\rm M} \mathcal{Y}$, and $\mathcal{W} \equiv_{\rm M} \widehat{\mathcal{X}} \times \widehat{\mathcal{Y}}$. To see $\mathcal{X} \geq_{\rm M} \widehat{\mathcal{X}}$, observe $\mathcal{X} \geq_{\rm M} \mathcal{W} \equiv_{\rm M} \widehat{\mathcal{X}} \times \widehat{\mathcal{Y}}$. $\mathcal{X}$ has meet-irreducible degree, so $\mathcal{X} \geq_{\rm M} \widehat{\mathcal{X}}$ or $\mathcal{X} \geq_{\rm M} \widehat{\mathcal{Y}}$. We cannot have $\mathcal{X} \geq_{\rm M} \widehat{\mathcal{Y}}$ because $\widehat{\mathcal{Y}} \geq_{\rm M} \mathcal{Y}$ and this would imply $\mathcal{W} \equiv_{\rm M} \mathcal{X} \times \mathcal{Y} \equiv_{\rm M} \mathcal{Y} >_{\rm M} \mathcal{W}$. Thus $\mathcal{X} \geq_{\rm M} \widehat{\mathcal{X}}$. Similarly $\widehat{\mathcal{X}} \not\geq_{\rm M} \widehat{\mathcal{Y}}$ for otherwise $\mathcal{W} \equiv_{\rm M} \widehat{\mathcal{Y}} \geq_{\rm M} \mathcal{Y} >_{\rm M} \mathcal{W}$. □

Corollary 5.3. *For any degree $\mathbf{w} \in \mathfrak{M}$ there are at most countably many meet-irreducible degrees that meet to $\mathbf{w}$.*

Proof. Fix a representative $\mathcal{W}$ for $\mathbf{w}$. In Lemma 5.2 we showed that if $\mathbf{x}$ is meet-irreducible and meets to $\mathbf{w}$, then $\mathbf{x}$ has a representative $\{f \in \mathcal{W} \mid \Phi^f(0) = 0\}$ for some program Φ. There are only countably many programs, so there can be at most countably many such $\mathbf{x}$. □

Notice that Corollary 5.3 is in contrast to the Muchnik case, in which a degree may have uncountably many meet-irreducibles that meet to it (see Lemma 4.9). Also notice that if $\mathbf{w}$ is closed (compact) and $\mathbf{x}$ is meet-irreducible and meets to $\mathbf{w}$, then Lemma 5.2 produces a closed (compact) representative for $\mathbf{x}$. Thus for a closed (compact) degree $\mathbf{w}$, the meet-irreducible degrees that meet to $\mathbf{w}$ are the same whether they are computed in $\mathfrak{M}$ or in $\mathfrak{M}_{\rm cl}$ ($\mathfrak{M}^{01}_{\rm cl}$).

Lemma 5.4. *Let $\mathcal{W}$ be a mass problem whose degree has countably many minimal meet-irreducible degrees meeting to it, and let $\langle\, \mathcal{X}_i \mid i \in \omega \,\rangle$ be a list of representatives for these degrees. Then there are mass problems $\langle\, \widehat{\mathcal{X}}_i \mid i \in \omega \,\rangle$ such that:*

(i) *$\widehat{\mathcal{X}}_i \subseteq \mathcal{W}$ is clopen in $\mathcal{W}$ for each i,*
(ii) *$\widehat{\mathcal{X}}_i \cap \widehat{\mathcal{X}}_j = \emptyset$ for $i \neq j$,*
(iii) *$\widehat{\mathcal{X}}_i \equiv_{\rm M} \mathcal{X}_i$ for each i,*
(iv) *$\widehat{\mathcal{X}}_i \not\geq_{\rm M} \mathcal{W} - \widehat{\mathcal{X}}_i$ for each i.*

Proof. Inductively construct the sequence $\langle\, \widehat{\mathcal{X}}_i \mid i \in \omega \,\rangle$. At the start of step $n+1$ we have $\langle\, \widehat{\mathcal{X}}_i \mid i \leq n \,\rangle$ satisfying (i)–(iv) for $i, j \leq n$, and we have indices $e_0, \ldots, e_n$ such that, for $i \leq n$, $\widehat{\mathcal{X}}_i = \big\{f \in \mathcal{W} - \bigcup_{j<i} \widehat{\mathcal{X}}_j \mid \Phi^f_{e_i}(0) = 0\big\}$ and $\mathcal{W} - \bigcup_{j\leq i} \widehat{\mathcal{X}}_j = \big\{f \in \mathcal{W} - \bigcup_{j<i} \widehat{\mathcal{X}}_j \mid \Phi^f_{e_i}(0) = 1\big\}$.

We first show $\mathcal{W} \equiv_{\rm M} \widehat{\mathcal{X}}_0 \times \cdots \times \widehat{\mathcal{X}}_n \times \big(\mathcal{W} - \bigcup_{i\leq n} \widehat{\mathcal{X}}_i\big)$. The meet is $\geq_{\rm M} \mathcal{W}$ because each term is a subset of $\mathcal{W}$. To see the reverse inequality, write the meet as $\bigcup_{i\leq n} i^\frown\widehat{\mathcal{X}}_i \cup \big(n+1\big)^\frown\big(\mathcal{W} - \bigcup_{i\leq n} \widehat{\mathcal{X}}_i\big)$.

Then apply the following reduction: For each $i \leq n$ in order, check if $\Phi_{e_i}^f(0)$ is 0 or 1. If it is 0, send f to $i^\frown f$. If it is 1, go to the next i. If $\Phi_{e_i}^f(0) = 1$ for each $i \leq n$, then send f to $(n+1)^\frown f$.

We now have $\mathcal{X}_{n+1} \geq_{\mathrm{M}} \mathcal{W} \equiv_{\mathrm{M}} \widehat{\mathcal{X}}_0 \times \cdots \times \widehat{\mathcal{X}}_n \times (\mathcal{W} - \bigcup_{i \leq n} \widehat{\mathcal{X}}_i)$. We cannot have $\mathcal{X}_{n+1} \geq_{\mathrm{M}} \widehat{\mathcal{X}}_i$ for any $i \leq n$ because $\widehat{\mathcal{X}}_i \equiv_{\mathrm{M}} \mathcal{X}_i$ and the $\mathcal{X}_i$'s are incomparable by minimality. However, $\mathcal{X}_{n+1}$ has meet-irreducible degree. Therefore $\mathcal{X}_{n+1} \geq_{\mathrm{M}} \mathcal{W} - \bigcup_{i \leq n} \widehat{\mathcal{X}}_i$. Moreover, by distributivity $[\mathcal{X}_{n+1}]$ meets to $[\mathcal{W} - \bigcup_{i \leq n} \widehat{\mathcal{X}}_i]$ because $[\mathcal{X}_{n+1}]$ meets to $[\mathcal{W}]$ and $[\mathcal{X}_{n+1}] \geq_{\mathrm{M}} [\mathcal{W} - \bigcup_{i \leq n} \widehat{\mathcal{X}}_i] \geq_{\mathrm{M}} [\mathcal{W}]$. Thus, as in Lemma 5.2, there is an $\widehat{\mathcal{X}}_{n+1} \subseteq \mathcal{W} - \bigcup_{i \leq n} \widehat{\mathcal{X}}_i$ clopen in $W - \bigcup_{i \leq n} \widehat{\mathcal{X}}_i$ and an e_{n+1} such that $\widehat{\mathcal{X}}_{n+1} = \{f \in \mathcal{W} - \bigcup_{i \leq n} \widehat{\mathcal{X}}_i \mid \Phi_{e_{n+1}}^f(0) = 0\}$, $\mathcal{W} - \bigcup_{i \leq n+1} \widehat{\mathcal{X}}_i = \{f \in \mathcal{W} - \bigcup_{i \leq n} \widehat{\mathcal{X}}_i \mid \Phi_{e_{n+1}}^f(0) = 1\}$, $\widehat{\mathcal{X}}_{n+1} \equiv_{\mathrm{M}} \mathcal{X}_{n+1}$, and $\widehat{\mathcal{X}}_{n+1} \not\geq_{\mathrm{M}} \mathcal{W} - \bigcup_{i \leq n+1} \widehat{\mathcal{X}}_i$. Clearly $\widehat{\mathcal{X}}_{n+1}$ is disjoint from $\widehat{\mathcal{X}}_i$ for $i \leq n$. $\widehat{\mathcal{X}}_{n+1}$ is clopen in $\mathcal{W}$ because it is clopen in $\mathcal{W} - \bigcup_{i \leq n} \widehat{\mathcal{X}}_i$ which is clopen in $\mathcal{W}$. Finally, $\widehat{\mathcal{X}}_{n+1} \not\geq_{\mathrm{M}} \mathcal{W} - \widehat{\mathcal{X}}_{n+1}$ because $\widehat{\mathcal{X}}_{n+1}$ has meet-irreducible degree, $\widehat{\mathcal{X}}_{n+1} \not\geq_{\mathrm{M}} \widehat{\mathcal{X}}_i$ for $i \leq n$, $\widehat{\mathcal{X}}_{n+1} \not\geq_{\mathrm{M}} \mathcal{W} - \bigcup_{i \leq n+1} \widehat{\mathcal{X}}_i$, and $\mathcal{W} - \widehat{\mathcal{X}}_{n+1} \equiv_{\mathrm{M}} \widehat{\mathcal{X}}_0 \times \cdots \times \widehat{\mathcal{X}}_n \times (\mathcal{W} - \bigcup_{i \leq n+1} \widehat{\mathcal{X}}_i)$. □

The next lemma implies that every subset of ω has a code. That is, if $\mathbf{w}$ is closed (compact) and $S \subseteq \widetilde{E}(\mathbf{w})$ then there is a closed (compact) $\mathbf{a}$ such that $F(\mathbf{a}) \cap \widetilde{E}(\mathbf{w}) = S$.

Lemma 5.5. *Let $\mathbf{w} \in \mathfrak{M}$ and let $\mathcal{W}$ be a representative for $\mathbf{w}$. Then for any $S \subseteq \widetilde{E}(\mathbf{w})$ there is an $\mathcal{A} \subseteq \mathcal{W}$ closed in $\mathcal{W}$ such that $F([\mathcal{A}]) \cap \widetilde{E}(\mathbf{w}) = S$.*

Proof. We only consider the case in which $\widetilde{E}(\mathbf{w})$ is infinite. By Corollary 5.3, $\widetilde{E}(\mathbf{w})$ is countable. Let $\langle \mathcal{X}_i \mid i \in \omega \rangle$ be a list of representatives for the degrees in $\widetilde{E}(\mathbf{w})$. Apply Lemma 5.4 to $\mathcal{W}$ and $\langle \mathcal{X}_i \mid i \in \omega \rangle$ to get a new set of representatives $\langle \widehat{\mathcal{X}}_i \mid i \in \omega \rangle$ disjoint and clopen in $\mathcal{W}$ with $\widehat{\mathcal{X}}_i \not\geq_{\mathrm{M}} \mathcal{W} - \widehat{\mathcal{X}}_i$ for each i. Put $\mathcal{A} = \mathcal{W} - \bigcup\{\widehat{\mathcal{X}}_i \mid [\widehat{\mathcal{X}}_i] \notin S\}$, and note that $\mathcal{A}$ is closed in $\mathcal{W}$. We show $\widehat{\mathcal{X}}_i \geq_{\mathrm{M}} \mathcal{A}$ if and only if $[\widehat{\mathcal{X}}_i] \in S$. If $[\widehat{\mathcal{X}}_i] \in S$ then $\widehat{\mathcal{X}}_i \subseteq \mathcal{A}$ and so $\widehat{\mathcal{X}}_i \geq_{\mathrm{M}} \mathcal{A}$. If $[\widehat{\mathcal{X}}_i] \notin S$ then $\mathcal{A} \subseteq \mathcal{W} - \widehat{\mathcal{X}}_i$ and so $\mathcal{A} \geq_{\mathrm{M}} \mathcal{W} - \widehat{\mathcal{X}}_i$. Thus $\widehat{\mathcal{X}}_i \not\geq_{\mathrm{M}} \mathcal{A}$ because $\widehat{\mathcal{X}}_i \not\geq_{\mathrm{M}} \mathcal{W} - \widehat{\mathcal{X}}_i$. □

5.2. **Finding a code for $\mathfrak{N}_2$ in $\mathfrak{M}_{\mathrm{cl}}$ and in $\mathfrak{M}_{\mathrm{cl}}^{01}$.** The following lemma is the $\mathfrak{M}_{\mathrm{cl}}$ analog to Lemma 3.7 and Lemma 4.9:

Lemma 5.6. *If $\mathcal{W}$ is an effectively discrete Turing antichain, then $\widetilde{E}([\mathcal{W}]) = \{[\{f\}] \mid f \in \mathcal{W}\}$.*

Proof. First, let $f \in \mathcal{W}$ and suppose $\mathcal{B} \geq_{\mathrm{M}} \mathcal{W}$ has meet-irreducible degree. We claim $\mathcal{B} \not\geq_{\mathrm{M}} \{f\}$ implies $\{f\} \not\geq_{\mathrm{M}} \mathcal{B}$. To see this, use the effectively discreteness of $\mathcal{W}$ to show $\mathcal{W} \equiv_{\mathrm{M}} \{f\} \times (\mathcal{W} - \{f\})$. If $\mathcal{B} \not\geq_{\mathrm{M}} \{f\}$, then it must be that $\mathcal{B} \geq_{\mathrm{M}} \mathcal{W} - \{f\}$ because $\mathcal{B}$ has meet-irreducible degree and $\mathcal{B} \geq_{\mathrm{M}} \{f\} \times (\mathcal{W} - \{f\})$. Hence $\{f\} \not\geq_{\mathrm{M}} \mathcal{B}$ because $\{f\} \not\geq_{\mathrm{M}} \mathcal{W} - \{f\}$.

Now, if $f \in \mathcal{W}$, it is clear that $[\{f\}]$ is meet-irreducible and meets to $[\mathcal{W}]$. To see that $[\{f\}]$ is minimal, suppose $\mathcal{B}$ is closed, has meet-irreducible degree, and $\mathcal{W} \leq_{\mathrm{M}} \mathcal{B} \leq_{\mathrm{M}} \{f\}$. The contrapositive of the claim tells us $\{f\} \leq_{\mathrm{M}} \mathcal{B}$. Thus $[\{f\}]$ is minimal, making $[\{f\}] \in \widetilde{E}([\mathcal{W}])$.

Conversely, suppose for a contradiction that $\mathcal{B}$ is closed and $[\mathcal{B}] \in \widetilde{E}([\mathcal{W}])$, but $\mathcal{B} \not\geq_{\mathrm{M}} \{f\}$ for all $f \in \mathcal{W}$. By the claim, we also have $\{f\} \not\geq_{\mathrm{M}} \mathcal{B}$ for all $f \in \mathcal{W}$. So if $\mathcal{W} \equiv_{\mathrm{M}} \mathcal{B} \times \mathcal{C}$ for some $\mathcal{C}$, we must have $\mathcal{W} \geq_{\mathrm{M}} \mathcal{C}$ because no Φ can send a member of $\mathcal{W}$ to a member of $\mathcal{B}$. This contradicts that $[\mathcal{B}]$ meets to $[\mathcal{W}]$. So if $\mathbf{b} \in \widetilde{E}([\mathcal{W}])$, we must have $\mathbf{b} \geq_{\mathrm{M}} [\{f\}]$ for some $f \in \mathcal{W}$. But $[\{f\}] \in \widetilde{E}([\mathcal{W}])$, hence $\mathbf{b} = [\{f\}]$ by minimality. □

We also need the compact version of Lemma 5.6 for $\mathfrak{M}_{\mathrm{cl}}^{01}$:

Lemma 5.7. *If $\mathcal{W} = \{g\} \cup \{\sigma_i{}^\frown f_i \mid i \in X\}$ is a g-spine that is a Turing antichain, then $\widetilde{E}([\mathcal{W}]) = \{[\{f_i\}] \mid i \in X\}$.*

Proof. One can check $\mathcal{W} \equiv_{\mathrm{M}} \{f_i\} \times (\mathcal{W} - \{\sigma_i{}^\frown f_i\})$ for each $i \in X$. So if $\mathcal{B} \geq_{\mathrm{M}} \mathcal{W}$ has meet-irreducible degree and $\mathcal{B} \not\geq_{\mathrm{M}} \{f_i\}$, we have both $\mathcal{B} \geq_{\mathrm{M}} \mathcal{W} - \{\sigma_i{}^\frown f_i\}$ and $\{f_i\} \not\geq_{\mathrm{M}} \mathcal{B}$. The proof that $[\{f_i\}] \in \widetilde{E}([\mathcal{W}])$ is then the same as in Lemma 5.6.

Conversely, suppose for a contradiction that $[\mathcal{B}] \in \widetilde{E}([\mathcal{W}])$ but $\mathcal{B} \not\geq_{\mathrm{M}} \{f_i\}$ for all $i \in X$. Therefore $\{f_i\} \not\geq_{\mathrm{M}} \mathcal{B}$ for all $i \in X$. Suppose then that $\mathcal{W} \equiv_{\mathrm{M}} \mathcal{B} \times \mathcal{C}$ for some $\mathcal{C}$, and let Φ be such that $\Phi(\mathcal{W}) \subseteq 0{}^\frown\mathcal{B} \cup 1{}^\frown\mathcal{C}$. We must have $\Phi^{\sigma_i{}^\frown f_i} \in 1{}^\frown\mathcal{C}$ for each $i \in X$ because otherwise $\{f_i\} \geq_{\mathrm{M}} \mathcal{B}$ for some i. We must also have $\Phi^g \in 1{}^\frown\mathcal{C}$: If not, $\Phi^g(0) = 0$ and there is some $\tau \subset g$ such that $\Phi^\tau(0) = 0$. Choose $i \in X$ with $i > |\tau|$. Then $\tau \subset \sigma_i$, and we have the contradiction $\Phi^{\sigma_i{}^\frown f_i}(0) = 0$. We must therefore have $\mathcal{W} \geq_{\mathrm{M}} \mathcal{C}$. This contradicts that $[\mathcal{B}]$ meets to $[\mathcal{W}]$. So if $\mathbf{b} \in \widetilde{E}([\mathcal{W}])$, we must have $\mathbf{b} \geq_{\mathrm{M}} [\{f_i\}]$ for some i. But $[\{f_i\}] \in \widetilde{E}([\mathcal{W}])$, hence $\mathbf{b} = [\{f_i\}]$ by minimality. □

Notice the difference between Lemma 4.9 and Lemma 5.7. If $\mathcal{A}$ is a g-spine that is a Turing antichain, then in $\mathfrak{M}^{01}_{\mathrm{w,cl}}$ we have $[\{g\}]_{\mathrm{w}} \in \widetilde{E}([\mathcal{A}]_{\mathrm{w}})$, but in $\mathfrak{M}^{01}_{\mathrm{cl}}$ we have $[\{g\}] \notin \widetilde{E}([\mathcal{A}])$.

Lemma 5.8. *There is a code for $\mathfrak{N}_2$ in $\mathfrak{M}_{\mathrm{cl}}$.*

Proof. As in Lemma 4.12. Use Lemma 5.6 in place of Lemma 4.9. □

Lemma 5.9. $\mathrm{Th}(\mathfrak{N}_2) \leq_1 \mathrm{Th}(\mathfrak{M}_{\mathrm{cl}})$.

Proof. As in Lemma 4.13. □

Lemma 5.10. *There is a code for $\mathfrak{N}_2$ in $\mathfrak{M}^{01}_{\mathrm{cl}}$.*

Proof. Let g, $\mathcal{W}'_0 = \{f_{0,i} \mid i \in \omega\}$, $\mathcal{W}'_1 = \{f_{1,i} \mid i \in \omega\}$, and $\mathcal{W}'_2 = \{f_{2,i} \mid i \in \omega\}$ be such that $\{g\} \cup \mathcal{W}'_0 \cup \mathcal{W}'_1 \cup \mathcal{W}'_2 \subseteq 2^\omega$ is independent. Then let

- $\mathcal{W}_0 = \mathrm{spine}(g, \mathcal{W}_0)$, $\mathcal{W}_1 = \mathrm{spine}(g, \mathcal{W}_1)$, $\mathcal{W}_2 = \mathrm{spine}(g, \mathcal{W}_2)$,
- $\mathcal{M} = \mathrm{spine}(g, \{f_{0,i} \oplus f_{1,i} \mid i \in \omega\} \cup \{f_{0,i} \oplus f_{2,i} \mid i \in \omega\})$,
- $\mathcal{L} = \mathrm{spine}(g, \{f_{0,i} \oplus f_{1,j} \mid i \leq j\})$,
- $\mathcal{P} = \mathrm{spine}(g, \{f_{0,i} \oplus f_{1,j} \oplus f_{2,k} \mid i + j = k\})$,
- $\mathcal{T} = \mathrm{spine}(g, \{f_{0,i} \oplus f_{1,j} \oplus f_{2,k} \mid i \times j = k\})$.

The above mass problems are g-spines that are Turing antichains. Put $\mathbf{w}_0 = [\mathcal{W}_0]$, $\mathbf{w}_1 = [\mathcal{W}_1]$, $\mathbf{w}_2 = [\mathcal{W}_2]$, $\mathbf{m} = [\mathcal{M}]$, $\mathbf{l} = [\mathcal{L}]$, $\mathbf{p} = [\mathcal{P}]$, $\mathbf{t} = [\mathcal{T}]$. The verification that these degrees satisfy Definition 4.6 is the same as the verification that the corresponding degrees defined in Lemma 3.11 satisfy case (ii) of Definition 3.8. Use Lemma 5.7 in place of Lemma 3.7. □

Lemma 5.11. $\mathrm{Th}(\mathfrak{N}_2) \leq_1 \mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{cl}})$.

Proof. As in Lemma 4.13. □

Theorem 5.12. $\mathrm{Th}(\mathfrak{M}_{\mathrm{w,cl}}) \equiv_1 \mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{w,cl}}) \equiv_1 \mathrm{Th}(\mathfrak{M}_{\mathrm{cl}}) \equiv_1 \mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{cl}}) \equiv_1 \mathrm{Th}(\mathfrak{N}_2)$.

Proof. First $\mathrm{Th}(\mathfrak{M}_{\mathrm{cl}})$, $\mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{cl}})$, $\mathrm{Th}(\mathfrak{M}_{\mathrm{w,cl}})$, $\mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{w,cl}}) \leq_1 \mathrm{Th}(\mathfrak{N}_2)$ by Lemma 2.2. Next we have $\mathrm{Th}(\mathfrak{N}_2) \leq_1 \mathrm{Th}(\mathfrak{M}_{\mathrm{w,cl}})$, $\mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{w,cl}})$ by Lemma 4.13 and Lemma 4.17. Finally $\mathrm{Th}(\mathfrak{N}_2) \leq_1 \mathrm{Th}(\mathfrak{M}_{\mathrm{cl}})$, $\mathrm{Th}(\mathfrak{M}^{01}_{\mathrm{cl}})$ by Lemma 5.9 and Lemma 5.11. □

6. A first-order sentence distinguishing $\mathfrak{M}_{\mathrm{cl}}$ and $\mathfrak{M}^{01}_{\mathrm{cl}}$ from $\mathfrak{M}_{\mathrm{w,cl}}$ and $\mathfrak{M}^{01}_{\mathrm{w,cl}}$

We have seen in Lemma 4.7 that $\mathfrak{M}_{\mathrm{w,cl}}$ and $\mathfrak{M}^{01}_{\mathrm{w,cl}}$ are countably meet-complete. In contrast, Dyment proved that in $\mathfrak{M}$ there are countable collections of degrees which do not have greatest lower bounds [3]. This result holds for $\mathfrak{M}_{\mathrm{cl}}$ and $\mathfrak{M}^{01}_{\mathrm{cl}}$ as well.

Definition 6.1. In a lattice $\mathfrak{L}$, a set $X \subseteq \mathfrak{L}$ is called *strongly meet-incomplete* if for any finite $\{\mathbf{y}_i \mid i \leq n\} \subseteq X$ there is an $\mathbf{x} \in X$ such that $\mathbf{x} \not\geq \mathbf{y}_1 \times \mathbf{y}_2 \times \cdots \times \mathbf{y}_n$.

Lemma 6.2 (Dyment [3]; See also [22])**.** *No countable strongly meet-incomplete $X \subseteq \mathfrak{M}$ has a greatest lower bound.*

The proof of Lemma 6.2 works in $\mathfrak{M}_{\mathrm{cl}}$, and it only requires a slight modification for $\mathfrak{M}^{01}_{\mathrm{cl}}$.

We have shown that if $\mathbf{w}_0$ is as in a code for $\mathfrak{N}_2$ in any of $\mathfrak{M}_{\mathrm{cl}}$, $\mathfrak{M}^{01}_{\mathrm{cl}}$, $\mathfrak{M}_{\mathrm{w,cl}}$, $\mathfrak{M}^{01}_{\mathrm{w,cl}}$, then $\widetilde{E}(\mathbf{w}_0)$ is countable. This observation gives us the following theorem:

Theorem 6.3. *Neither $\mathfrak{M}_{\mathrm{cl}}$ nor $\mathfrak{M}^{01}_{\mathrm{cl}}$ is elementarily equivalent to either $\mathfrak{M}_{\mathrm{w,cl}}$ or $\mathfrak{M}^{01}_{\mathrm{w,cl}}$.*

Proof. Let φ be the first-order sentence that says "for all w_0, if w_0 is as in a code for $\mathfrak{N}_2$, then $\widetilde{E}(w_0)$ has a greatest lower bound." The sentence φ is true in both $\mathfrak{M}_{\mathrm{w,cl}}$ and $\mathfrak{M}^{01}_{\mathrm{w,cl}}$ because such an $\widetilde{E}(\mathbf{w}_0)$ is countable and these lattices are countably meet-complete. On the other hand, φ fails in both $\mathfrak{M}_{\mathrm{cl}}$ and $\mathfrak{M}^{01}_{\mathrm{cl}}$. If $\mathbf{w}_0$ is as in the code for $\mathfrak{N}_2$ produced in either Lemma 5.8 or Lemma 5.10, then $\widetilde{E}(\mathbf{w}_0) = \{[\{f\}]_i \mid i \in \omega\}$ where $\{f_i \mid i \in \omega\}$ is a Turing antichain. It is then easy to check that $\widetilde{E}(\mathbf{w}_0)$ is strongly meet-incomplete and hence has no greatest lower bound. □

Acknowledgements

Many thanks to my advisor Richard Shore for many helpful discussions. Thanks as well to Andrew Lewis, André Nies, and Andrea Sorbi for their gracious acknowledgement of my work on Theorem 3.13 during their presentation of it at CiE 2009. Lastly, thanks to my anonymous reviewers and their helpful suggestions which improved the clarity of this work.

References

1. Caterina Bianchini and Andrea Sorbi, *A note on closed degrees of difficulty of the Medvedev lattice*, Mathematical Logic Quarterly **42** (1996), no. 1, 127–133.
2. Elena Z. Dyment, *Certain properties of the Medvedev lattice*, Mathematics of the USSR Sbornik **30** (1976), 321–340.
3. ———, *Exact bounds of denumerable collections of degrees of difficulty*, Matematicheskie Zametki **28** (1980), no. 6, 899–910.
4. Wilfrid Hodges, *A Shorter Model Theory*, Cambridge University Press, 1997.
5. V. A. Jankov, *The calculus of the weak law of excluded middle*, Mathematics of the USSR Izvestiya **2** (1968), no. 5, 997–1004.
6. Manuel Lerman, *Degrees of Unsolvability*, Springer–Verlag, 1983.
7. Andrew E.M. Lewis, André Nies, and Andrea Sorbi, *The first order theories of the Medvedev and Muchnik lattices*, CiE (Klaus Ambos-Spies, Benedikt Löwe, and Wolfgang Merkle, eds.), Lecture Notes in Computer Science, vol. 5635, Springer, 2009, pp. 324–331.
8. Andrew E.M. Lewis, Richard A. Shore, and Andrea Sorbi, *Topological aspects of the Medvedev lattice*, to appear.
9. Yuri T. Medvedev, *Degrees of difficulty of the mass problems*, Doklady Akademii Nauk SSSR (NS), vol. 104, 1955, pp. 501–504.
10. ———, *Finite problems*, Doklady Akademii Nauk SSSR (NS), vol. 142, 1962, pp. 1015–1018.
11. Albert A. Muchnik, *On strong and weak reducibilities of algorithmic problems*, Sibirskii Matematicheskii Zhurnal **4** (1963), 1328–1341.
12. André Nies, Richard A. Shore, and Theodore A. Slaman, *Interpretability and definability in the recursively enumerable degrees*, Proceedings of the London Mathematical Society **77** (1998), no. 02, 241–291.
13. Paul Shafer, *Characterizing the join-irreducible Medvedev degrees*, to appear in the Notre Dame Journal of Formal Logic.
14. Richard A. Shore, *The theory of the degrees below 0'*, Journal of the London Mathematical Society **24** (1981), 1–14.
15. Stephen G. Simpson, *First-order theory of the degrees of recursive unsolvability*, Annals of Mathematics **105** (1977), no. 1, 121–139.
16. ———, *Subsystems of Second Order Arithmetic*, Cambridge University Press, 2009.
17. Elena Z. Skvortsova, *A faithful interpretation of the intuitionistic propositional calculus by means of an initial segment of the Medvedev lattice*, Sibirskii Matematicheskii Zhurnal **29** (1988), no. 1, 171–178.
18. Robert I. Soare, *Recursively Enumerable Sets and Degrees*, Springer–Verlag, 1987.
19. Andrea Sorbi, *Some remarks on the algebraic structure of the Medvedev lattice*, Journal of Symbolic Logic **55** (1990), no. 2, 831–853.

20. ———, *Embedding Brouwer algebras in the Medvedev lattice*, Notre Dame Journal of Formal Logic **32** (1991), no. 2, 266–275.
21. ———, *Some quotient lattices of the Medvedev lattice*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik **37** (1991), no. 9–12, 167–182.
22. ———, *The Medvedev lattice of degrees of difficulty*, Computability, Enumerability, Unsolvability: Directions in Recursion Theory (S. B. Cooper, T. A. Slaman, and S. S. Wainer, eds.), London Mathematical Society Lecture Notes, vol. 224, Cambridge University Press, 1996, pp. 289–312.
23. Andrea Sorbi and Sebastiaan A. Terwijn, *Intermediate logics and factors of the Medvedev lattice*, Annals of Pure and Applied Logic **155** (2008), no. 2, 69–85.

Department of Mathematics, Malott Hall, Cornell University, Ithaca, NY 14853, USA
E-mail address: pshafer@math.cornell.edu
URL: http://www.math.cornell.edu/~pshafer/