CODING TRUE ARITHMETIC IN THE MEDVEDEV DEGREES OF Π^0_1 CLASSES

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Abstract. Let $\mathcal{E}_\mathrm{s}$ denote the lattice of Medvedev degrees of non-empty Π^0_1 subsets of 2^ω, and let $\mathcal{E}_\mathrm{w}$ denote the lattice of Muchnik degrees of non-empty Π^0_1 subsets of 2^ω. We prove that the first-order theory of $\mathcal{E}_\mathrm{s}$ as a partial order is recursively isomorphic to the first-order theory of true arithmetic. Our coding of arithmetic in $\mathcal{E}_\mathrm{s}$ also shows that the Σ^0_3-theory of $\mathcal{E}_\mathrm{s}$ as a lattice and the Σ^0_4-theory of $\mathcal{E}_\mathrm{s}$ as a partial order are undecidable. Moreover, we show that the degree of $\mathcal{E}_\mathrm{s}$ as a lattice is $\mathbf{0}'''$ in the sense that $\mathbf{0}'''$ computes a presentation of $\mathcal{E}_\mathrm{s}$ and that every presentation of $\mathcal{E}_\mathrm{s}$ computes $\mathbf{0}'''$. Finally, we show that the Σ^0_3-theory of $\mathcal{E}_\mathrm{w}$ as a lattice and the Σ^0_4-theory of $\mathcal{E}_\mathrm{w}$ as a partial order are undecidable.

1. Introduction

1.1. **Mass problems and reducibilities.** A *mass problem* is a set $X \subseteq \omega^\omega$ thought of as an abstract mathematical problem, namely the problem of finding a member of X. Medvedev introduced his notion of reducibility among the mass problems as a formalization of Kolmogorov's idea of a "calculus of problems" [24]. For sets $X, Y \subseteq \omega^\omega$, $X \leq_\mathrm{s} Y$ (read X *Medvedev reduces* or *strongly reduces* to Y) if and only if there is a Turing functional Φ such that $(\forall g \in Y)(\Phi(g) \in X)$. Under the interpretation of subsets of ω^ω as mathematical problems, $X \leq_\mathrm{s} Y$ means that problem Y is at least as hard as problem X in a strongly intuitionistic sense: solutions to Y can be converted to solutions to X by a uniform effective procedure.

Medvedev reducibility induces a degree structure on $\mathcal{P}(\omega^\omega)$ in the same way that Turing reducibility induces a degree structure on ω^ω. For sets $X, Y \subseteq \omega^\omega$, $X \equiv_\mathrm{s} Y$ (read X is *Medvedev equivalent* or *strongly equivalent* to Y) if and only if $X \leq_\mathrm{s} Y$ and $Y \leq_\mathrm{s} X$. $\mathcal{D}_\mathrm{s}$ denotes the *Medvedev degrees*, that is, the set of all $\equiv_\mathrm{s}$-equivalence classes $\deg_\mathrm{s}(X)$ for $X \subseteq \omega^\omega$. The preordering $\leq_\mathrm{s}$ of $\mathcal{P}(\omega^\omega)$ induces a partial ordering of $\mathcal{D}_\mathrm{s}$, also named $\leq_\mathrm{s}$. Muchnik introduced a non-uniform variant of Medvedev reducibility [26]. For sets $X, Y \subseteq \omega^\omega$, $X \leq_\mathrm{w} Y$ (read X *Muchnik reduces* or *weakly reduces* to Y) if and only if $(\forall g \in Y)(\exists f \in X)(f \leq_\mathrm{T} g)$. *Muchnik equivalence* (or *weak equivalence*) $\equiv_\mathrm{w}$ and the *Muchnik degrees* $\mathcal{D}_\mathrm{w}$ are defined analogously to $\equiv_\mathrm{s}$ and $\mathcal{D}_\mathrm{s}$ but with $\leq_\mathrm{w}$ in place of $\leq_\mathrm{s}$.

$\mathcal{D}_\mathrm{s}$ and $\mathcal{D}_\mathrm{w}$ extend the Turing degrees $\mathcal{D}_\mathrm{T}$. The natural maps $\deg_\mathrm{T}(f) \mapsto \deg_\mathrm{s}(\{f\})$ and $\deg_\mathrm{T}(f) \mapsto \deg_\mathrm{w}(\{f\})$ are upper-semilattice embeddings of $\mathcal{D}_\mathrm{T}$ into $\mathcal{D}_\mathrm{s}$ and $\mathcal{D}_\mathrm{w}$ respectively. Moreover, the range of each of these embeddings is definable in the corresponding structure. This fact is due to Dyment for $\mathcal{D}_\mathrm{s}$ ([12] Corollary 2.1), and the proof for $\mathcal{D}_\mathrm{w}$ is simpler (see also [41] Theorem 2.2). $\mathcal{D}_\mathrm{s}$ and $\mathcal{D}_\mathrm{w}$ enjoy a much richer algebraic structure than $\mathcal{D}_\mathrm{T}$ does. Most importantly, $\mathcal{D}_\mathrm{s}$ and $\mathcal{D}_\mathrm{w}$ are both distributive lattices. In fact, $\mathcal{D}_\mathrm{s}$ is a Brouwer algebra, and $\mathcal{D}_\mathrm{w}$ is both a Brouwer algebra and a Heyting algebra. Heyting and Brouwer algebras provide semantics for propositional logic, and the interpretation of $\mathcal{D}_\mathrm{s}$ and $\mathcal{D}_\mathrm{w}$ as semantics for propositional logic was an original motivation for their study. This interpretation continues to drive much of the research in this area. Sorbi's survey [41] is a good introduction to $\mathcal{D}_\mathrm{s}$ and $\mathcal{D}_\mathrm{w}$.

A classic problem in computability theory is to determine the complexity of the first-order theory of a given degree structure, such as $\mathcal{D}_\mathrm{T}$, $\mathcal{D}_\mathrm{s}$, or $\mathcal{D}_\mathrm{w}$. The benchmarks are theories of arithmetic, the comparisons are made via recursive isomorphisms, and the results typically express that the

This research was partially supported by NSF grants DMS-0554855 and DMS-0852811.

first-order theories of the degree structures are as complicated as possible. The original result of this sort, due to Simpson, is that the first-order theory of $\mathcal{D}_{\mathrm{T}}$ is recursively isomorphic to the second-order theory of true arithmetic [35]. Lewis, Nies, and Sorbi and independently the author have determined that the first-order theories of $\mathcal{D}_{\mathrm{s}}$ and $\mathcal{D}_{\mathrm{w}}$ are both recursively isomorphic to the third-order theory of true arithmetic [21, 32].

1.2. **Substructures of $\mathcal{D}_{\mathrm{s}}$ and $\mathcal{D}_{\mathrm{w}}$.** Various substructures arise in the study of degree structures, and the complexities of their first-order theories naturally come into question. In the Turing degrees, two popular substructures are $\mathcal{D}_{\mathrm{T}}(\leq_{\mathrm{T}} \mathbf{0}')$, the Turing degrees below $\mathbf{0}'$, and $\mathcal{R}$, the Turing degrees of r.e. sets. Both $\mathcal{D}_{\mathrm{T}}(\leq_{\mathrm{T}} \mathbf{0}')$ and $\mathcal{R}$ have first-order theories that are recursively isomorphic to the first-order theory of true arithmetic. The $\mathcal{D}_{\mathrm{T}}(\leq_{\mathrm{T}} \mathbf{0}')$ case is due to Shore [33]. The $\mathcal{R}$ case is due to unpublished work of Harrington and Slaman (see also [27]). For degree structures on the mass problems, substructures naturally arise by restricting the family of mass problems under consideration to natural topological classes. For instance, restricting to the degrees of closed subsets of ω^ω yields $\mathcal{D}_{\mathrm{s,cl}}$, the closed Medvedev degrees, and $\mathcal{D}_{\mathrm{w,cl}}$, the closed Muchnik degrees. Restricting to the degrees of compact subsets of ω^ω (or equivalently restricting to closed subsets of 2^ω) yields $\mathcal{D}^{01}_{\mathrm{s},cl}$, the compact Medvedev degrees, and $\mathcal{D}^{01}_{\mathrm{w,cl}}$, the compact Muchnik degrees. $\mathcal{D}_{\mathrm{s,cl}}$ and $\mathcal{D}^{01}_{\mathrm{s},cl}$ are sublattices of $\mathcal{D}_{\mathrm{s}}$, and $\mathcal{D}_{\mathrm{w,cl}}$ and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ are sublattices of $\mathcal{D}_{\mathrm{w}}$. These sublattices have received attention in for example [3, 22, 31, 32, 41]. The author has determined that the four structures $\mathcal{D}_{\mathrm{s,cl}}$, $\mathcal{D}^{01}_{\mathrm{s},cl}$, $\mathcal{D}_{\mathrm{w,cl}}$, and $\mathcal{D}^{01}_{\mathrm{w,cl}}$ all have first-order theories that are recursively isomorphic to the second-order theory of true arithmetic [32].

In this paper, we consider $\mathcal{E}_{\mathrm{s}}$, the sublattice of $\mathcal{D}_{\mathrm{s}}$ consisting of the Medvedev degrees of non-empty Π^0_1 subsets of 2^ω. We also consider $\mathcal{E}_{\mathrm{w}}$, the sublattice of $\mathcal{D}_{\mathrm{w}}$ consisting of the Muchnik degrees of non-empty Π^0_1 subsets of 2^ω, though to a much lesser extent. $\mathcal{E}_{\mathrm{s}}$ and its sister-structure $\mathcal{E}_{\mathrm{w}}$ are the effective counterparts of $\mathcal{D}^{01}_{\mathrm{s},cl}$ and $\mathcal{D}^{01}_{\mathrm{w,cl}}$. They have enjoyed considerable attention from many authors, beginning with Simpson's suggestion to the Foundations of Mathematics discussion group that $\mathcal{E}_{\mathrm{w}}$ is analogous to $\mathcal{R}$ but with more natural examples [36]. This analogy with $\mathcal{R}$ drives much of the research on $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{w}}$. For example, every non-minimum member of $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{w}}$ is join-reducible [4], reflecting Sacks's splitting theorem for $\mathcal{R}$ [28], and $\mathcal{E}_{\mathrm{s}}$ is dense [7], reflecting Sacks's density theorem for $\mathcal{R}$ [30]. The question of whether $\mathcal{E}_{\mathrm{w}}$ is dense remains open. See the recent surveys by Simpson [39] and Hinman [14] for an overview of $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{w}}$.

1.3. **Undecidability in $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{w}}$.** Our main result is that the first-order theory of $\mathcal{E}_{\mathrm{s}}$ is recursively isomorphic to the first-order theory of true arithmetic. This result holds in both the language of lattices and in the language of partial orders because, in any lattice, the lattice operations are arithmetically definable from the partial order. In light of the analogies between $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{R}$, our main result can be seen as a companion to the result that the first-order theory of $\mathcal{R}$ is recursively isomorphic to the first-order theory of true arithmetic. We are able to prove that the first-order theory of $\mathcal{E}_{\mathrm{w}}$ is undecidable, but, beyond that result, the question of the exact complexity of the first-order theory of $\mathcal{E}_{\mathrm{w}}$ remains wide open. Cole and Simpson conjecture that the first-order theory of $\mathcal{E}_{\mathrm{w}}$ is recursively isomorphic to $\mathcal{O}^{(\omega)}$ (the ω^{th} Turing jump of Kleene's $\mathcal{O}$), the obvious upper bound [11].

We also consider the decidability of fragments of the first-order theories of $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{w}}$. Here we need to be careful to specify whether we are working in the language of lattices or in the language of partial orders. Binns has shown that the Σ^0_1-theories of $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{w}}$ as lattices are identical and decidable [4]. Cole and Kihara have shown that the Σ^0_2-theory of $\mathcal{E}_{\mathrm{s}}$ as a partial order is decidable [10]. The corresponding result for $\mathcal{E}_{\mathrm{w}}$ is not known. The decidability of the Σ^0_2-theories of $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{w}}$ as a lattices and the Σ^0_3-theories of $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{w}}$ as partial orders are not known. Our method of coding arithmetic in distributive lattices proves that the Σ^0_3-theories of $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{w}}$ as lattices and the Σ^0_4-theories of $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{w}}$ as a partial orders are all undecidable.

There has been a huge amount of difficult work on the decidability of various fragments of the first-order theories of $\mathcal{D}_{\mathrm{T}}$ and $\mathcal{R}$. We summarize the results for $\mathcal{R}$ for comparison (see [34] for a survey of this area). The Σ^0_1-theory of $\mathcal{R}$ as an upper-semilattice is decidable [29]. The decidability of the Σ^0_2-theory of $\mathcal{R}$ as either a partial order or an upper semi-lattice is unknown. However, the Σ^0_3-theory of $\mathcal{R}$ as a partial order is undecidable [19]. Moreover, if one extends the partial infimum function on $\mathcal{R}$ (as an upper-semilattice) to any total function, then the Σ^0_2-theory of the resulting structure is undecidable [25]. These two undecidability results for $\mathcal{R}$ suggest by analogy that the Σ^0_2-theories of $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{w}}$ as lattices and the Σ^0_3-theories of $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{w}}$ as partial orders may all be undecidable. The following table summarizes the current state of knowledge concerning the decidability of various fragments of the first-order theories of $\mathcal{R}$, $\mathcal{E}_{\mathrm{s}}$, and $\mathcal{E}_{\mathrm{w}}$.

	$\boldsymbol{\Sigma^0_1}$	$\boldsymbol{\Sigma^0_2}$	$\boldsymbol{\Sigma^0_3}$	$\boldsymbol{\Sigma^0_4}$
$\mathcal{R}$ as a partial order	decidable	?	undecidable	undecidable
$\mathcal{R}$ as an upper-semilattice	decidable	?	undecidable	undecidable
$\mathcal{E}_{\mathrm{s}}$ as a partial order	decidable	decidable	?	undecidable
$\mathcal{E}_{\mathrm{s}}$ as a lattice	decidable	?	undecidable	undecidable
$\mathcal{E}_{\mathrm{w}}$ as a partial order	decidable	?	?	undecidable
$\mathcal{E}_{\mathrm{w}}$ as a lattice	decidable	?	undecidable	undecidable

We also prove that $\mathcal{E}_{\mathrm{s}}$ is as complicated as possible in terms of degree of presentation. Specifically, we prove that the degree of $\mathcal{E}_{\mathrm{s}}$ as a lattice is $\mathbf{0}'''$. This means that $\mathbf{0}'''$ computes a presentation of $\mathcal{E}_{\mathrm{s}}$ as a lattice and that $\mathbf{0}'''$ is computable in every presentation of $\mathcal{E}_{\mathrm{s}}$ as a lattice. A corollary is that $\mathcal{E}_{\mathrm{s}}$ has no recursive presentation as a partial order. The natural presentation of $\mathcal{E}_{\mathrm{w}}$ has Turing degree $\mathcal{O}$ [11], so it is reasonable to expect that $\mathcal{E}_{\mathrm{w}}$ has degree $\mathcal{O}$, though this question remains open. For comparison, it follows from the results of [27] (though it is not stated explicitly) that the degree of $\mathcal{R}$ as an upper-semilattice is $\mathbf{0}^{(4)}$.

This paper is organized as follows. Section 2 provides the necessary background material. Section 3 presents our scheme for coding arithmetic in distributive lattices. Section 4 presents the theory of meet-irreducibles in $\mathcal{E}_{\mathrm{s}}$ necessary to implement our coding in $\mathcal{E}_{\mathrm{s}}$. Section 5 implements our coding in $\mathcal{E}_{\mathrm{s}}$, thereby proving our results concerning the complexity of the first-order theory of $\mathcal{E}_{\mathrm{s}}$. Section 6 proves that the degree of $\mathcal{E}_{\mathrm{s}}$ as a lattice is $\mathbf{0}'''$. Section 7 proves our undecidability results concerning the first-order theory of $\mathcal{E}_{\mathrm{w}}$.

2. Background

Here we present the relevant background concerning classical computability theory, distributive lattices, Π^0_1 classes and their Medvedev and Muchnik degrees, and arithmetic. Much of the notation should be familiar from the standard sources, such as [20] and [40]. Unfortunately, notation for the Medvedev degrees is far from standardized. We follow [39] in the hope that its notation will become standard.

2.1. **Computability theory.** Let $n \in \omega$, $\sigma, \tau \in \omega^{<\omega}$, $f, g \in \omega^\omega$, and $X, Y \subseteq \omega^\omega$. Then

- $f \upharpoonright n$ is the initial segment of f of length n,
- $|\sigma|$ is the length of σ,
- $\sigma \subseteq \tau$ means that σ is an initial segment of τ,
- $\sigma \subset f$ means that σ is an initial segment of f,
- $\sigma^\frown f$ is the concatenation of σ and f:

$$(\sigma^\frown f)(n) = \begin{cases} \sigma(n) & \text{if } n < |\sigma| \\ f(n - |\sigma|) & \text{if } n \geq |\sigma|, \end{cases}$$

  with $n^\frown f$ abbreviating $(n)^\frown f$ for sequences (n) of length 1,

- $f \oplus g$ is the function defined by

$$(f \oplus g)(n) = \begin{cases} f(m) & \text{if } n = 2m \\ g(m) & \text{if } n = 2m+1, \end{cases}$$

- $\sigma^\frown X = \{\sigma^\frown f \mid f \in X\}$, and
- $X \otimes Y = \{f \oplus g \mid f \in X \wedge g \in Y\}$.

The function $\langle \cdot, \cdot \rangle \colon \omega \times \omega \to \omega$ is a fixed recursive bijection. Φ_e denotes the e^{th} Turing functional. Φ always denotes a Turing functional, and if $f \in \omega^\omega$, then $\Phi(f)$ is the partial function computed when Φ uses f as its oracle. For $\sigma \in \omega^{<\omega}$, $\Phi(\sigma)$ is the partial function that, on input $n \in \omega$, is computed by running Φ on input n for at most $|\sigma|$ steps and using σ to answer oracle queries. The restriction on the running time of $\Phi(\sigma)$ ensures that oracle queries are only made of numbers $< |\sigma|$. Consequently, if $\Phi(\sigma)(n)\downarrow$, then $\Phi(f)(n) = \Phi(\sigma)(n)$ for all $f \supset \sigma$.

Let $A, B \subseteq \omega$. $A \leq_1 B$ if and only if there is a one-to-one recursive function f such that $\forall n(n \in A \leftrightarrow f(n) \in B)$. A and B are *recursively isomorphic* if and only if there is such an f that is a bijection. The Myhill isomorphism theorem states that A and B are recursively isomorphic if and only if $A \equiv_1 B$, that is, if and only if $A \leq_1 B$ and $B \leq_1 A$ (see [40] Section I.5).

2.2. **Distributive lattices.** The usual options for lattice notation conflict with either the logical notation ($\vee$ and $\wedge$) or the arithmetic notation ($+$ and $\times$). To avoid this conflict, we follow [39] and write sup for join and inf for meet.

A lattice $\mathcal{L}$ is *distributive* if and only if sup and inf distribute over each other:

- $(\forall x, y, z \in \mathcal{L})(\sup(x, \inf(y, z)) = \inf(\sup(x, y), \sup(x, z))$, and
- $(\forall x, y, z \in \mathcal{L})(\inf(x, \sup(y, z)) = \sup(\inf(x, y), \inf(x, z))$.

An element x of a lattice $\mathcal{L}$ is *meet-reducible* if and only if $(\exists y, z \in \mathcal{L})((y > x) \wedge (z > x) \wedge (x = \inf(y, z)))$. Otherwise x is *meet-irreducible.* We frequently use the following well-known characterization without mention:

Lemma 2.1 (see [2] Section III.2)**.** *If $\mathcal{L}$ is a distributive lattice, then $x \in \mathcal{L}$ is meet-irreducible if and only if $(\forall y, z \in \mathcal{L})(x \geq \inf(y, z) \to x \geq y \vee x \geq z)$.*

Proof. Suppose x is meet-irreducible and $x \geq \inf(y, z)$. Then

$$x = \sup(x, \inf(y, z)) = \inf(\sup(x, y), \sup(x, z)).$$

Thus $x = \sup(x, y)$ or $x = \sup(x, z)$, which means $x \geq y$ or $x \geq z$. Conversely, if x is meet-reducible, then by definition there are $y, z > x$ with $x = \inf(y, z)$. □

Dualizing gives the definitions of join-reducible and join-irreducible, and it gives a characterization of join-irreducible in distributive lattices.

Sometimes we want to ignore the lattice operations of a lattice $\mathcal{L}$ and consider $\mathcal{L}$ as a partial order. When we do, we write $(\mathcal{L}; \leq_{\mathcal{L}})$ to indicate that we are considering only the partial order structure on $\mathcal{L}$. In particular, $\mathrm{Th}(\mathcal{L})$ denotes the first-order theory of $\mathcal{L}$ as a lattice, and $\mathrm{Th}(\mathcal{L}; \leq_{\mathcal{L}})$ denotes the first-order theory of $\mathcal{L}$ as a partial order.

2.3. **Π^0_1 classes and their Medvedev and Muchnik degrees.** The Π^0_1 *classes* are the Π^0_1 subsets of ω^ω, where a set $X \subseteq \omega^\omega$ is Π^0_1 if and only if it is of the form $X = \{f \in \omega^\omega \mid \forall n \varphi(f, n)\}$ for some recursive predicate φ. The Π^0_1 classes have been persistent objects of study throughout computability theory, due in no small part to their applications to recursive mathematics and reverse mathematics. The surveys by Cenzer [6] and by Cenzer and Remmel [8] provide an extensive overview of the theory of the Π^0_1 classes, as does the forthcoming book by Cenzer and Remmel [9].

A useful characterization of the Π^0_1 classes is as the sets of paths through recursive trees. A *tree* is a set $T \subseteq \omega^{<\omega}$ closed under initial segments: $(\forall \sigma, \tau \in \omega^{<\omega})(\sigma \in T \wedge \tau \subseteq \sigma \to \tau \in T)$. A function $f \in \omega^\omega$ is a *path* through T if and only if $(\forall n \in \omega)(f \upharpoonright n \in T)$. If T is a tree, then $[T]$ denotes the set of all paths through T. A set $X \subseteq \omega^\omega$ is a Π^0_1 class if and only if $X = [T]$ for some recursive tree T (see [8] Lemma 2.2). Recall the usual product topology on ω^ω. Basic open sets are of the form $I(\sigma) = \{f \in \omega^\omega \mid \sigma \subset f\}$ for $\sigma \in \omega^{<\omega}$. A set $X \subseteq \omega^\omega$ is closed in this topology if and only if $X = [T]$ for some (not necessarily recursive) tree T. For this reason, the Π^0_1 classes are sometimes called the effectively closed sets.

For sets $X, Y \subseteq \omega^\omega$, $X \leq_{\mathrm{s}} Y$ if and only if there is a Turing functional Φ such that $(\forall g \in Y)(\Phi(g) \in X)$, a condition which we abbreviate by $\Phi(Y) \subseteq X$. Similarly, $X \leq_{\mathrm{w}} Y$ if and only if $(\forall g \in Y)(\exists f \in X)(f \leq_{\mathrm{T}} g)$. We consider $\leq_{\mathrm{s}}$ and $\leq_{\mathrm{w}}$ restricted to non-empty Π^0_1 subsets of 2^ω. Henceforth the term "Π^0_1 class" refers exclusively to non-empty Π^0_1 subsets of 2^ω, and all trees are subsets of $2^{<\omega}$. Every Π^0_1 class is a closed subset of the compact space 2^ω and is therefore compact. The compactness of the Π^0_1 classes is crucial to many of our arguments. As a first example, compactness allows us to express $\leq_{\mathrm{s}}$ arithmetically.

Lemma 2.2. *$[T_0] \leq_{\mathrm{s}} [T_1]$ is Σ^0_3 relative to the trees T_0 and T_1.*

Proof. For a given Turing functional Φ, we show that

$$\Phi([T_1]) \subseteq [T_0] \text{ if and only if } (\forall n \in \omega)(\exists s \in \omega)(\forall \sigma \in 2^s)(\sigma \in T_1 \to \Phi(\sigma) \restriction n \in T_0),$$

where $\Phi(\sigma) \restriction n \in T_0$ includes the provision that $(\forall i < n)(\Phi(\sigma)(i)\downarrow)$. It then follows that

$$[T_0] \leq_{\mathrm{s}} [T_1] \text{ if and only if } (\exists e \in \omega)(\forall n \in \omega)(\exists s \in \omega)(\forall \sigma \in 2^s)(\sigma \in T_1 \to \Phi_e(\sigma) \restriction n \in T_0),$$

which gives our Σ^0_3 definition of $\leq_{\mathrm{s}}$.

For the forward direction, let $n \in \omega$ be given. Let $\Sigma = \{\sigma \in 2^{<\omega} \mid \Phi(\sigma) \restriction n \in T_0\}$. The condition $\Phi([T_1]) \subseteq [T_0]$ implies that $[T_1] \subseteq \bigcup_{\sigma \in \Sigma} I(\sigma)$. By compactness, there is a finite $\Sigma_0 \subseteq \Sigma$ such that $[T_1] \subseteq \bigcup_{\sigma \in \Sigma_0} I(\sigma)$ and an $s \in \omega$ such that $(\forall \sigma \in 2^s)(\sigma \in T_1 \to (\exists \sigma_0 \in \Sigma_0)(\sigma_0 \subseteq \sigma))$. Then $(\forall \sigma \in 2^s)(\sigma \in T_1 \to \Phi(\sigma) \restriction n \in T_0)$.

For the reverse direction, consider $f \in [T_1]$. Given any $n \in \omega$, let $s \in \omega$ be such that $(\forall \sigma \in 2^s)(\sigma \in T_1 \to \Phi(\sigma) \restriction n \in T_0)$. Then $\Phi(f \restriction s) \restriction n \in T_0$, so $\Phi(f) \restriction n \in T_0$. Thus $\forall n(\Phi(f) \restriction n \in T_0)$. Hence $\Phi(f) \in [T_0]$, and therefore $\Phi([T_1]) \subseteq [T_0]$. ☐

Let $\mathcal{E}_{\mathrm{s}} = \{\deg_{\mathrm{s}}(X) \mid X \text{ is a } \Pi^0_1 \text{ class}\}$. $\mathcal{E}_{\mathrm{s}}$ is partially ordered by $\leq_{\mathrm{s}}$. If X and Y are both Π^0_1 classes, then so are $X \otimes Y$ and $0^\frown X \cup 1^\frown Y$. Given trees T_0 and T_1, let $T_0 \otimes T_1 = \{\sigma \oplus \tau \mid \sigma \in T_0 \wedge \tau \in T_1 \wedge |\tau| \leq |\sigma| \leq |\tau| + 1\}$. Then $T_0 \otimes T_1$ and $0^\frown T_0 \cup 1^\frown T_1$ are both trees, $[T_0] \otimes [T_1] = [T_0 \otimes T_1]$, and $0^\frown [T_0] \cup 1^\frown [T_1] = [0^\frown T_0 \cup 1^\frown T_1]$. Define $\sup(\deg_{\mathrm{s}}(X), \deg_{\mathrm{s}}(Y)) = \deg_{\mathrm{s}}(X \otimes Y)$ and $\inf(\deg_{\mathrm{s}}(X), \deg_{\mathrm{s}}(Y)) = \deg_{\mathrm{s}}(0^\frown X \cup 1^\frown Y)$. One readily checks that $\sup(\deg_{\mathrm{s}}(X), \deg_{\mathrm{s}}(Y))$ and $\inf(\deg_{\mathrm{s}}(X), \deg_{\mathrm{s}}(Y))$ are, respectively, the $\leq_{\mathrm{s}}$-least upper bound and $\leq_{\mathrm{s}}$-greatest lower bound of $\deg_{\mathrm{s}}(X)$ and $\deg_{\mathrm{s}}(Y)$ and that sup and inf distribute over each other. Thus $\mathcal{E}_{\mathrm{s}}$ is a distributive lattice. $\mathcal{E}_{\mathrm{s}}$ has a least element $\mathbf{0}_{\mathrm{s}} = \deg_{\mathrm{s}}(2^\omega)$, and a Π^0_1 class has least degree if and only if it has a recursive member. $\mathcal{E}_{\mathrm{s}}$ also has a greatest element $\mathbf{1}_{\mathrm{s}}$ (see [37] Lemma 3.20). Two examples of Π^0_1 classes with greatest degree are $\mathrm{DNR}_2 = \{f \in 2^\omega \mid \forall e(f(e) \neq \Phi_e(e))\}$ (DNR stands for *diagonally non-recursive*) and the class of all (appropriately Gödel numbered) complete consistent extensions of Peano arithmetic.

Let $\mathcal{E}_{\mathrm{w}} = \{\deg_{\mathrm{w}}(X) \mid X \text{ is a } \Pi^0_1 \text{ class}\}$. $\mathcal{E}_{\mathrm{w}}$ is a distributive lattice with order $\leq_{\mathrm{w}}$ and with joins and meets computed as in $\mathcal{E}_{\mathrm{s}}$. Notice, however, that $0^\frown X \cup 1^\frown Y \equiv_{\mathrm{w}} X \cup Y$, which is not in general true with $\equiv_{\mathrm{s}}$ in place of $\equiv_{\mathrm{w}}$. $\mathcal{E}_{\mathrm{w}}$ has a least element $\mathbf{0}_{\mathrm{w}}$, a greatest element $\mathbf{1}_{\mathrm{w}}$, and the above examples of Π^0_1 classes with least or greatest Medvedev degree also have least or greatest Muchnik degree. However, it is not the case that every Π^0_1 class with greatest Muchnik degree also has greatest Medvedev degree.

A sequence of trees $\{T_n\}_{n \in \omega}$ is *uniformly recursive* if and only if the set $\{\langle n, \sigma \rangle \mid \sigma \in T_n\}$ is recursive. A *recursive sequence of* Π^0_1 *classes* is a sequence of Π^0_1 classes $\{X_n\}_{n \in \omega}$ for which there is a uniformly recursive sequence of trees $\{T_n\}_{n \in \omega}$ such that $X_n = [T_n]$ for each $n \in \omega$. For convenience, we also allow indexing over recursive sets A and consider recursive sequences of Π^0_1

classes of the form $\{X_n\}_{n\in A}$. Though not strictly necessary for our results, a convenient fact is that there is a recursive sequence all Π^0_1 classes (with many repetitions).

Lemma 2.3 (see [9] Chapter XV and [8] Section 2.7)**.** *There is a uniformly recursive sequence of infinite trees $\{T_e\}_{e\in\omega}$ such that for every Π^0_1 class X there is an $e \in \omega$ such that $X = [T_e]$.*

Proof. In fact, [8] Lemma 2.2 proves that every Π^0_1 class is of the form $[T]$ for a primitive recursive tree. Let $\{P_e\}_{e\in\omega}$ be a recursive sequence of all primitive recursive functions. Then define T'_e to be the tree $T'_e = \{\sigma \in 2^{<\omega} \mid (\forall\tau \subseteq \sigma)(P_e(\tau) = 1)\}$. If P_e is the characteristic function of a tree, then T'_e is that tree. Thus if X is a Π^0_1 class, then $X = [T'_e]$ for some $e \in \omega$. We just need to make a final adjustment to ensure that every tree in the sequence is infinite. To this end, let

$$T_e = \{\sigma \in 2^{<\omega} \mid \sigma \in T'_e \vee (\forall m \leq |\sigma|)(\sigma \restriction m \notin T'_e \to (\forall\tau \in 2^m)(\tau \notin T'_e))\}.$$

If T'_e is infinite, then $T_e = T'_e$. Otherwise, T_e consists of T'_e along with all strings that extend a string in T'_e of maximum length.

□

2.4. **Arithmetic.** In Section 3, we code structures that model PA^- (Peano arithmetic without induction) in distributive lattices. For reference, we present the axioms of PA^- as they appear in [18].

Definition 2.4 (see [18] Section 2.1)**.** PA^- is the theory axiomatized by the following sentences.

(i) $\forall x, y, z((x + y) + z = x + (y + z))$
(ii) $\forall x, y(x + y = y + x)$
(iii) $\forall x, y, z((x \times y) \times z = x \times (y \times z))$
(iv) $\forall x, y(x \times y = y \times x)$
(v) $\forall x, y, z(x \times (y + z) = (x \times y) + (x \times z))$
(vi) $\forall x(x + 0 = x \wedge x \times 0 = 0)$
(vii) $\forall x(x \times 1 = x)$
(viii) $\forall x, y, z(x < y \wedge y < z \to x < z)$
(ix) $\forall x\neg(x < x)$
(x) $\forall x, y(x < y \vee x = y \vee y < x)$
(xi) $\forall x, y, z(x < y \to x + z < y + z)$
(xii) $\forall x, y, z(0 < z \wedge x < y \to x \times z < y \times z)$
(xiii) $\forall x, y(x < y \to \exists z(x + z = y))$
(xiv) $0 < 1 \wedge \forall x(0 < x \to x = 1 \vee 1 < x)$
(xv) $\forall x(x = 0 \vee 0 < x)$

To reduce the quantifier complexity of axiom (xiii) for when we analyze the fragments of $\mathrm{Th}(\mathcal{E}_\mathrm{s})$, we introduce the *monus* symbol "$\dot{-}$" and Skolemize. We call the resulting theory $\mathrm{PA}^{\dot{-}}$.

Definition 2.5. $\mathrm{PA}^{\dot{-}}$ is the theory whose axioms are the same as PA^- but with axiom (xiii) replaced by the axiom $\forall x, y(x < y \to x + (y \dot{-} x) = y)$.

The standard relational model of arithmetic is the structure $\mathcal{N} = (\omega; <, +, \times, 0, 1)$, where $<$ is a 2-ary relation on ω, $+$ and $\times$ are 3-ary relations on ω, and 0 and 1 are constants in ω interpreted as the usual less-than, plus, times, zero, and one respectively. $\mathrm{Th}(\mathcal{N})$ denotes the first-order theory of $\mathcal{N}$. We use the relational versions of $+$ and $\times$ instead of the usual functional versions because our coding techniques most naturally code relations. Any formula in which $+$ and $\times$ are relation symbols can be trivially translated into an equivalent formula in which $+$ and $\times$ are function symbols. Translations in the other direction require *unnesting*. In general, a formula is said to be *unnested* if and only if every atomic subformula is of the form $x = y$, $c = y$, $f(x_1, \ldots, x_n) = y$, or $R(x_1, \ldots, x_n)$, where x, y, and the x_i for $i \leq n$ are variables, c is a constant symbol, f is a function symbol, and R is a relation symbol. Every formula can be recursively translated into an equivalent unnested formula (see [15] section 2.6). When unnesting is applied to a first-order formula in the functional language of arithmetic, we get an equivalent formula in which every atomic subformula is of the form $x = y$, $0 = y$, $1 = y$, $x < y$, $x + y = z$, or $x \times y = z$. That is, we get an equivalent formula in the relational language of arithmetic. Therefore the relational and functional versions of $\mathrm{Th}(\mathcal{N})$ are recursively isomorphic.

We also make use of the structure $\mathcal{N}^{\dot{-}} = (\omega; <, +, \times, \dot{-}, 0, 1)$, where $<$, $+$, $\times$, 0, and 1 are as for $\mathcal{N}$, and $\dot{-}$ is the 3-ary relation on ω corresponding to the function

$$x \dot{-} y = \begin{cases} x - y & \text{if } x \geq y \\ 0 & \text{if } x < y. \end{cases}$$

Clearly, $\mathcal{N} \models \mathrm{PA}^-$, $\mathcal{N}^{\dot{-}} \models \mathrm{PA}^{\dot{-}}$, and $\mathrm{PA}^{\dot{-}} \vdash \mathrm{PA}^-$.

Let $\mathcal{M} \models \mathrm{PA}^-$. An *initial segment* of $\mathcal{M}$ is a $<$-downward-closed substructure $\mathcal{M}'$ of $\mathcal{M}$: $(\forall x \in \mathcal{M}')(\forall y \in \mathcal{M})((\mathcal{M} \models y < x) \rightarrow (y \in \mathcal{M}'))$. An *initial interval* of $\mathcal{M}$ is a subset of $\mathcal{M}$ of the form $\{y \in \mathcal{M} \mid \mathcal{M} \models y < x \vee y = x\}$ for some $x \in \mathcal{M}$. The following fact ensures that our coding in the next section correctly codes structures isomorphic to $\mathcal{N}$.

Lemma 2.6 (see [18] Theorem 2.2)**.** *If $\mathcal{M} \models \mathrm{PA}^-$, then there is an initial segment of $\mathcal{M}$ that is isomorphic to $\mathcal{N}$. In particular, $\mathcal{N}$ is the unique model of* PA^-*, up to isomorphism, in which every initial interval is finite.*

For the undecidability of Σ^0_3-$\mathrm{Th}(\mathcal{E}_\mathrm{s})$, we also need the following fact.

Lemma 2.7 (see [18] Corollary 2.9)**.** *If φ is a Σ^0_1 sentence and $\mathcal{N} \models \varphi$, then* $\mathrm{PA}^- \vdash \varphi$.

3. Coding arithmetic in distributive lattices

We present our scheme for coding arithmetic in distributive lattices. Although the definitions below make sense in any lattice, they were designed with the particular goal of coding $\mathcal{N}$ into $\mathcal{D}_\mathrm{s}$, $\mathcal{D}_\mathrm{w}$, and their sublattices in mind. For example, meet-irreducible elements play a major role in the coding. One may dualize the coding to replace meet-irreducible by join-irreducible, but this would not suffice for our purposes because all non-zero elements of $\mathcal{E}_\mathrm{s}$ are join-reducible [4]. The coding presented here has been slightly modified from the original version developed in [32] in order to reduce the quantifier complexity of coded relations.

3.1. **Coding relations.**

Definition 3.1. For elements s and w of a lattice, *s meets to w* if and only if $\exists y(y > w \wedge \inf(s, y) = w)$.

Definition 3.2. For a lattice $\mathcal{L}$ and a $w \in \mathcal{L}$,

$$E(w) = \{s \in \mathcal{L} \mid s \text{ is meet-irreducible} \wedge s \text{ meets to } w\}.$$

The next two lemmas prove important properties of E in distributive lattices.

Lemma 3.3. *If $\mathcal{L}$ is a distributive lattice, then $E(w)$ is an antichain for every $w \in \mathcal{L}$.*

Proof. Suppose for a contradiction that there are $s, s' \in E(w)$ with $s > s'$. Let $y > w$ be such that $\inf(s, y) = w$. Then $s' \geq y$ because s' is meet-irreducible, $s' \geq \inf(s, y)$, and $s' \not\geq s$. Therefore $s > s' \geq y > w$, giving the contradiction $\inf(s, y) = y > w$. □

Lemma 3.4. *If $\mathcal{L}$ is a distributive lattice and $\{s_i\}_{i<n} \subseteq \mathcal{L}$ is a finite antichain of meet-irreducible elements, then $E(\inf_{i<n} s_i) = \{s_i\}_{i<n}$.*

Proof. Let $w = \inf_{i<n} s_i$. First we show that $s_i \in E(w)$ for each $i < n$. Fix $i < n$ and let $t_i = \inf\{s_j \mid j < n \wedge j \neq i\}$. Clearly $t_i \geq w$ and $\inf(s_i, t_i) = w$. Moreover, $s_i \ngeq t_i$ because otherwise the meet-irreducibility of s_i implies that $s_i \geq s_j$ for some $j \neq i$, contradicting that $\{s_i\}_{i<n}$ is an antichain. Thus in fact $t_i > w$, so t_i witnesses that s_i meets to w. Hence $s_i \in E(w)$. Conversely, if $x \in E(w)$, then x is meet-irreducible and $x \geq w$. Thus $x \geq s_i$ for some $i < n$, so $x = s_i$ because $E(w)$ is an antichain by Lemma 3.3. Thus $E(w) = \{s_i\}_{i<n}$. □

Given an element w of a lattice, we think of w as code for the set $E(w)$. The symbol "E" stands for "elements," as in the elements of the set coded by w.[1]

Now we code 2-ary and 3-ary relations on $E(w_0)$ for an element w_0 of a lattice $\mathcal{L}$. The same scheme can code n-ary relations for any $n \in \omega$, but we only need to code 2-ary and 3-ary relations to code $\mathcal{N}$. The intuition behind the following definition is that if $s_0, u_0 \in E(w_0)$, then $\sup(s_0, u_0)$ should code the pair (s_0, u_0). However, this coding makes the pairs (s_0, u_0) and (u_0, s_0) indistinguishable because $\sup(s_0, u_0) = \sup(u_0, s_0)$. To solve this problem, we fix additional parameters $w_1, w_2, m \in \mathcal{L}$. Once $w_0, w_1, w_2, m \in \mathcal{L}$ are fixed, any $c \in \mathcal{L}$ can be interpreted as coding a 2-ary relation R^2_c on $E(w_0)$ and a 3-ary relation R^3_c on $E(w_0)$.

Definition 3.5. Let $\mathcal{L}$ be a lattice and fix elements $w_0, w_1, w_2, m \in \mathcal{L}$. Then any $c \in \mathcal{L}$ defines a 2-ary relation R^2_c on $E(w_0)$ and a 3-ary relation R^3_c on $E(w_0)$ by

$$\begin{aligned} R^2_c(s_0, u_0) \text{ if and only if } & s_0 \in E(w_0) \wedge u_0 \in E(w_0) \\ & \wedge \exists u_1(u_1 \text{ meets to } w_1 \wedge \sup(u_0, u_1) \geq m \wedge \sup(s_0, u_1) \geq c) \\ R^3_c(s_0, u_0, v_0) \text{ if and only if } & s_0 \in E(w_0) \wedge u_0 \in E(w_0) \wedge v_0 \in E(w_0) \\ & \wedge \exists u_1 \exists v_2(u_1 \text{ meets to } w_1 \wedge v_2 \text{ meets to } w_2 \\ & \qquad \wedge \sup(u_0, u_1) \geq m \wedge \sup(v_0, v_2) \geq m \wedge \sup(s_0, u_1, v_2) \geq c). \end{aligned}$$

3.2. **Coding arithmetic.** With Definition 3.5 in hand, we can define codes for models of various theories. For PA^- we have the following definitions.

Definition 3.6. In a lattice $\mathcal{L}$, a *code (for a structure in the language of arithmetic)* is a sequence of elements

$$\vec{w} = (w_0, w_1, w_2, m, \ell, p, t, z, o)$$

from $\mathcal{L}$ interpreted as coding the structure

$$\mathcal{M}_{\vec{w}} = (E(w_0); R^2_\ell, R^3_p, R^3_t, z, o)$$

where R^2_ℓ, R^3_p, and R^3_t are the relations on $E(w_0)$ defined from ℓ, p, and t, respectively, as in Definition 3.5.

In Definition 3.6, w is for "ω," m is for "match," ℓ is for "less," p is for "plus," t is for "times," z is for "zero," and o is for "one."

If $\vec{w}$ is a code in a lattice $\mathcal{L}$, then sentences in the language of arithmetic are interpreted in $\mathcal{M}_{\vec{w}}$ in the obvious way.

Definition 3.7. Let φ be a sentence in the language of arithmetic. The *translation* of φ is the formula $\varphi'(w_0, w_1, w_2, m, \ell, p, t, z, o)$ (with the displayed variables free) in the language of lattices obtained from φ by making the following replacements.

- Replace $<$ by the formula defining R^2_ℓ,
- replace $+$ by the formula defining R^3_p,
- replace $\times$ by the formula defining R^3_t,

[1]In [32], $E(w)$ was called $\widetilde{E}(w)$ (see [32] Definition 4.4) and its definition required that the $s \in \widetilde{E}(w)$ also be minimal with respect being meet-irreducible and meeting to w. The minimality requirement is unnecessary by Lemma 3.3.

- replace 0 by z,
- replace 1 by o,
- replace $\exists x$ by the formula expressing $\exists x \in E(w_0)$, and
- replace $\forall x$ by the formula expressing $\forall x \in E(w_0)$.

If $\vec{w}$ is a code in a lattice $\mathcal{L}$, then $\mathcal{M}_{\vec{w}} \models \varphi$ means that $\mathcal{L} \models \varphi'(\vec{w})$.

Definition 3.8. In a lattice $\mathcal{L}$, a *code for a model of* PA^- is a code $\vec{w}$ such that $\mathcal{M}_{\vec{w}} \models \mathrm{PA}^-$.

If φ is a first-order sentence in the language of arithmetic, then its translation φ' is a first-order formula in the language of lattices. Thus for such a sentence φ, the property "$\vec{w}$ is a code such that $\mathcal{M}_{\vec{w}} \models \varphi$" is first-order. The property "$\vec{w}$ is a code for a model of PA^-" is therefore expressible by a first-order formula in the language of lattices.

To code $\mathcal{N}$, we add extra conditions to Definition 3.8 ensuring that the coded structure is isomorphic to $\mathcal{N}$. Ultimately, these extra conditions express that every initial interval of the coded structure is finite, which suffices by Lemma 2.6. The following definitions allows us to compare the cardinalities of initial intervals of coded models of PA^-.

Definition 3.9. Let $\mathcal{L}$ be a lattice and let $\vec{w}$ be a code for a model of PA^-. An $a \in \mathcal{L}$ *codes an initial interval of* $\mathcal{M}_{\vec{w}}$ if and only if $(\exists s \in E(w_0))(\forall b \in \mathcal{L})(b \in E(a) \leftrightarrow R^2_\ell(b,s) \vee b = s)$.

Definition 3.10. For a lattice $\mathcal{L}$ and elements $r, q \in \mathcal{L}$, $E(r)$ *matches* $E(q)$ if and only if there is a $z \in \mathcal{L}$ such that

(i) $(\forall x \in E(q))(\exists! y \in E(r))(\sup(x,y) \in E(z))$, and
(ii) $(\forall x \in E(r))(\exists! y \in E(q))(\sup(x,y) \in E(z))$.

Clearly if $E(r)$ matches $E(q)$, then $|E(r)| = |E(q)|$. The next definition enforces a weak converse of this fact.

Definition 3.11. A lattice $\mathcal{L}$ has the *finite matching property* if and only if whenever $q, q' \in \mathcal{L}$ are such that $|E(q)| = |E(q')| = n$ for some $n \in \omega$ then there is an $r \in \mathcal{L}$ such that $E(r)$ matches both $E(q)$ and $E(q')$.

We can now define a code for $\mathcal{N}$ in a lattice $\mathcal{L}$ and prove that codes for $\mathcal{N}$ always code structures isomorphic to $\mathcal{N}$ provided that $\mathcal{L}$ is distributive, that $\mathcal{L}$ has the finite matching property, and that some code in $\mathcal{L}$ codes a structure isomorphic to $\mathcal{N}$. It follows that $\mathrm{Th}(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{L})$.

Definition 3.12. In a lattice $\mathcal{L}$, a *code for* $\mathcal{N}$ is a code $\vec{w}$ such that

(i) $\vec{w}$ is a code for a model of PA^-,
(ii) $(\forall s \in E(w_0))(\exists a \in \mathcal{L})(\forall b \in \mathcal{L})(b \in E(a) \leftrightarrow R^2_\ell(b,s) \vee b = s)$ (that is, every initial interval of $\mathcal{M}_{\vec{w}}$ is coded by some $a \in \mathcal{L}$), and
(iii) For every $a \in \mathcal{L}$ that codes an initial interval of $\mathcal{M}_{\vec{w}}$ and every code $\vec{w}'$ that satisfies items (i) and (ii) above, there is an $a' \in \mathcal{L}$ that codes an initial interval of $\mathcal{M}_{\vec{w}'}$ and an $r \in \mathcal{L}$ such that $E(r)$ matches both $E(a)$ and $E(a')$.

Again, the property "$\vec{w}$ is a code for $\mathcal{N}$" can be expressed by a first-order formula in the language of lattices.

Lemma 3.13. *Let $\mathcal{L}$ be a distributive lattice with the finite matching property, and let $\vec{w}$ be a code such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$. Then $\vec{w}$ is a code for $\mathcal{N}$ and, moreover, $\mathcal{M}_{\vec{w}'} \cong \mathcal{N}$ for every $\vec{w}'$ that is a code for $\mathcal{N}$ in $\mathcal{L}$.*

Proof. First let $\vec{w}$ be as in the statement of the lemma and show that $\vec{w}$ satisfies Definition 3.12. Item (i) is satisfied by the assumption $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$. For item (ii), let $s \in E(w_0)$ and notice that $\{b \mid R^2_\ell(b,s) \vee b = s\}$ is finite because it is an initial interval of a structure isomorphic to $\mathcal{N}$ and that it is an antichain because it is a subset of $E(w_0)$ which is an antichain by Lemma 3.3. Thus

$a = \inf\{b \mid R^2_\ell(b,s) \vee b = s\}$ witnesses item (ii) for s because $E(a) = \{b \mid R^2_\ell(b,s) \vee b = s\}$ by Lemma 3.4. For item (iii), let $a \in \mathcal{L}$ code an initial interval of $\mathcal{M}_{\vec{w}}$ and let $\vec{w}'$ be a code satisfying items (i) and (ii) of Definition 3.12. $|E(a)| = n$ for some $n \in \omega$ because $E(a)$ is an initial interval of a structure isomorphic to $\mathcal{N}$. $\mathcal{M}_{\vec{w}'} \models \mathrm{PA}^-$, so by Lemma 2.6 there is an initial interval of $\mathcal{M}_{\vec{w}'}$ of cardinality n and, by item (ii), there is an $a' \in \mathcal{L}$ coding this initial interval. Thus $|E(a)| = |E(a')| = n$, so by the finite matching property there is an $r \in \mathcal{L}$ such that $E(r)$ matches both $E(a)$ and $E(a')$. Thus $\vec{w}$ is indeed a code for $\mathcal{N}$.

Now suppose that $\vec{w}'$ is a code for $\mathcal{N}$ in $\mathcal{L}$. We show that $\mathcal{M}_{\vec{w}'} \cong \mathcal{N}$. By Definition 3.12 item (i), $\mathcal{M}_{\vec{w}'} \models \mathrm{PA}^-$. So by Lemma 2.6, it suffices to show that every initial interval $\mathcal{M}_{\vec{w}'}$ is finite. Thus let $s' \in E(w'_0)$, let $\{b' \mid R^2_{\ell'}(b',s') \vee b' = s'\}$ be the corresponding initial interval, and, by Definition 3.12 item (ii), let $a' \in \mathcal{L}$ code this initial interval. By Definition 3.12 item (iii) there is an $a \in \mathcal{L}$ coding an initial interval of $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$ such that $|E(a)| = |E(a')|$. $E(a)$ is finite, hence the initial interval $\{b' \mid R^2_{\ell'}(b',s') \vee b' = s'\}$ is finite. □

Lemma 3.14. *Let $\mathcal{L}$ be a distributive lattice with the finite matching property such that there exists a code $\vec{w}$ such that $\mathcal{M}_{\vec{w}} \cong \mathcal{N}$. Then* $\mathrm{Th}(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{L}; \leq_{\mathcal{L}})$.

Proof. Let φ be a sentence in the language of arithmetic. Let θ be the sentence

$$\theta = \exists \vec{w}(\vec{w} \text{ is a code for } \mathcal{N} \wedge \mathcal{M}_{\vec{w}} \models \varphi)$$

in the language of lattices. By Lemma 3.13, there are codes for $\mathcal{N}$ in $\mathcal{L}$ and every code for $\mathcal{N}$ in $\mathcal{L}$ codes a structure isomorphic to $\mathcal{N}$. Thus $\mathcal{N} \models \varphi$ if and only if $\mathcal{L} \models \theta$. This proves $\mathrm{Th}(\mathcal{N}) \leq_1 \mathrm{Th}(\mathcal{L})$. We always have $\mathrm{Th}(\mathcal{L}) \leq_1 \mathrm{Th}(\mathcal{L}; \leq_{\mathcal{L}})$ because the lattice operations sup and inf are first-order definable from the partial order. □

3.3. **Counting quantifiers.** An analysis of the quantifier complexity of our coding scheme shows that to determine the truth of existential sentences in $\mathcal{N}$ we only need to determine the truth of Π^0_3 sentences in $\mathcal{L}$.

We switch to coding models of $\mathrm{PA}^{\dot-}$ because the axioms of $\mathrm{PA}^{\dot-}$ are all of the form $\forall \vec{x}\psi(\vec{x})$ for quantifier-free ψ. Here *code* now means a code for a structure in the language of $\mathcal{N}^{\dot-}$. A code is now a sequence

$$\vec{w} = (w_0, w_1, w_2, m, \ell, p, t, d, z, o)$$

(with "d" for "difference") interpreted as coding the structure

$$\mathcal{M}^{\dot-}_{\vec{w}} = (E(w_0); R^2_\ell, R^3_p, R^3_t, R^3_d, z, o).$$

As in Definition 3.7, sentences in the language of $\mathcal{N}^{\dot-}$ translate to formulas in the language of lattices. The new $\dot-$ relation is replaced by the formula defining R^3_d in the translation. A *code for a model of* $\mathrm{PA}^{\dot-}$ is a code $\vec{w}$ such that $\mathcal{M}^{\dot-}_{\vec{w}} \models \mathrm{PA}^{\dot-}$.

In the language of lattices, "s is meet-irreducible" is a Π^0_1 property and "s meets to w" is a Σ^0_1 property, so "$s \in E(w)$" is a Δ^0_2 property. Hence $R^2_c(s_0, u_1)$ and $R^3_c(s_0, u_1, v_2)$ are both Δ^0_2 properties of s_0, u_1, v_2, and the coding parameters w_0, w_1, w_2, m, and c. Therefore, our coding translates atomic formulas in the language of $\mathcal{N}^{\dot-}$ to Δ^0_2 properties of lattices. Every Boolean combination of Δ^0_2 properties is again a Δ^0_2 property, so our coding also translates quantifier-free formulas in the language of $\mathcal{N}^{\dot-}$ to Δ^0_2 properties of lattices. Thus if $\varphi = \exists \vec{x}\psi(\vec{x})$ is a sentence in the language of $\mathcal{N}^{\dot-}$ where ψ is quantifier-free, then the translation $\varphi'(\vec{w})$ may be taken to be a Σ^0_2 formula in the language of lattices. Similarly, if $\varphi = \forall \vec{x}\psi(\vec{x})$, then the translation $\varphi'(\vec{w})$ is Π^0_2. Thus "$\mathcal{M}^{\dot-}_{\vec{w}} \models \mathrm{PA}^{\dot-}$" can be expressed by a Π^0_2 formula in the language of lattices. The axioms of $\mathrm{PA}^{\dot-}$ need to be unnested before they are translated, but this can be done in such a way that they all remain of the form $\forall \vec{x}\psi(\vec{x})$ for quantifier-free ψ.

In a lattice, the relations $\sup(x,y) = z$ and $\inf(x,y) = z$ are definable by Π^0_1 formulas in the language of partial orders. This translation increases the quantifier-complexities calculated in the previous paragraph by one alternation. Existential sentences in the language of $\mathcal{N}^{\dot-}$ translate to Σ^0_3 formulas in the language of partial orders, and universal sentences in the language of $\mathcal{N}^{\dot-}$ translate to Π^0_3 formulas in the language of partial orders. The property "$\mathcal{M}^{\dot-}_{\vec{w}} \models \mathrm{PA}^{\dot-}$" is a Π^0_3 property of $\vec{w}$ in the language of partial orders.

Lemma 3.15. *Let $\mathcal{L}$ be a lattice, and let $\vec{w}$ be a code such that $\mathcal{M}^{\dot-}_{\vec{w}} \cong \mathcal{N}^{\dot-}$. Then Σ^0_3-$\mathrm{Th}(\mathcal{L})$ and Σ^0_4-$\mathrm{Th}(\mathcal{L}; \leq_{\mathcal{L}})$ are undecidable.*

Proof. We prove

$$\{\exists\vec{x}\psi(\vec{x}) \mid \psi \text{ is quantifier-free} \wedge \mathcal{N} \models \exists\vec{x}\psi(\vec{x})\} \leq_1 \Pi^0_3\text{-}\mathrm{Th}(\mathcal{L}).$$

It is well-known that the problem of determining whether $\mathcal{N} \models \exists\vec{x}\psi(\vec{x})$ for quantifier-free ψ is undecidable.[2] Clearly Σ^0_3-$\mathrm{Th}(\mathcal{L}) \equiv_1 \Pi^0_3$-$\mathrm{Th}(\mathcal{L})$.

Let $\varphi = \exists\vec{x}\psi(\vec{x})$ be a sentence in the language of arithmetic where ψ is quantifier-free. Let θ be the sentence

$$\theta = \forall\vec{w}(\mathcal{M}^{\dot-}_{\vec{w}} \models \mathrm{PA}^{\dot-} \to \mathcal{M}^{\dot-}_{\vec{w}} \models \varphi)$$

in the language of lattices. As calculated above, $\mathcal{M}^{\dot-}_{\vec{w}} \models \mathrm{PA}^{\dot-}$ is a Π^0_2 property of $\vec{w}$, and $\mathcal{M}^{\dot-}_{\vec{w}} \models \varphi$ is a Σ^0_2 property of $\vec{w}$. Thus θ is a Π^0_3 sentence in the language of lattices. We need to show $\mathcal{N} \models \varphi$ if and only if $\mathcal{L} \models \theta$. Suppose $\mathcal{N} \models \varphi$. Then $\mathrm{PA}^{\dot-} \vdash \varphi$ by Lemma 2.7, which implies that $\mathcal{L} \models \theta$. Suppose $\mathcal{N} \not\models \varphi$. Then by assumption there is a code $\vec{w}$ such that $\mathcal{M}^{\dot-}_{\vec{w}} \cong \mathcal{N}^{\dot-}$. For this $\vec{w}$, $\mathcal{M}^{\dot-}_{\vec{w}} \models \mathrm{PA}^{\dot-}$ but $\mathcal{M}^{\dot-}_{\vec{w}} \not\models \varphi$, which implies $\mathcal{L} \not\models \theta$.

The proof that Σ^0_4-$\mathrm{Th}(\mathcal{L}; \leq_{\mathcal{L}})$ is undecidable is the same. The above sentence θ is Π^0_4 in the language of partial orders. □

4. Meet-irreducibles in $\mathcal{E}_{\mathrm{s}}$ and r.e. separating degrees

In this section we present facts about meet-irreducibles in $\mathcal{E}_{\mathrm{s}}$ that allow us to implement our coding in $\mathcal{E}_{\mathrm{s}}$. We begin with a characterization of the meet-irreducibles.

Lemma 4.1 ([1] Corollary 3.5)**.** *Let Q be a Π^0_1 class. Then $\deg_{\mathrm{s}}(Q)$ is meet-irreducible if and only if for every clopen $C \subseteq 2^\omega$ either $Q \cap C \equiv_{\mathrm{s}} Q$ or $Q \cap C^c \equiv_{\mathrm{s}} Q$.*

Proof. We prove the contrapositive in both directions. First, suppose $C \subseteq 2^\omega$ is clopen, $Q \cap C \not\equiv_{\mathrm{s}} Q$, and $Q \cap C^c \not\equiv_{\mathrm{s}} Q$. $Q \cap C \geq_{\mathrm{s}} Q$ and $Q \cap C^c \geq_{\mathrm{s}} Q$ by the identity functional, so it must be that $Q \cap C >_{\mathrm{s}} Q$ and $Q \cap C^c >_{\mathrm{s}} Q$. C is clopen, so there is a finite set of strings $\{\sigma_i\}_{i<n} \subseteq 2^{<\omega}$ such that $C = \bigcup_{i<n} I(\sigma_i)$. Then $0^\frown(Q \cap C) \cup 1^\frown(Q \cap C^c) \leq_{\mathrm{s}} Q$ by the functional

$$f \mapsto \begin{cases} 0^\frown f & \text{if } (\exists i < n)(\sigma_i \subset f) \\ 1^\frown f & \text{otherwise.} \end{cases}$$

So $\deg_{\mathrm{s}}(Q \cap C) >_{\mathrm{s}} \deg_{\mathrm{s}}(Q)$, $\deg_{\mathrm{s}}(Q \cap C^c) >_{\mathrm{s}} \deg_{\mathrm{s}}(Q)$, and $\inf(\deg_{\mathrm{s}}(Q \cap C), \deg_{\mathrm{s}}(Q \cap C^c)) \leq_{\mathrm{s}} \deg_{\mathrm{s}}(Q)$. Hence $\deg_{\mathrm{s}}(Q)$ is meet-reducible.

Conversely, suppose $\deg_{\mathrm{s}}(Q)$ is meet-reducible, and let X and Y be Π^0_1 classes such that $X >_{\mathrm{s}} Q$, $Y >_{\mathrm{s}} Q$, and $Q \equiv_{\mathrm{s}} 0^\frown X \cup 1^\frown Y$. Let Φ be such that $\Phi(Q) \subseteq 0^\frown X \cup 1^\frown Y$. Consider the set $\widehat{X} = \{f \in Q \mid \Phi(f)(0) = 0\}$. $\Phi(f)$ is total for all $f \in Q$, so we can write $\widehat{X} = Q \cap \{f \in 2^\omega \mid \Phi(f)(0) \neq 1\}$ (where $\Phi(f)(0) \neq 1$ includes the possibility that $\Phi(f)(0)$ diverges), which is the intersection of two closed subsets of 2^ω. Hence $\widehat{X}$ is compact. Let $\Sigma = \{\sigma \in 2^{<\omega} \mid \Phi(\sigma)(0) = 0\}$. Then $\widehat{X} \subseteq \bigcup_{\sigma\in\Sigma} I(\sigma)$, so by compactness there is a finite $\Sigma_0 \subseteq \Sigma$ such that $\widehat{X} \subseteq \bigcup_{\sigma\in\Sigma_0} I(\sigma)$. Let $C = \bigcup_{\sigma\in\Sigma_0} I(\sigma)$ be this

[2] For example, undecidability is implied by Matiyasevich's solution to Hilbert's tenth problem [23]. It is a standard fact in computability theory that determining whether $\mathcal{N} \models \exists\vec{x}\psi(\vec{x})$ is undecidable if ψ is allowed bounded quantifiers, but allowing bounded quantifiers in ψ increases the quantifier complexity of the translated formula.

clopen set. Φ witnesses that $Q \cap C \geq_{\mathrm{s}} 0^\frown X$ and that $Q \cap C^c \geq_{\mathrm{s}} 1^\frown Y$. As $0^\frown X \equiv_{\mathrm{s}} X >_{\mathrm{s}} Q$ and $1^\frown Y \equiv_{\mathrm{s}} Y >_{\mathrm{s}} Q$, we have the desired clopen set $C \subseteq 2^\omega$ such that $Q \cap C \not\equiv_{\mathrm{s}} Q$ and $Q \cap C^c \not\equiv_{\mathrm{s}} Q$. ☐

Degrees of r.e. separating classes are the main examples of meet-irreducibles in $\mathcal{E}_{\mathrm{s}}$.

Definition 4.2. For $A, B \subseteq \omega$, define

$$S(A,B) = \{f \in 2^\omega \mid \forall n((n \in A \to f(n) = 1) \wedge (n \in B \to f(n) = 0))\}.$$

An $f \in S(A,B)$ is said to *separate* A from B. $S \subseteq 2^\omega$ is an *r.e. separating class* if and only if there are disjoint r.e. sets $A, B \subseteq \omega$ such that $S = S(A,B)$.

From the definition, an r.e. separating class is always a Π^0_1 class. An $\mathbf{s} \in \mathcal{E}_{\mathrm{s}}$ is an *r.e. separating degree* if and only if $\mathbf{s} = \deg_{\mathrm{s}}(S)$ for an r.e. separating class S.

Lemma 4.3 ([7] Lemma 6)**.** *If S is an r.e. separating class and $C \subseteq 2^\omega$ is a clopen set such that $S \cap C \neq \emptyset$, then $S \cap C \equiv_{\mathrm{s}} S$.*

Proof. Let $S = S(A,B)$ be an r.e. separating class and let $C \subseteq 2^\omega$ be a clopen set such that $S \cap C \neq \emptyset$. $S \leq_{\mathrm{s}} S \cap C$ by the identity functional. To see $S \geq_{\mathrm{s}} S \cap C$, let $I(\sigma)$ be such that $I(\sigma) \subseteq C$ and $S \cap I(\sigma) \neq \emptyset$. For any $f \in 2^\omega$, let f_σ be the function obtained from f by replacing the initial segment of f of length $|\sigma|$ by σ:

$$f_\sigma(n) = \begin{cases} \sigma(n) & \text{if } n < |\sigma| \\ f(n) & \text{if } n \geq |\sigma|. \end{cases}$$

The condition $S \cap I(\sigma) \neq \emptyset$ implies that σ separates $\{n \in A \mid n < |\sigma|\}$ from $\{n \in B \mid n < |\sigma|\}$. Thus if f separates A from B, then so does f_σ. Therefore the functional $f \mapsto f_\sigma$ witnesses $S \geq_{\mathrm{s}} S \cap C$. ☐

Lemma 4.1 and Lemma 4.3 imply that every r.e. separating degree is meet-irreducible. It is important to note (as in [7]) that the r.e. separating classes are closed under $\otimes$ and consequently that the r.e. separating degrees are closed under sup: if $S(A_0, B_0)$ and $S(A_1, B_1)$ are r.e. separating classes then $S(A_0, B_0) \otimes S(A_1, B_1) = S(A_0 \oplus A_1, B_0 \oplus B_1)$. Thus the sup of two r.e. separating degrees is meet-irreducible. In fact, the sup of any r.e. separating degree and any meet-irreducible degree is again meet-irreducible.

Lemma 4.4. *Let $\mathbf{q} \in \mathcal{E}_{\mathrm{s}}$ be meet-irreducible and let $\mathbf{s} \in \mathcal{E}_{\mathrm{s}}$ be an r.e. separating degree. Then $\sup(\mathbf{q}, \mathbf{s})$ is meet-irreducible.*

Proof. Suppose $\sup(\mathbf{q}, \mathbf{s}) \geq_{\mathrm{s}} \inf(\mathbf{x}, \mathbf{y})$ for some $\mathbf{x}, \mathbf{y} \in \mathcal{E}_{\mathrm{s}}$. We show $\sup(\mathbf{q}, \mathbf{s}) \geq_{\mathrm{s}} \mathbf{x}$ or $\sup(\mathbf{q}, \mathbf{s}) \geq_{\mathrm{s}} \mathbf{y}$. Let Q, X, and Y be Π^0_1 classes such that $\deg_{\mathrm{s}}(Q) = \mathbf{q}$, $\deg_{\mathrm{s}}(X) = \mathbf{x}$, and $\deg_{\mathrm{s}}(Y) = \mathbf{y}$, and let S be an r.e. separating class such that $\deg_{\mathrm{s}}(S) = \mathbf{s}$. Let Φ be such that $\Phi(Q \otimes S) \subseteq 0^\frown X \cup 1^\frown Y$. By compactness, choose a $\sigma \in 2^{<\omega}$ such that $S \cap I(\sigma) \neq \emptyset$ and an $n \in \omega$ such that

$$(\forall \tau \in 2^n)((\exists f \in Q)(\tau \subset f) \to \Phi(\tau \oplus \sigma)(0)\downarrow).$$

Let $C = \bigcup\{I(\tau) \mid \tau \in 2^n \wedge \Phi(\tau \oplus \sigma)(0) = 0\}$. Then C is clopen, and Φ witnesses that $(Q \cap C) \otimes (S \cap I(\sigma)) \geq_{\mathrm{s}} 0^\frown X \equiv_{\mathrm{s}} X$ and that $(Q \cap C^c) \otimes (S \cap I(\sigma)) \geq_{\mathrm{s}} 1^\frown Y \equiv_{\mathrm{s}} Y$. Since $S \cap I(\sigma) \equiv_{\mathrm{s}} S$ by Lemma 4.3 and either $Q \cap C \equiv_{\mathrm{s}} Q$ or $Q \cap C^c \equiv_{\mathrm{s}} Q$ by Lemma 4.1, we have either $Q \otimes S \geq_{\mathrm{s}} X$ or $Q \otimes S \geq_{\mathrm{s}} Y$ as desired. ☐

Our proof that $\mathcal{E}_{\mathrm{s}}$ has the finite matching property uses the following lemma of Cole and Kihara. It is the main tool in their proof that the Σ^0_2-theory of $\mathcal{E}_{\mathrm{s}}$ as a partial order is decidable.

Lemma 4.5 ([10] Lemma 1)**.** *Let $\{\mathbf{q}_i\}_{i<n} \subseteq \mathcal{E}_{\mathrm{s}}$ and let $m \in \omega$. Then there is a set $\{\mathbf{r}_i\}_{i<m} \subseteq \mathcal{E}_{\mathrm{s}}$ such that*

$$\Big(\forall I \subseteq m\Big)\Big(\forall J, K \subseteq n\Big)\Big[J \cap K = \emptyset \wedge \sup_{i \in J} \mathbf{q}_i \not\geq_{\mathrm{s}} \inf_{i \in K} \mathbf{q}_i \to \sup\Big(\sup_{i \in J} \mathbf{q}_i, \sup_{i \in I} \mathbf{r}_i\Big) \not\geq_{\mathrm{s}} \inf\Big(\inf_{i \in K} \mathbf{q}_i, \inf_{i \notin I} \mathbf{r}_i\Big)\Big]$$

(where $\sup_{i \in \emptyset} \mathbf{x}_i = \mathbf{0}_{\mathrm{s}}$ and $\inf_{i \in \emptyset} \mathbf{x}_i = \mathbf{1}_{\mathrm{s}}$).

Cole and Kihara note that the $\{\mathbf{r}_i\}_{i<m}$ that they construct in Lemma 4.5 are all r.e. separating degrees. Their proof of Lemma 4.5 is an elaboration of Cenzer and Hinman's proof that $\mathcal{E}_\mathrm{s}$ is dense [7]. Cenzer and Hinman prove that if $\mathbf{p}, \mathbf{q} \in \mathcal{E}_\mathrm{s}$ are such that $\mathbf{q} \nleq_\mathrm{s} \mathbf{p}$, then there is an r.e. separating degree $\mathbf{r} \in \mathcal{E}_\mathrm{s}$ such that $\inf(\mathbf{q}, \mathbf{r}) \nleq_\mathrm{s} \mathbf{p}$ and $\mathbf{q} \nleq_\mathrm{s} \sup(\mathbf{p}, \mathbf{r})$. Thus if $\mathbf{p} <_\mathrm{s} \mathbf{q}$, then $\mathbf{p} <_\mathrm{s} \inf(\sup(\mathbf{p}, \mathbf{r}), \mathbf{q}) <_\mathrm{s} \mathbf{q}$, yielding density. To make Lemma 4.5 somewhat easier to read and apply, we note that we only need the following special case.

Lemma 4.6. *Let $\{\mathbf{q}_i\}_{i<n} \subseteq \mathcal{E}_\mathrm{s} \setminus \{\mathbf{1}_\mathrm{s}\}$ and let $m \in \omega$. Then there is a set of r.e. separating degrees $\{\mathbf{r}_i\}_{i<m} \subseteq \mathcal{E}_\mathrm{s}$ such that*

(i) $(\forall i, i' < n)(\forall j < m)(\mathbf{q}_i \ngeq_\mathrm{s} \mathbf{q}_{i'} \rightarrow \sup(\mathbf{q}_i, \mathbf{r}_j) \ngeq_\mathrm{s} \mathbf{q}_{i'})$ *and*
(ii) $(\forall i < n)(\forall j, j' < m)(j \neq j' \rightarrow \sup(\mathbf{q}_i, \mathbf{r}_j) \ngeq_\mathrm{s} \mathbf{r}_{j'})$.

We can now show that $\mathcal{E}_\mathrm{s}$ has the finite matching property.

Lemma 4.7. *$\mathcal{E}_\mathrm{s}$ has the finite matching property. That is, if $\mathbf{q}, \mathbf{q}' \in \mathcal{E}_\mathrm{s}$ are such that $|E(\mathbf{q})| = |E(\mathbf{q}')| = n$ for some $n \in \omega$, then there is an $\mathbf{r} \in \mathcal{E}_\mathrm{s}$ such that $E(\mathbf{r})$ matches both $E(\mathbf{q})$ and $E(\mathbf{q}')$.*

Proof. If $n = 0$, then let $\mathbf{r} = \mathbf{q}$. Any degree $\mathbf{z}$ vacuously witnesses that $E(\mathbf{r})$ matches $E(\mathbf{q})$ and that $E(\mathbf{r})$ matches $E(\mathbf{q}')$. So suppose $n > 0$, let $E(\mathbf{q}) = \{\mathbf{q}_i\}_{i<n}$, and let $E(\mathbf{q}') = \{\mathbf{q}'_i\}_{i<n}$. Apply Lemma 4.6 to $\{\mathbf{q}_i\}_{i<n} \cup \{\mathbf{q}'_i\}_{i<n}$ with $m = n$, noting that $\{\mathbf{q}_i\}_{i<n}$ and $\{\mathbf{q}'_i\}_{i<n}$ are both antichains by Lemma 3.3, to get r.e. separating degrees $\{\mathbf{r}_i\}_{i<n}$ such that

(i) $\sup(\mathbf{q}_i, \mathbf{r}_j) \ngeq_\mathrm{s} \mathbf{q}_k$ and $\sup(\mathbf{q}'_i, \mathbf{r}_j) \ngeq_\mathrm{s} \mathbf{q}'_k$ whenever $i, j, k < n$ are such that $i \neq k$, and
(ii) $\sup(\mathbf{q}_i, \mathbf{r}_j) \ngeq_\mathrm{s} \mathbf{r}_k$ and $\sup(\mathbf{q}'_i, \mathbf{r}_j) \ngeq_\mathrm{s} \mathbf{r}_k$ whenever $i, j, k < n$ are such that $j \neq k$.

(Lemma 4.6 applies because, by definition, $\mathbf{1}_\mathrm{s}$ does not meet to any degree and so cannot be in $E(\mathbf{q})$ or $E(\mathbf{q}')$.)

Put $\mathbf{r} = \inf_{i<n} \mathbf{r}_i$, $\mathbf{z} = \inf_{i<n} \sup(\mathbf{q}_i, \mathbf{r}_i)$, and $\mathbf{z}' = \inf_{i<n} \sup(\mathbf{q}'_i, \mathbf{r}_i)$. We show that $\mathbf{z}$ witnesses that $E(\mathbf{r})$ matches $E(\mathbf{q})$. The proof that $\mathbf{z}'$ witnesses that $E(\mathbf{r})$ matches $E(\mathbf{q}')$ is similar. Item (ii) implies that $\{\mathbf{r}_i\}_{i<n}$ and $\{\sup(\mathbf{q}_i, \mathbf{r}_i)\}_{i<n}$ are both antichains. Lemma 4.4 implies that $\sup(\mathbf{q}_i, \mathbf{r}_i)$ is meet-irreducible for each $i < n$. Therefore $E(\mathbf{r}) = \{\mathbf{r}_i\}_{i<n}$ and $E(\mathbf{z}) = \{\sup(\mathbf{q}_i, \mathbf{r}_i)\}_{i<n}$ by Lemma 3.4. Suppose $\sup(\mathbf{q}_i, \mathbf{r}_j) \geq_\mathrm{s} \mathbf{z}$ for some $i, j < n$. Then $\sup(\mathbf{q}_i, \mathbf{r}_j) \geq_\mathrm{s} \sup(\mathbf{q}_k, \mathbf{r}_k)$ for some $k < n$ because $\sup(\mathbf{q}_i, \mathbf{r}_j)$ is meet-irreducible by Lemma 4.4. Item (i) implies that $i = k$, and item (ii) implies that $j = k$. Thus for each $i < n$, $\mathbf{r}_i$ is the unique $\mathbf{y} \in E(\mathbf{r})$ such that $\sup(\mathbf{q}_i, \mathbf{y}) \in E(\mathbf{z})$, and $\mathbf{q}_i$ is the unique $\mathbf{y} \in E(\mathbf{q})$ such that $\sup(\mathbf{r}_i, \mathbf{y}) \in E(\mathbf{z})$. Thus $\mathbf{z}$ witnesses that $E(\mathbf{r})$ matches $E(\mathbf{q})$. □

We need one last fact about the r.e. separating classes to implement our coding in $\mathcal{E}_\mathrm{s}$. Let $\{f_n\}_{n\in\omega} \subseteq 2^\omega$ be a sequence of functions, and let $m \in \omega$. Define $\bigoplus_{n\in\omega} f_n$ and $\bigoplus_{n\in\omega\setminus\{m\}} f_n$ by

$$\left(\bigoplus_{n\in\omega} f_n\right)(\langle i, j\rangle) = f_i(j) \text{ and}$$

$$\left(\bigoplus_{n\in\omega\setminus\{m\}} f_n\right)(\langle i, j\rangle) = \begin{cases} f_i(j) & \text{if } i \neq m \\ 0 & \text{if } i = m. \end{cases}$$

Definition 4.8. A sequence of functions $\{f_n\}_{n\in\omega} \subseteq 2^\omega$ is *strongly independent* if and only if $\forall m(f_m \nleq_\mathrm{T} \bigoplus_{n\in\omega\setminus\{m\}} f_n)$. A sequence of Π^0_1 classes $\{X_n\}_{n\in\omega}$ is *strongly independent* if and only if $\{f_n\}_{n\in\omega}$ is strongly independent whenever $\forall n(f_n \in X_n)$.

Lemma 4.9 ([17] Theorem 4.1)**.** *There is a recursive sequence $\{S_n\}_{n\in\omega}$ r.e. separating classes that is strongly independent.*

5. Interpreting true arithmetic in $\mathcal{E}_\mathrm{s}$

In this section we prove that $\mathrm{Th}(\mathcal{E}_\mathrm{s};\leq_\mathrm{s}) \equiv_1 \mathrm{Th}(\mathcal{N})$ and that Π^0_3-$\mathrm{Th}(\mathcal{E}_\mathrm{s})$ and Π^0_4-$\mathrm{Th}(\mathcal{E}_\mathrm{s};\leq_\mathrm{s})$ are undecidable. By Lemma 3.14, Lemma 3.15, and Lemma 4.7 it suffices to find a code $\vec{\mathbf{w}}$ in $\mathcal{E}_\mathrm{s}$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$. This section is analogous to [32] Section 5, in which it is proved that the first-order theories of $\mathcal{D}_{\mathrm{s,cl}}$ and $\mathcal{D}^{01}_{\mathrm{s},cl}$ are recursively isomorphic to true second-order arithmetic.

Definition 5.1. Let Q be a Π^0_1 class with no recursive member. Let A be an infinite recursive set, and let $\{\sigma_n\}_{n\in A}$ be a recursive sequence of pairwise incomparable strings such that $\bigcup_{n\in A} I(\sigma_n) = 2^\omega \setminus Q$ (for example, let T be a recursive tree such that $Q = [T]$ and let $\{\sigma_n\}_{n\in A}$ be the strings $\sigma \notin T$ of minimal length). Let $\{S_n\}_{n\in A}$ be an infinite recursive sequence of Π^0_1 classes. Define $\mathrm{spine}(Q, \{S_n\}_{n\in A})$ to be the Π^0_1 class

$$\mathrm{spine}(Q, \{S_n\}_{n\in A}) = Q \cup \bigcup_{n\in A} \sigma_n{}^\frown S_n.$$

The next lemma gives the analog of Lemma 3.4 for spines.

Lemma 5.2. *Let Q be a Π^0_1 class with no recursive member. Let $\{S_n\}_{n\in A}$ be an infinite recursive sequence of r.e. separating classes (indexed by a recursive set A) that is an antichain and is such that $Q \nleq_\mathrm{s} S_n$ for all $n \in A$. Let $\mathbf{w} = \deg_\mathrm{s}(\mathrm{spine}(Q, \{S_n\}_{n\in A}))$.*

(i) *If $\mathbf{x} \in \mathcal{E}_\mathrm{s}$ meets to $\mathbf{w}$, then $\mathbf{x} \leq_\mathrm{s} \deg_\mathrm{s}(S_n)$ for some $n \in A$.*
(ii) *$E(\mathbf{w}) = \{\deg_\mathrm{s}(S_n) \mid n \in A\}$.*

Proof. Let $W = \mathrm{spine}(Q, \{S_n\}_{n\in A})$.

(i) Let $\mathbf{x} \in \mathcal{E}_\mathrm{s}$ be such that $\mathbf{x}$ meets to $\mathbf{w}$. Suppose for a contradiction that $\mathbf{x} \nleq_\mathrm{s} \deg_\mathrm{s}(S_n)$ for all $n \in A$. Let X be a Π^0_1 class such that $\mathbf{x} = \deg_\mathrm{s}(X)$, and let Y be a Π^0_1 class such that $\deg_\mathrm{s}(Y)$ witnesses that $\mathbf{x}$ meets to $\mathbf{w}$. That is, $Y >_\mathrm{s} W$ and $W \equiv_\mathrm{s} 0^\frown X \cup 1^\frown Y$. Let Φ be such that $\Phi(Q \cup \bigcup_{n\in A} \sigma_n{}^\frown S_n) \subseteq 0^\frown X \cup 1^\frown Y$.

Claim.
(a) $\Phi(\sigma_n{}^\frown S_n) \subseteq 1^\frown Y$ *for all $n \in A$ and*
(b) $\Phi(Q) \subseteq 1^\frown Y$.

Proof of claim. If item (a) fails, then for some $n \in A$ there is a clopen $C \subseteq 2^\omega$ such that $(\sigma_n{}^\frown S_n) \cap C \neq \emptyset$ and $\Phi((\sigma_n{}^\frown S_n) \cap C) \subseteq 0^\frown X$. So $(\sigma_n{}^\frown S_n) \cap C \geq_\mathrm{s} 0^\frown X \equiv_\mathrm{s} X$. The class $\sigma_n{}^\frown S_n$ is an r.e. separating class because S_n is, so $(\sigma_n{}^\frown S_n) \cap C \equiv_\mathrm{s} \sigma_n{}^\frown S_n \equiv_\mathrm{s} S_n$, where the first equivalence is by Lemma 4.3. Thus the contradiction $X \leq_\mathrm{s} S_n$.

If item (b) fails, then there is an $f \in Q$ and a $\sigma \subset f$ such that $\Phi(\sigma)(0)\downarrow = 0$. Since $I(\sigma) \nsubseteq Q$, there is an $n \in A$ such that $\sigma_n \supseteq \sigma$. Hence $\Phi(\sigma_n{}^\frown S_n) \nsubseteq 1^\frown Y$, contradicting item (a). □

The claim shows that $\Phi(Q \cup \bigcup_{n\in\omega} \sigma_n{}^\frown S_n) \subseteq 1^\frown Y$. Thus $Y \leq_\mathrm{s} W$, which contradicts $Y >_\mathrm{s} W$.

(ii) Let $n \in A$. To see that $\deg_\mathrm{s}(S_n) \in E(\mathbf{w})$, let $Y = Q \cup \bigcup_{i\in A\setminus\{n\}} \sigma_i{}^\frown S_i$.

Claim. $S_n \ngeq_\mathrm{s} Y$

Proof of claim. Suppose for a contradiction that Φ is such that $\Phi(S_n) \subseteq Y$. If there is an $i \in A\setminus\{n\}$ such that $\Phi(S_n) \cap (\sigma_i{}^\frown S_i) \neq \emptyset$, then there is a clopen $C \subseteq 2^\omega$ such that $S_n \cap C \neq \emptyset$ and $\Phi(S_n \cap C) \subseteq \sigma_i{}^\frown S_i$. Hence $S_n \equiv_\mathrm{s} S_n \cap C$ by Lemma 4.3, and $S_n \cap C \geq_\mathrm{s} \sigma_i{}^\frown S_i \equiv_\mathrm{s} S_i$. This contradicts that $\{S_n\}_{n\in A}$ is an antichain. Thus $\Phi(S_n) \cap (\sigma_i{}^\frown S_i) = \emptyset$ for all $n \in A$. Therefore $\Phi(S_n) \subseteq Q$. This contradicts $Q \nleq_\mathrm{s} S_n$. □

It is easy to check that $W \equiv_\mathrm{s} 0^\frown S_n \cup 1^\frown Y$, so, by the claim, $\deg_\mathrm{s}(Y)$ witnesses that $\deg_\mathrm{s}(S_n)$ meets to $\mathbf{w}$. The degree $\deg_\mathrm{s}(S_n)$ is meet-irreducible because it is an r.e. separating degree. Thus $\deg_\mathrm{s}(S_n) \in E(\mathbf{w})$.

We have shown that $\{\deg_s(S_n) \mid n \in A\} \subseteq E(\mathbf{w})$. To see equality, let $\mathbf{x} \in E(\mathbf{w})$. By item (i), $\mathbf{x} \leq_s \deg_s(S_n)$ for some $n \in A$. $E(\mathbf{w})$ is an antichain by Lemma 3.3 and $\deg_s(S_n) \in E(\mathbf{w})$, so it must be that $\mathbf{x} = \deg_s(S_n)$. □

We now have all the ingredients to find a code for $\mathcal{N}$ in $\mathcal{E}_s$.

Lemma 5.3. *There is a code $\vec{\mathbf{w}}$ in $\mathcal{E}_s$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$.*

Proof. By Lemma 4.9, let Q be an r.e. separating class and let $\{S_{0,n}\}_{n\in\omega}$, $\{S_{1,n}\}_{n\in\omega}$, and $\{S_{2,n}\}_{n\in\omega}$ be recursive sequences of r.e. separating classes such that $\{Q\} \cup \{S_{0,n}\}_{n\in\omega} \cup \{S_{1,n}\}_{n\in\omega} \cup \{S_{2,n}\}_{n\in\omega}$ is strongly independent. Then let

$$
\begin{array}{rll}
\mathbf{w}_0 = \deg_s(W_0) & \text{for} & W_0 = \operatorname{spine}(Q, \{S_{0,n}\}_{n\in\omega}),\\
\mathbf{w}_1 = \deg_s(W_1) & \text{for} & W_1 = \operatorname{spine}(Q, \{S_{1,n}\}_{n\in\omega}),\\
\mathbf{w}_2 = \deg_s(W_2) & \text{for} & W_2 = \operatorname{spine}(Q, \{S_{2,n}\}_{n\in\omega}),\\
\mathbf{m} = \deg_s(M) & \text{for} & M = \operatorname{spine}(Q, \{S_{0,n} \otimes S_{1,n}\}_{n\in\omega} \cup \{S_{0,n} \otimes S_{2,n}\}_{n\in\omega}),\\
\boldsymbol{\ell} = \deg_s(L) & \text{for} & L = \operatorname{spine}(Q, \{S_{0,i} \otimes S_{1,j} \mid i < j\}),\\
\mathbf{p} = \deg_s(P) & \text{for} & P = \operatorname{spine}(Q, \{S_{0,i} \otimes S_{1,j} \otimes S_{2,k} \mid i + j = k\}),\\
\mathbf{t} = \deg_s(T) & \text{for} & T = \operatorname{spine}(Q, \{S_{0,i} \otimes S_{1,j} \otimes S_{2,k} \mid i \times j = k\}),\\
\mathbf{d} = \deg_s(D) & \text{for} & D = \operatorname{spine}(Q, \{S_{0,i} \otimes S_{1,j} \otimes S_{2,k} \mid i \dot{-} j = k\}),\\
\mathbf{z} = \deg_s(S_{0,0}), & \text{and} & \\
\mathbf{o} = \deg_s(S_{0,1}). & &
\end{array}
$$

By Lemma 5.2 item (ii), $E(\mathbf{w_0}) = \{\deg_s(S_{0,n})\}_{n\in\omega}$. The map $\deg_s(S_{0,n}) \mapsto n$ is the isomorphism witnessing $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$. Clearly $\mathbf{z} \mapsto 0$ and $\mathbf{o} \mapsto 1$. We show that the map preserves $<$. The proofs that the map preserves $+$, $\times$, and $\dot{-}$ are similar. Let $i, j \in \omega$. If $i < j$, then $\deg_s(S_{1,j})$ meets to $\mathbf{w}_1$ by Lemma 5.2 item (ii), and it is easy to see that $\sup(\deg_s(S_{0,j}), \deg_s(S_{1,j})) \geq_s \mathbf{m}$ and that $\sup(\deg_s(S_{0,i}), \deg_s(S_{1,j})) \geq_s \boldsymbol{\ell}$. Thus $R^2_{\boldsymbol{\ell}}(\deg_s(S_{0,i}), \deg_s(S_{0,j}))$. Conversely, suppose that $R^2_{\boldsymbol{\ell}}(\deg_s(S_{0,i}), \deg_s(S_{0,j}))$. Let $\mathbf{u}_1 \in \mathcal{E}_s$ be such that $\mathbf{u}_1$ meets to $\mathbf{w}_1$, $\sup(\deg_s(S_{0,j}), \mathbf{u}_1) \geq_s \mathbf{m}$, and $\sup(\deg_s(S_{0,i}), \mathbf{u}_1) \geq_s \boldsymbol{\ell}$. Since $\mathbf{u}_1$ meets to $\mathbf{w}_1$, it must be that $\mathbf{u}_1 \leq_s \deg_s(S_{1,k})$ for some $k \in \omega$ by Lemma 5.2 item (i). Thus $\sup(\deg_s(S_{0,j}), \deg_s(S_{1,k})) \geq_s \mathbf{m}$. However, if $k \neq j$, then no member of $S_{0,j} \otimes S_{1,k}$ computes any member of M by strong independence. Thus $\mathbf{u}_1 \leq_s \deg_s(S_{1,j})$, which implies that $\sup(\deg_s(S_{0,i}), \deg_s(S_{1,j})) \geq_s \boldsymbol{\ell}$. Again by strong independence, if $i \not< j$, then no member of $S_{0,i} \otimes S_{1,j}$ computes any member of L. Hence $i < j$. □

Higuchi also used spines of recursive sequences of independent r.e. separating classes to prove that $\mathcal{E}_s$ is not a Brouwer algebra [13].

Theorem 5.4. $\operatorname{Th}(\mathcal{E}_s; \leq_s) \equiv_1 \operatorname{Th}(\mathcal{N})$.

Proof. We first prove $\operatorname{Th}(\mathcal{E}_s; \leq_s) \leq_1 \operatorname{Th}(\mathcal{N})$. Let $\{T_e\}_{e\in\omega}$ be a uniformly recursive sequence of trees representing all Π^0_1 classes as in Lemma 2.3. Given a sentence θ in the language of partial orders, produce an equivalent sentence in the language of partial orders by replacing every atomic formula $x = y$ by the formula $x \leq y \wedge y \leq x$. Then produce a sentence φ in the language of arithmetic by replacing every atomic formula $x \leq y$ by the Σ^0_3 formula from Lemma 2.2 expressing $[T_x] \leq_s [T_y]$. Then $\mathcal{E}_s \models \theta$ if and only if $\mathcal{N} \models \varphi$.

For $\operatorname{Th}(\mathcal{N}) \leq_1 \operatorname{Th}(\mathcal{E}_s; \leq_s)$, by Lemma 5.3 let $\vec{\mathbf{w}}$ be a code in $\mathcal{E}_s$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$. Removing the degree $\mathbf{d}$ from the code $\vec{\mathbf{w}}$ gives a code $\vec{\mathbf{v}}$ such that $\mathcal{M}_{\vec{\mathbf{v}}} \cong \mathcal{N}$. $\mathcal{E}_s$ has the finite matching property by Lemma 4.7, thus $\operatorname{Th}(\mathcal{N}) \leq_1 \operatorname{Th}(\mathcal{E}_s; \leq_s)$ by Lemma 3.14. □

Theorem 5.5. *Σ^0_3-$\operatorname{Th}(\mathcal{E}_s)$ and Σ^0_4-$\operatorname{Th}(\mathcal{E}_s; \leq_s)$ are undecidable.*

Proof. There is a code $\vec{\mathbf{w}}$ in $\mathcal{E}_{\mathrm{s}}$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$ by Lemma 5.3. The results then follow from Lemma 3.15. ☐

Fragments of first-order theories were not considered in [32]. The refined coding scheme used here also shows that Σ^0_3-Th($\mathcal{L}$) and Σ^0_4-Th($\mathcal{L}; \leq_{\mathcal{L}}$) are undecidable for $\mathcal{L} = \mathcal{D}_{\mathrm{s}}$, $\mathcal{D}_{\mathrm{w}}$, $\mathcal{D}_{\mathrm{s,cl}}$, $\mathcal{D}_{\mathrm{w,cl}}$, $\mathcal{D}^{01}_{\mathrm{s},cl}$, and $\mathcal{D}^{01}_{\mathrm{w,cl}}$.

6. The degree of $\mathcal{E}_{\mathrm{s}}$ is $\mathbf{0}'''$

In this section, we consider the complexities of presentations of $\mathcal{E}_{\mathrm{s}}$.

Definition 6.1. A *presentation of* $\mathcal{E}_{\mathrm{s}}$ *as a partial order* consists of a relation $\leq_{\mathcal{P}} \subseteq \omega \times \omega$ such that the structure $\mathcal{P} = (\omega; \leq_{\mathcal{P}})$ is isomorphic to $(\mathcal{E}_{\mathrm{s}}; \leq_{\mathrm{s}})$. A *presentation of* $\mathcal{E}_{\mathrm{s}}$ *as a lattice* consists of a relation $\leq_{\mathcal{L}} \subseteq \omega \times \omega$ and functions $\sup_{\mathcal{L}} \colon \omega \times \omega \to \omega$ and $\inf_{\mathcal{L}} \colon \omega \times \omega \to \omega$ such that the structure $\mathcal{L} = (\omega; \leq_{\mathcal{L}}, \sup_{\mathcal{L}}, \inf_{\mathcal{L}})$ is isomorphic to $\mathcal{E}_{\mathrm{s}}$.

We measure the complexities of presentations by their Turing degrees.

Definition 6.2. The *degree* of a presentation $\mathcal{P}$ of $\mathcal{E}_{\mathrm{s}}$ as a partial order is $\deg_{\mathrm{T}}(\mathcal{P}) = \deg_{\mathrm{T}}(\leq_{\mathcal{P}})$. The *degree* of a presentation $\mathcal{L}$ of $\mathcal{E}_{\mathrm{s}}$ as a lattice is $\deg_{\mathrm{T}}(\mathcal{L}) = \deg_{\mathrm{T}}(\leq_{\mathcal{L}} \oplus \sup_{\mathcal{L}} \oplus \inf_{\mathcal{L}})$.

Equivalently, the degree of a presentation is the Turing degree of its atomic diagram, suitably Gödel numbered.

Lemma 6.3. *There is a presentation $\mathcal{L}$ of $\mathcal{E}_{\mathrm{s}}$ as a lattice with* $\deg_{\mathrm{T}}(\mathcal{L}) \leq_{\mathrm{T}} \mathbf{0}'''$.

Proof. Let $\{T_e\}_{e\in\omega}$ be a uniformly recursive sequence of trees representing all Π^0_1 classes as in Lemma 2.3. Since $[T_i] \leq_{\mathrm{s}} [T_j]$ is a Σ^0_3 property of $\langle i, j\rangle$ by Lemma 2.2, we can use $\mathbf{0}'''$ to make a new sequence of trees $\{T'_e\}_{e\in\omega}$ such that $\{[T'_e]\}_{e\in\omega}$ contains exactly one representative for each degree in $\mathcal{E}_{\mathrm{s}}$. Inductively, let T'_e be T_i for the least $i \in \omega$ such that $(\forall j < e)([T_i] \not\equiv_{\mathrm{s}} [T'_j])$. Again using $\mathbf{0}'''$, for $i, j \in \omega$ define $i \leq_{\mathcal{L}} j$ if and only if $[T'_i] \leq_{\mathrm{s}} [T'_j]$, define $\sup_{\mathcal{L}}(i, j)$ to be the $k \in \omega$ such that $[T'_k] \equiv_{\mathrm{s}} [T'_i \otimes T'_j]$, and define $\inf_{\mathcal{L}}(i, j)$ to be the $k \in \omega$ such that $[T'_k] \equiv_{\mathrm{s}} [0^\frown T'_i \cup 1^\frown T'_j]$. Then $\mathcal{L} \cong \mathcal{E}_{\mathrm{s}}$ and $\deg_{\mathrm{T}}(\mathcal{L}) \leq_{\mathrm{T}} \mathbf{0}'''$. ☐

We prepare to show that every presentation of $\mathcal{E}_{\mathrm{s}}$ as a lattice computes $\mathbf{0}'''$. Let $\{X_n\}_{n\in\omega}$ be a recursive sequence of Π^0_1 classes, and let $m \in \omega$. Define $\bigotimes_{n\in\omega} X_n$ and $\bigotimes_{n\in\omega\setminus\{m\}} X_n$ by

$$\bigotimes_{n\in\omega} X_n = \Big\{ \bigoplus_{n\in\omega} f_n \;\Big|\; \forall n(f_n \in X_n) \Big\} \text{ and}$$
$$\bigotimes_{n\in\omega\setminus\{m\}} X_n = \Big\{ \bigoplus_{n\in\omega\setminus\{m\}} f_n \;\Big|\; \forall n(n \neq m \to f_n \in X_n) \Big\}.$$

The predicates $\forall n(f_n \in X_n)$ and $\forall n(n \neq m \to f_n \in X_n)$ are Π^0_1 because the sequence $\{X_n\}_{n\in\omega}$ is recursive. Hence $\bigotimes_{n\in\omega} X_n$ and $\bigotimes_{n\in\omega\setminus\{m\}} X_n$ are Π^0_1 classes. If $\{S(A_n, B_n)\}_{n\in\omega}$ is a recursive sequence of r.e. separating classes, then one checks that

$$\bigotimes_{n\in\omega} S(A_n, B_n) = S\Big(\bigoplus_{n\in\omega} A_n, \bigoplus_{n\in\omega} B_n\Big) \text{ and}$$
$$\bigotimes_{n\in\omega\setminus\{m\}} S(A_n, B_n) = S\Big(\bigoplus_{n\in\omega\setminus\{m\}} A_n, \Big(\bigoplus_{n\in\omega\setminus\{m\}} B_n\Big) \cup \{\langle m, k\rangle \mid k \in \omega\}\Big).$$

These two Π^0_1 classes are in fact r.e. separating classes because any Π^0_1 class that is a separating class must be an r.e. separating class. If T is a recursive tree such that $[T] = S(A, B)$ for $A, B \subseteq \omega$, then $A = \{n \mid (\exists s > n)(\forall \sigma \in 2^s)(\sigma \in T \to \sigma(n) = 1)\}$ and $B = \{n \mid (\exists s > n)(\forall \sigma \in 2^s)(\sigma \in T \to \sigma(n) = 0)\}$, both of which are r.e.

Lemma 6.4. *Let Q be an r.e. separating class, and let $\varphi(e,m,k,\ell)$ be a recursive predicate. Then there is a recursive sequence of Π^0_1 classes $\{X_{\langle e,m\rangle}\}_{\langle e,m\rangle\in\omega}$ such that for all $e,m\in\omega$*

$$\deg_{\mathrm{s}}\big(X_{\langle e,m\rangle}\big)=\begin{cases}\mathbf{0}_{\mathrm{s}} & \text{if } \forall k\exists\ell\varphi(e,m,k,\ell)\\ \deg_{\mathrm{s}}(Q) & \text{if } \exists k\forall\ell\neg\varphi(e,m,k,\ell).\end{cases}$$

Proof. Let A and B be disjoint r.e. sets such that $Q=S(A,B)$. Let $\{A_s\}_{s\in\omega}$ and $\{B_s\}_{s\in\omega}$ be recursive stage enumerations of A and B respectively. For $e,m\in\omega$, let $X_{\langle e,m\rangle}$ be the r.e. separating class $X_{\langle e,m\rangle}=S\big(C_{\langle e,m\rangle},D_{\langle e,m\rangle}\big)$ where

$$C_{\langle e,m\rangle}=\{\langle k,x\rangle\mid\exists s(x\in A_s\wedge(\forall\ell<s)(\neg\varphi(e,m,k,\ell)))\}\text{ and}$$
$$D_{\langle e,m\rangle}=\{\langle k,x\rangle\mid\exists s(x\in B_s\wedge(\forall\ell<s)(\neg\varphi(e,m,k,\ell)))\}.$$

For all $k\in\omega$, the k^{th} column of $C_{\langle e,m\rangle}$ is a subset of A, and the k^{th} column of $D_{\langle e,m\rangle}$ is a subset of B. Thus $C_{\langle e,m\rangle}$ and $D_{\langle e,m\rangle}$ are disjoint. The sequences $\{C_{\langle e,m\rangle}\}_{\langle e,m\rangle\in\omega}$ and $\{D_{\langle e,m\rangle}\}_{\langle e,m\rangle\in\omega}$ are uniformly r.e., which implies that the sequence $\{X_{\langle e,m\rangle}\}_{\langle e,m\rangle\in\omega}$ is a recursive sequence of Π^0_1 classes.

To see that $X_{\langle e,m\rangle}$ has the desired degree, first suppose that $\forall k\exists\ell\varphi(e,m,k,\ell)$. In this case, the set $C_{\langle e,m\rangle}$ is recursive. To determine if $\langle k,x\rangle\in C_{\langle e,m\rangle}$, search for the least ℓ such that $\varphi(e,m,k,\ell)$, which must exist by assumption. Once ℓ is found, enumerate A up to stage ℓ. Then $\langle k,x\rangle\in C_{\langle e,m\rangle}$ if and only if $x\in A_\ell$. $X_{\langle e,m\rangle}$ contains the characteristic function of $C_{\langle e,m\rangle}$, which we have just shown is recursive, so $\deg_{\mathrm{s}}\big(X_{\langle e,m\rangle}\big)=\mathbf{0}_{\mathrm{s}}$. On the other hand, if $\exists k\forall\ell\neg\varphi(e,m,k,\ell)$, then fix a witnessing k. In this case, the k^{th} column of $C_{\langle e,m\rangle}$ is A, and the k^{th} column of $D_{\langle e,m\rangle}$ is B. Given $f\in 2^\omega$, let f_k be the function $f_k(x)=f(\langle k,x\rangle)$. If f separates $C_{\langle e,m\rangle}$ from $D_{\langle e,m\rangle}$, then f_k separates A from B. Thus the functional $f\mapsto f_k$ witnesses $X_{\langle e,m\rangle}\geq_{\mathrm{s}}Q$. The functional $f\mapsto g$ where $g(\langle i,x\rangle)=f(x)$ always witnesses $Q\geq_{\mathrm{s}}X_{\langle e,m\rangle}$. Hence $\deg_{\mathrm{s}}\big(X_{\langle e,m\rangle}\big)=\deg_{\mathrm{s}}(Q)$. ☐

Lemma 6.5. *If $\mathcal{L}$ is a presentation of $\mathcal{E}_{\mathrm{s}}$ as a lattice, then $\mathbf{0}'''\leq_{\mathrm{T}}\deg_{\mathrm{T}}(\mathcal{L})$.*

Proof. Let $\mathcal{L}=(\omega;\leq_{\mathcal{L}},\sup_{\mathcal{L}},\inf_{\mathcal{L}})$ be a presentation of $\mathcal{E}_{\mathrm{s}}$. Let $f\colon\mathcal{E}_{\mathrm{s}}\to\mathcal{L}$ be an isomorphism. Fix a Σ^0_3-complete set $C\subseteq\omega$. We show how to compute C from $\leq_{\mathcal{L}}\oplus\sup_{\mathcal{L}}\oplus\inf_{\mathcal{L}}$.

By Lemma 4.9, let Q be an r.e. separating class and let $\{S_{0,n}\}_{n\in\omega}$ and $\{S_{1,n}\}_{n\in\omega}$ be recursive sequences of r.e. separating classes such that $\{Q\}\cup\{S_{0,n}\}_{n\in\omega}\cup\{S_{1,n}\}_{n\in\omega}$ is strongly independent. Then let

$$\begin{aligned}
\mathbf{w}_0&=\deg_{\mathrm{s}}(W_0) &&\text{for} && W_0=\operatorname{spine}(Q,\{S_{0,n}\}_{n\in\omega}),\\
\mathbf{w}_1&=\deg_{\mathrm{s}}(W_1) &&\text{for} && W_1=\operatorname{spine}(Q,\{S_{1,n}\}_{n\in\omega}),\\
\mathbf{v}&=\deg_{\mathrm{s}}(V) &&\text{for} && V=\bigotimes_{n\in\omega}S_{0,n},\\
\mathbf{r}&=\deg_{\mathrm{s}}(R) &&\text{for} && R=\operatorname{spine}(Q,\{R_n\}_{n\in\omega}),\ \text{where } R_n=\bigotimes_{m\in\omega\setminus\{n\}}S_{0,m},\\
\mathbf{m}&=\deg_{\mathrm{s}}(M) &&\text{for} && M=\operatorname{spine}(Q,\{S_{0,n}\otimes S_{1,n}\}_{n\in\omega}),\ \text{and}\\
\mathbf{p}&=\deg_{\mathrm{s}}(P) &&\text{for} && P=\operatorname{spine}(Q,\{S_{0,n}\otimes S_{1,n+1}\}_{n\in\omega}).
\end{aligned}$$

Let $\{Z_e\}_{e\in\omega}$ be a recursive sequence containing all Π^0_1 classes as in Lemma 2.3. Let $D\subseteq\omega$ be the set

$$D=\{e\mid\exists n(n\in C\wedge Z_e\leq_{\mathrm{s}}S_{0,n}\wedge V\leq_{\mathrm{s}}Z_e\otimes R_n)\}.$$

D is Σ^0_3 because C is Σ^0_3, the sequences $\{Z_e\}_{e\in\omega}$, $\{S_{0,n}\}_{n\in\omega}$, and $\{R_n\}_{n\in\omega}$ are recursive, and $\leq_{\mathrm{s}}$ is Σ^0_3 by Lemma 2.2. Let $\varphi(e,m,k,\ell)$ be a recursive predicate such that $D=\{e\mid\exists m\forall k\exists\ell\varphi(e,m,k,\ell)\}$.

By Lemma 6.4, let $\{X_{\langle e,m\rangle}\}_{\langle e,m\rangle\in\omega}$ be a recursive sequence of Π^0_1 classes such that for all $e, m \in \omega$

$$\deg_s\big(X_{\langle e,m\rangle}\big) = \begin{cases} \mathbf{0}_s & \text{if } \forall k\exists\ell\varphi(e,m,k,\ell) \\ \deg_s(Q) & \text{if } \exists k\forall\ell\neg\varphi(e,m,k,\ell). \end{cases}$$

Let $\mathbf{x} = \deg_s(X)$ for $X = \operatorname{spine}\big(Q, \{Z_e \otimes X_{\langle e,m\rangle}\}_{\langle e,m\rangle\in\omega}\big)$.

The procedure for determining whether $n \in C$ from $\leq_\mathcal{L} \oplus \sup_\mathcal{L} \oplus \inf_\mathcal{L}$ uses the fixed parameters $f(\mathbf{w}_0)$, $f(\mathbf{w}_1)$, $f(\mathbf{r})$, $f(\mathbf{v})$, $f(\mathbf{m})$, $f(\mathbf{p})$, $f(\deg_s(S_{0,0}))$, and $f(\mathbf{x})$. Given $n \in \omega$ search $\mathcal{L}$ for elements $a_{i,j}$ for $i < 2$ and $1 \leq j \leq n$ and for an element b satisfying the conditions

(i) $a_{i,j}$ meets to $f(\mathbf{w}_i)$ for all $i < 2$ and all $1 \leq j \leq n$,
(ii) $\sup_\mathcal{L}(a_{0,j}, a_{1,j}) \geq_\mathcal{L} f(\mathbf{m})$ for all $1 \leq j \leq n$,
(iii) $\sup_\mathcal{L}(a_{0,j}, a_{1,j+1}) \geq_\mathcal{L} f(\mathbf{p})$ for all $0 \leq j \leq n-1$ (where $a_{0,0} = f(\deg_s(S_{0,0}))$),
(iv) b meets to $f(\mathbf{r})$, and
(v) $\sup_\mathcal{L}(a_{0,n}, b) \geq_\mathcal{L} f(\mathbf{v})$.

When the search is completed, output "yes" if $f(\mathbf{x}) \leq_\mathcal{L} a_{0,n}$ and output "no" otherwise.

First, observe that the above search is recursive in $\leq_\mathcal{L} \oplus \sup_\mathcal{L} \oplus \inf_\mathcal{L}$ because the "meets to" relation is r.e. in $\leq_\mathcal{L} \oplus \sup_\mathcal{L} \oplus \inf_\mathcal{L}$. Furthermore, the search will always terminate because the elements $a_{i,j} = f(\deg_s(S_{i,j}))$ for all $i < 2$ and all $1 \leq j \leq n$ and the element $b = f(\deg_s(R_n))$ satisfy conditions (i)–(v), and the search will eventually find them. Conditions (i) and (iv) follow from Lemma 5.2 item (ii), which says that the meet-irreducibles that meet to $\mathbf{w}_i$ are exactly the $\deg_s(S_{i,j})$ and that the meet-irreducibles that meet to $\mathbf{r}$ are exactly the $\deg_s(R_n)$. Notice that Q and $\{R_j\}_{j\in\omega}$ satisfy the hypothesis of Lemma 5.2 because $\{Q\} \cup \{S_{0,j}\}_{j\in\omega}$ is strongly independent. Conditions (ii) and (iii) are easy to see. For condition (v), it is also easy to see that $S_{0,n} \otimes R_n \equiv_s V$.

We need to show that the procedure outputs "yes" on input n if and only if $n \in C$. Let $a_{i,j}$ for $i < 2$ and $1 \leq j \leq n$ and b be the elements found in the search performed on input n.

Claim. *For all $i < 2$ and all $1 \leq j \leq n$, $a_{i,j} \leq_\mathcal{L} f(\deg_s(S_{i,j}))$.*

Proof of claim. For each $i < 2$ and each $1 \leq j \leq n$, let $A_{i,j}$ be a Π^0_1 class such that $\deg_s(A_{i,j}) = f^{-1}(a_{i,j})$. By condition (i) of the search and Lemma 5.2 item (i), $A_{0,1} \leq_s S_{0,m}$ and $A_{1,1} \leq_s S_{1,k}$ for some $m, k \in \omega$. Condition (iii) implies that $S_{0,0} \otimes S_{1,k} \geq_s P$, which is false by strong independence unless $k = 1$. So $A_{1,1} \leq_s S_{1,1}$. Knowing this, condition (ii) implies that $S_{0,m} \otimes S_{1,1} \geq_s M$, which is false by strong independence unless $m = 1$. So $A_{0,1} \leq_s S_{0,1}$. Now proceed by induction. Let $1 \leq j < n$ and assume that $A_{0,j} \leq_s S_{0,j}$ and that $A_{1,j} \leq_s S_{1,j}$. Just as in the argument for the base case, $A_{0,j+1} \leq_s S_{0,m}$ and $A_{1,j+1} \leq_s S_{1,k}$ for some $m, k \in \omega$. $S_{0,j} \otimes S_{1,k} \geq_s P$ by condition (iii), which implies that $k = j+1$. $S_{0,m} \otimes S_{1,j+1} \geq_s M$ by condition (ii), which implies that $m = j+1$. ☐

At the end of the search, $a_{0,n} \leq_\mathcal{L} f(\deg_s(S_{0,n}))$ by the claim, b meets to $f(\mathbf{r})$ by condition (iv), and $\sup_\mathcal{L}(a_{0,n}, b) \geq_\mathcal{L} f(\mathbf{v})$ by condition (v). By Lemma 5.2 item (i), $b \leq_\mathcal{L} f(\deg_s(R_m))$ for some $m \in \omega$. However, if $m \neq n$, then $S_{0,n} \leq_s R_m$, in which case $S_{0,n} \otimes R_m \equiv_s R_m \not\geq_s V$. Thus it must be that $b \leq_\mathcal{L} f(\deg_s(R_n))$.

Suppose $n \in C$. Since $\{Z_e\}_{e\in\omega}$ lists all the Π^0_1 classes, there is an $e \in \omega$ such that $\deg_s(Z_e) = f^{-1}(a_{0,n})$. This e satisfies $\exists n(n \in C \wedge Z_e \leq_s S_{0,n} \wedge V \leq_s Z_e \otimes R_n)$. Thus $e \in D$, which means $\exists m\forall k\exists\ell\varphi(e,m,k,\ell)$. If m is such that $\forall k\exists\ell\varphi(e,m,k,\ell)$, then we have that $\deg_s(X_{\langle e,m\rangle}) = \mathbf{0}_s$ and $Z_e \otimes X_{\langle e,m\rangle} \equiv_s Z_e$. Thus $X \leq_s Z_e$, which means $f(\mathbf{x}) \leq_\mathcal{L} a_{0,n}$. Thus "yes" was the output.

Suppose $n \notin C$. We show $X \not\leq_s S_{0,n}$.

Claim. *For all $e, m \in \omega$, $S_{0,n} \not\geq_s Z_e \otimes X_{\langle e,m\rangle}$.*

Proof of claim. If $\deg_s(X_{\langle e,m\rangle}) = \deg_s(Q)$, then $S_{0,n} \not\geq_s Z_e \otimes X_{\langle e,m\rangle}$ because $S_{0,n} \not\geq_s Q$ by strong independence. If $\deg_s(X_{\langle e,m\rangle}) = \mathbf{0}_s$, then $\forall k\exists\ell\varphi(e,m,k,\ell)$. Therefore $e \in D$, so there is an n' such that $n' \in C$, $Z_e \leq_s S_{0,n'}$, and $V \leq_s Z_e \otimes R_{n'}$. Notice that $n \neq n'$ because $n \notin C$ and $n' \in C$.

Therefore $S_{0,n} \leq_{\mathrm{s}} R_{n'}$. So if $S_{0,n} \geq_{\mathrm{s}} Z_e$, then $R_{n'} \geq_{\mathrm{s}} Z_e$. So $V \nleq_{\mathrm{s}} R_{n'} \equiv_{\mathrm{s}} Z_e \otimes R_{n'}$, a contradiction. Hence $S_{0,n} \ngeq_{\mathrm{s}} Z_e \otimes X_{\langle e,m\rangle}$. □

Suppose for a contradiction that Φ is such that $\Phi(S_{0,n}) \subseteq X$. If there are $n, m \in \omega$ such that $\Phi(S_{0,n}) \cap \big(\sigma_{\langle e,m\rangle}{}^\frown\big(Z_e \otimes X_{\langle e,m\rangle}\big)\big) \neq \emptyset$, then there is a clopen $C \subseteq 2^\omega$ such that $S_{0,n} \cap C \neq \emptyset$ and $\Phi(S_{0,n} \cap C) \subseteq \sigma_{\langle e,m\rangle}{}^\frown\big(Z_e \otimes X_{\langle e,m\rangle}\big)$. $S_{0,n} \equiv_{\mathrm{s}} S_{0,n} \cap C$ by Lemma 4.3, and $S_{0,n} \cap C \geq_{\mathrm{s}} \big(\sigma_{\langle e,m\rangle}{}^\frown\big(Z_e \otimes X_{\langle e,m\rangle}\big)\big) \equiv_{\mathrm{s}} Z_e \otimes X_{\langle e,m\rangle}$. This contradicts the claim. Thus $\Phi(S_{0,n}) \cap \big(\sigma_{\langle e,m\rangle}{}^\frown\big(Z_e \otimes X_{\langle e,m\rangle}\big)\big) = \emptyset$ for all $e, m \in \omega$. Therefore $\Phi(S_{0,n}) \subseteq Q$. This contradicts $Q \nleq_{\mathrm{s}} S_n$. Hence $X \nleq_{\mathrm{s}} S_{0,n}$. It follows that $f(\mathbf{x}) \nleq_{\mathcal{L}} a_{0,n}$ because $a_{0,n} \leq_{\mathcal{L}} f(\deg_{\mathrm{s}}(S_{0,n}))$. Thus "no" was the output. □

Theorem 6.6. *The degree of $\mathcal{E}_{\mathrm{s}}$ as a lattice is $\mathbf{0}'''$. That is, there is a presentation of $\mathcal{E}_{\mathrm{s}}$ as a lattice recursive in $\mathbf{0}'''$ and $\mathbf{0}'''$ is recursive in every presentation of $\mathcal{E}_{\mathrm{s}}$ as a lattice.*

Proof. Lemma 6.3 proves that there is a presentation recursive in $\mathbf{0}'''$, and Lemma 6.5 proves that $\mathbf{0}'''$ is recursive in every presentation. □

Corollary 6.7. *$\mathcal{E}_{\mathrm{s}}$ has no presentation as a partial order recursive in $\mathbf{0}'$.*

Proof. In any lattice, the relations $\sup(x, y) = z$ and $\inf(x, y) = z$ are definable from the partial order by Π^0_1 formulas. Thus if $\mathcal{E}_{\mathrm{s}}$ had a presentation as a partial order recursive in $\mathbf{0}'$, it would have a presentation as a lattice recursive in $\mathbf{0}''$. This contradicts the theorem. □

Of course the same argument shows that $\mathcal{E}_{\mathrm{s}}$ cannot have a presentation as a partial order recursive in any degree $\mathbf{d}$ such that $\mathbf{d}' <_{\mathrm{T}} \mathbf{0}'''$.

7. Undecidability in $\mathcal{E}_{\mathrm{w}}$

In this section, we code $\mathcal{N}^{\dot{-}}$ in $\mathcal{E}_{\mathrm{w}}$, thereby showing that Σ^0_3-Th$(\mathcal{E}_{\mathrm{w}})$ and Σ^0_4-Th$(\mathcal{E}_{\mathrm{w}}; \leq_{\mathrm{w}})$ are undecidable. In place of separating classes, our coding of $\mathcal{N}^{\dot{-}}$ in $\mathcal{E}_{\mathrm{w}}$ uses Simpson's Σ^0_3 embedding lemma and his embedding of $\mathcal{R}$ into $\mathcal{E}_{\mathrm{w}}$.

Lemma 7.1 (Σ^0_3 *embedding lemma* [38] Lemma 3.3)**.** *Let $S \subseteq \omega^\omega$ be Σ^0_3 and let $P \subseteq 2^\omega$ be a Π^0_1 class. Then there is a Π^0_1 class $Q \subseteq 2^\omega$ such that $Q \equiv_{\mathrm{w}} S \cup P$.*

In the Muchnik case, $\inf(\deg_{\mathrm{w}}(S), \deg_{\mathrm{w}}(P)) = \deg_{\mathrm{w}}(S \cup P)$ for any $S, P \subseteq \omega^\omega$. For this reason, the Σ^0_3 embedding lemma may be phrased as "if S is Σ^0_3 and P is a Π^0_1 class then $\inf(\deg_{\mathrm{w}}(S), \deg_{\mathrm{w}}(P)) \in \mathcal{E}_{\mathrm{w}}$." For our purposes, P is always DNR_2, so $\deg_{\mathrm{w}}(P) = \deg_{\mathrm{w}}(\mathrm{DNR}_2) = \mathbf{1}_{\mathrm{w}}$, the greatest element of $\mathcal{E}_{\mathrm{w}}$.

If A is an r.e. set, then $\{A\}$ is a Σ^0_3 (in fact a Π^0_2) subset of 2^ω. One of Simpson's original applications of his Σ^0_3 embedding lemma is to show that the map $\deg_{\mathrm{T}}(A) \mapsto \inf(\deg_{\mathrm{w}}(\{A\}), \mathbf{1}_{\mathrm{w}})$ is an upper-semilattice embedding of $\mathcal{R}$ into $\mathcal{E}_{\mathrm{w}}$ preserving the least and greatest elements [38]. To show that this map is indeed an embedding, Simpson uses the following variant of the Arslanov completeness criterion, which we also employ.

Lemma 7.2 (see [16] Lemma 4.1 and [40] Theorem V.5.1)**.** *If A is an r.e. set, then* $\mathrm{DNR}_2 \leq_{\mathrm{w}} \{A\}$ *if and only if $A \equiv_{\mathrm{T}} 0'$.*

Proof. It is easy to compute a function in DNR_2 from $0'$. Conversely, if A computes a function in DNR_2, then A computes a function f such that $\forall e(W_{f(e)} \neq W_e)$, where here $\{W_e\}_{e\in\omega}$ is the standard enumeration of the r.e. sets (such an f is called *fixed-point free*; see [16] Lemma 4.1). Thus $A \equiv_{\mathrm{T}} 0'$ by the Arslanov completeness criterion (see [40] Theorem V.5.1). □

For comparison, it is not known whether $\mathcal{R}$ embeds into $\mathcal{E}_{\mathrm{s}}$. See [5] for further results concerning embedding distributive lattices in $\mathcal{E}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{w}}$.

For us, the key property of the degrees $\inf(\deg_{\mathrm{w}}(\{A\}), \mathbf{1}_{\mathrm{w}})$ for r.e. sets A is that they are all meet-irreducible in $\mathcal{E}_{\mathrm{w}}$ (of course these degrees are generally meet-reducible in $\mathcal{D}_{\mathrm{w}}$).

Lemma 7.3. *If A is an r.e. set, then* $\inf(\deg_w(\{A\}), \mathbf{1}_w)$ *is meet-irreducible in* $\mathcal{E}_w$.

Proof. Suppose $\mathbf{x}, \mathbf{y} \in \mathcal{E}_w$ are such that $\inf(\deg_w(\{A\}), \mathbf{1}_w) \geq_w \inf(\mathbf{x}, \mathbf{y})$. Either $\deg_w(\{A\}) \geq_w \mathbf{x}$ or $\deg_w(\{A\}) \geq_w \mathbf{y}$ because $\deg_w(\{A\})$ is the degree of a singleton. As $\mathbf{1}_w \geq_w \mathbf{x}$ and $\mathbf{1}_w \geq_w \mathbf{y}$, either $\inf(\deg_w(\{A\}), \mathbf{1}_w) \geq_w \mathbf{x}$ or $\inf(\deg_w(\{A\}), \mathbf{1}_w) \geq_w \mathbf{y}$. □

If $\{A_n\}_{n \in B}$ is a uniformly r.e. sequence of r.e. sets indexed by a recursive set B (i.e., the set $\{\langle n, m\rangle \mid n \in B \wedge m \in A_n\}$ is r.e.), then $\{A_n\}_{n \in B}$ is a Σ^0_3 subset of 2^ω and it follows that $\inf\big(\deg_w\big(\{A_n\}_{n \in B}\big), \mathbf{1}_w\big) \in \mathcal{E}_w$. In place of Lemma 4.9, we use the following simpler fact.

Lemma 7.4 (see [40] Section VII.2)**.** *There is a uniformly r.e. sequence of r.e. sets $\{A_n\}_{n \in \omega}$ that is strongly independent.*

Notice that Lemma 7.4 is also a consequence of Lemma 4.9. If $\{S(A_n, B_n)\}_{n \in \omega}$ is a recursive sequence of r.e. separating classes that is strongly independent, then $\{A_n\}_{n \in \omega}$ and $\{B_n\}_{n \in \omega}$ are both uniformly r.e. sequences of r.e. sets that are strongly independent.

Now we have the following analog of Lemma 5.2.

Lemma 7.5. *Let $\{A_n\}_{n \in B}$ be an infinite uniformly r.e. sequence of r.e. sets (indexed by a recursive set B) that is a $\leq_T$-antichain. Let* $\mathbf{w} = \inf\big(\deg_w\big(\{A_n\}_{n \in B}\big), \mathbf{1}_w\big)$.

(i) *If $\mathbf{x} \in \mathcal{E}_w$ meets to $\mathbf{w}$, then $\mathbf{x} \leq_w \inf(\deg_w(\{A_n\}), \mathbf{1}_w)$ for some $n \in B$.*
(ii) $E(\mathbf{w}) = \{\inf(\deg_w(\{A_n\}), \mathbf{1}_w) \mid n \in B\}$.

Proof. (i) Let $\mathbf{x} \in \mathcal{E}_w$ be such that $\mathbf{x}$ meets to $\mathbf{w}$, and suppose that $\mathbf{x} \not\leq_w \inf(\deg_w(\{A_n\}), \mathbf{1}_w)$ for all $n \in B$ for a contradiction. Since $\mathbf{x} \leq_w \mathbf{1}_w$, it must be that $\mathbf{x} \not\leq_w \deg_w(\{A_n\})$ for all $n \in B$. Let $\mathbf{y} \in \mathcal{E}_w$ witness that $\mathbf{x}$ meets to $\mathbf{w}$. That is, $\mathbf{y} >_w \mathbf{w}$ and $\inf(\mathbf{x}, \mathbf{y}) = \mathbf{w}$. Let X and Y be Π^0_1 classes such that $\mathbf{x} = \deg_w(X)$ and $\mathbf{y} = \deg_w(Y)$. Then $X \cup Y \leq_w \{A_n\}$ for all $n \in B$. Thus $Y \leq_w \{A_n\}$ for all $n \in B$ because $X \not\leq_w \{A_n\}$ for all $n \in B$. Therefore $Y \leq_w \{A_n\}_{n \in B}$, which implies that $\mathbf{y} \leq_w \mathbf{w}$, a contradiction.

(ii) Let $n \in B$. To see that $\inf(\deg_w(\{A_n\}), \mathbf{1}_w) \in E(\mathbf{w})$, let $\mathbf{y} = \inf\big(\deg_w\big(\{A_i\}_{i \in B \setminus \{n\}}\big), \mathbf{1}_w\big)$. It is easy to check that $\inf(\inf(\deg_w(\{A_n\}), \mathbf{1}_w), \mathbf{y}) = \mathbf{w}$. Moreover, $\inf(\deg_w(\{A_n\}), \mathbf{1}_w) \not\geq_w \mathbf{y}$. This is because $\{A_n\} \not\geq_w \{A_i\}_{i \in B \setminus \{n\}}$ as $\{A_i\}_{i \in B}$ is a $\leq_T$-antichain and because $\{A_n\} \not\geq_w \mathrm{DNR}_2$ by Lemma 7.2 (note that $A_n <_T 0'$ because $\{A_i\}_{i \in B}$ is a $\leq_T$-antichain). Thus $\mathbf{y} >_w \mathbf{w}$, and therefore $\mathbf{y}$ witnesses that $\inf(\deg_w(\{A_n\}), \mathbf{1}_w)$ meets to $\mathbf{w}$. The degree $\inf(\deg_w(\{A_n\}), \mathbf{1}_w)$ is meet-irreducible in $\mathcal{E}_w$ by Lemma 7.3. Thus $\inf(\deg_w(\{A_n\}), \mathbf{1}_w) \in E(\mathbf{w})$.

We have shown that $\{\inf(\deg_w(\{A_n\}), \mathbf{1}_w) \mid n \in B\} \subseteq E(\mathbf{w})$. To see equality, let $\mathbf{x} \in E(\mathbf{w})$. By item (i), $\mathbf{x} \leq_w \inf(\deg_w(\{A_n\}), \mathbf{1}_w)$ for some $n \in B$. $E(\mathbf{w})$ is an antichain by Lemma 3.3 and $\inf(\deg_w(\{A_n\}), \mathbf{1}_w) \in E(\mathbf{w})$, so it must be that $\mathbf{x} = \inf(\deg_w(\{A_n\}), \mathbf{1}_w)$. □

We are now able to code $\mathcal{N}^{\dot{-}}$ in $\mathcal{E}_w$.

Lemma 7.6. *There is a code $\vec{\mathbf{w}}$ in $\mathcal{E}_w$ such that* $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$.

Proof. The proof is very similar to the proof of Lemma 5.3. By Lemma 7.4, let $\{A_{0,n}\}_{n \in \omega}$, $\{A_{1,n}\}_{n \in \omega}$, and $\{A_{2,n}\}_{n \in \omega}$ be uniformly r.e. sequences of r.e. sets such that $\{A_{0,n}\}_{n \in \omega} \cup \{A_{1,n}\}_{n \in \omega} \cup$

$\{A_{2,n}\}_{n\in\omega}$ is strongly independent. Let

$$
\begin{aligned}
\mathbf{w}_0 &= \inf(\deg_{\mathrm{w}}(W_0), \mathbf{1}_{\mathrm{w}}) &&\text{for} && W_0 = \{A_{0,n}\}_{n\in\omega},\\
\mathbf{w}_1 &= \inf(\deg_{\mathrm{w}}(W_1), \mathbf{1}_{\mathrm{w}}) &&\text{for} && W_1 = \{A_{1,n}\}_{n\in\omega},\\
\mathbf{w}_2 &= \inf(\deg_{\mathrm{w}}(W_2), \mathbf{1}_{\mathrm{w}}) &&\text{for} && W_2 = \{A_{2,n}\}_{n\in\omega},\\
\mathbf{m} &= \inf(\deg_{\mathrm{w}}(M), \mathbf{1}_{\mathrm{w}}) &&\text{for} && M = \{A_{0,n}\oplus A_{1,n}\}_{n\in\omega}\cup\{A_{0,n}\oplus A_{2,n}\}_{n\in\omega},\\
\boldsymbol{\ell} &= \inf(\deg_{\mathrm{w}}(L), \mathbf{1}_{\mathrm{w}}) &&\text{for} && L = \{A_{0,i}\oplus A_{1,j} \mid i<j\},\\
\mathbf{p} &= \inf(\deg_{\mathrm{w}}(P), \mathbf{1}_{\mathrm{w}}) &&\text{for} && P = \{A_{0,i}\oplus A_{1,j}\oplus A_{2,k} \mid i+j=k\},\\
\mathbf{t} &= \inf(\deg_{\mathrm{w}}(T), \mathbf{1}_{\mathrm{w}}) &&\text{for} && T = \{A_{0,i}\oplus A_{1,j}\oplus A_{2,k} \mid i\times j=k\},\\
\mathbf{d} &= \inf(\deg_{\mathrm{w}}(D), \mathbf{1}_{\mathrm{w}}) &&\text{for} && D = \{A_{0,i}\oplus A_{1,j}\oplus A_{2,k} \mid i\dot{-} j=k\},\\
\mathbf{z} &= \inf(\deg_{\mathrm{w}}(\{A_{0,0}\}), \mathbf{1}_{\mathrm{w}}), &&\text{and}\\
\mathbf{o} &= \inf(\deg_{\mathrm{w}}(\{A_{0,1}\}), \mathbf{1}_{\mathrm{w}}).
\end{aligned}
$$

To aid readability, let $\mathbf{a}_{i,j} = \deg_{\mathrm{w}}(\{A_{i,j}\})$ for all $i<3$ and $j\in\omega$. By Lemma 7.5 item (ii), $E(\mathbf{w}_0) = \{\inf(\mathbf{a}_{0,n}, \mathbf{1}_{\mathrm{w}})\}_{n\in\omega}$. The map $\inf(\mathbf{a}_{0,n}, \mathbf{1}_{\mathrm{w}}) \mapsto n$ is the isomorphism witnessing $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}} \cong \mathcal{N}^{\dot{-}}$. Clearly $\mathbf{z}\mapsto 0$ and $\mathbf{o}\mapsto 1$. We show that the map preserves $<$. The proofs that the map preserves $+$, $\times$, and $\dot{-}$ are similar. Let $i,j\in\omega$. If $i<j$, then $\inf(\mathbf{a}_{1,j}, \mathbf{1}_{\mathrm{w}})$ meets to $\mathbf{w}_1$ by Lemma 7.5 item (ii), and by distributivity

$$
\begin{aligned}
\sup(\inf(\mathbf{a}_{0,j}, \mathbf{1}_{\mathrm{w}}), \inf(\mathbf{a}_{1,j}, \mathbf{1}_{\mathrm{w}})) &= \inf(\sup(\mathbf{a}_{0,j}, \mathbf{a}_{1,j}), \mathbf{1}_{\mathrm{w}})\\
&= \inf(\deg_{\mathrm{w}}(\{A_{0,j}\oplus A_{1,j}\}), \mathbf{1}_{\mathrm{w}})\\
&\geq_{\mathrm{w}} \mathbf{m}, \text{ and}\\
\sup(\inf(\mathbf{a}_{0,i}, \mathbf{1}_{\mathrm{w}}), \inf(\mathbf{a}_{1,j}, \mathbf{1}_{\mathrm{w}})) &= \inf(\sup(\mathbf{a}_{0,i}, \mathbf{a}_{1,j}), \mathbf{1}_{\mathrm{w}})\\
&= \inf(\deg_{\mathrm{w}}(\{A_{0,i}\oplus A_{1,j}\}), \mathbf{1}_{\mathrm{w}})\\
&\geq_{\mathrm{w}} \boldsymbol{\ell}.
\end{aligned}
$$

Thus $R^2_{\boldsymbol{\ell}}(\inf(\mathbf{a}_{0,i}, \mathbf{1}_{\mathrm{w}}), \inf(\mathbf{a}_{0,j}, \mathbf{1}_{\mathrm{w}}))$. Conversely, suppose that $R^2_{\boldsymbol{\ell}}(\inf(\mathbf{a}_{0,i}, \mathbf{1}_{\mathrm{w}}), \inf(\mathbf{a}_{0,j}, \mathbf{1}_{\mathrm{w}}))$. Let $\mathbf{u}_1\in\mathcal{E}_{\mathrm{w}}$ be such that $\mathbf{u}_1$ meets to $\mathbf{w}_1$, $\sup(\inf(\mathbf{a}_{0,j}, \mathbf{1}_{\mathrm{w}}), \mathbf{u}_1)\geq_{\mathrm{w}}\mathbf{m}$, and $\sup(\inf(\mathbf{a}_{0,i}, \mathbf{1}_{\mathrm{w}}), \mathbf{u}_1)\geq_{\mathrm{w}}\boldsymbol{\ell}$. Since $\mathbf{u}_1$ meets to $\mathbf{w}_1$, it must be that $\mathbf{u}_1\leq_{\mathrm{w}}\inf(\mathbf{a}_{1,k}, \mathbf{1}_{\mathrm{w}})$ for some $k\in\omega$ by Lemma 7.5 item (i). Thus $\sup(\inf(\mathbf{a}_{0,j}, \mathbf{1}_{\mathrm{w}}), \inf(\mathbf{a}_{1,k}, \mathbf{1}_{\mathrm{w}}))\geq_{\mathrm{w}}\mathbf{m}$, so by distributivity

$$\inf(\deg_{\mathrm{w}}(\{A_{0,j}\oplus A_{1,k}\}), \mathbf{1}_{\mathrm{w}}) = \inf(\sup(\mathbf{a}_{0,j}, \mathbf{a}_{1,k}), \mathbf{1}_{\mathrm{w}}) = \sup(\inf(\mathbf{a}_{0,j}, \mathbf{1}_{\mathrm{w}}), \inf(\mathbf{a}_{1,k}, \mathbf{1}_{\mathrm{w}}))\geq_{\mathrm{w}}\mathbf{m}.$$

However, if $k\neq j$, then $\{A_{0,j}\oplus A_{1,k}\}\not\geq_{\mathrm{w}} M$ by strong independence and $\{A_{0,j}\oplus A_{1,k}\}\not\geq_{\mathrm{w}}\mathrm{DNR}_2$ by Lemma 7.2. This implies that $\inf(\deg_{\mathrm{w}}(\{A_{0,j}\oplus A_{1,k}\}), \mathbf{1}_{\mathrm{w}})\not\geq_{\mathrm{w}}\mathbf{m}$, so it must be that $k=j$. Thus $\mathbf{u}_1\leq_{\mathrm{w}}\inf(\mathbf{a}_{1,j}, \mathbf{1}_{\mathrm{w}})$, which implies that $\sup(\inf(\mathbf{a}_{0,i}, \mathbf{1}_{\mathrm{w}}), \inf(\mathbf{a}_{1,j}, \mathbf{1}_{\mathrm{w}}))\geq_{\mathrm{w}}\boldsymbol{\ell}$. Then

$$\inf(\deg_{\mathrm{w}}(\{A_{0,i}\oplus A_{1,j}\}), \mathbf{1}_{\mathrm{w}}) = \inf(\sup(\mathbf{a}_{0,i}, \mathbf{a}_{1,j}), \mathbf{1}_{\mathrm{w}}) = \sup(\inf(\mathbf{a}_{0,i}, \mathbf{1}_{\mathrm{w}}), \inf(\mathbf{a}_{1,j}, \mathbf{1}_{\mathrm{w}}))\geq_{\mathrm{w}}\boldsymbol{\ell}.$$

So if $i\not< j$, then $\{A_{0,i}\oplus A_{1,j}\}\not\geq_{\mathrm{w}} L$ by strong independence and $\{A_{0,i}\oplus A_{1,j}\}\not\geq_{\mathrm{w}}\mathrm{DNR}_2$ by Lemma 7.2, giving the contradiction $\inf(\deg_{\mathrm{w}}(\{A_{0,i}\oplus A_{1,j}\}), \mathbf{1}_{\mathrm{w}})\not\geq_{\mathrm{w}}\boldsymbol{\ell}$. Hence $i<j$. □

Theorem 7.7. Σ^0_3-Th$(\mathcal{E}_{\mathrm{w}})$ *and* Σ^0_4-Th$(\mathcal{E}_{\mathrm{w}};\leq_{\mathrm{w}})$ *are undecidable.*

Proof. There is a code $\vec{\mathbf{w}}$ in $\mathcal{E}_{\mathrm{w}}$ such that $\mathcal{M}^{\dot{-}}_{\vec{\mathbf{w}}}\cong\mathcal{N}^{\dot{-}}$ by Lemma 7.6. The results then follow from Lemma 3.15. □

Clearly then Th$(\mathcal{E}_{\mathrm{w}};\leq_{\mathrm{w}})$ is undecidable. Unfortunately we do not yet know how to prove anything like the finite matching property for $\mathcal{E}_{\mathrm{w}}$ to obtain Th$(\mathcal{N})\leq_1$ Th$(\mathcal{E}_{\mathrm{w}};\leq_{\mathrm{w}})$. The proof of the finite matching property for $\mathcal{E}_{\mathrm{s}}$ (Lemma 4.7 above) appeals to a lemma of Cole and Kihara that grew out of Cenzer and Hinman's proof that $\mathcal{E}_{\mathrm{s}}$ is dense. By analogy, perhaps progress must be made on the density of $\mathcal{E}_{\mathrm{w}}$ before further progress is made on the complexity of Th$(\mathcal{E}_{\mathrm{w}};\leq_{\mathrm{w}})$.

Acknowledgements

Thanks to my advisor Richard A. Shore for introducing me to the Medvedev degrees and Muchnik degrees and for many helpful discussions. This research was partially supported by NSF grants DMS-0554855 and DMS-0852811.

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