EFFECTIVE POWERS OF ω OVER Δ_2 COHESIVE SETS AND INFINITE Π_1 SETS WITHOUT Δ_2 COHESIVE SUBSETS

PAUL SHAFER

ABSTRACT. A *cohesive power* of a computable structure is an effective ultrapower of the structure, where a cohesive set plays the role of an ultrafilter. Let ω, ζ, and η denote the respective order-types of the natural numbers, the integers, and the rationals. We study cohesive powers of computable copies of ω over Δ_2 cohesive sets. We show that there is a computable copy $\mathcal{L}$ of ω such that, for every Δ_2 cohesive set C, the cohesive power of $\mathcal{L}$ over C has order-type $\omega + \eta$. This improves an earlier result of Dimitrov, Harizanov, Morozov, Shafer, A. Soskova, and Vatev by generalizing from Σ_1 cohesive sets to Δ_2 cohesive sets and by computing a single copy of ω that has the desired cohesive power over all Δ_2 cohesive sets. Furthermore, our result is optimal in the sense that Δ_2 cannot be replaced by Π_2. More generally, we show that if $X \subseteq \mathbb{N} \setminus \{0\}$ is a Boolean combination of Σ_2 sets, thought of as a set of finite order-types, then there is a computable copy $\mathcal{L}$ of ω where the cohesive power of $\mathcal{L}$ over any Δ_2 cohesive set has order-type $\omega + \boldsymbol{\sigma}(X \cup \{\omega + \zeta\eta + \omega^*\})$. If X is finite and non-empty, then there is also a computable copy $\mathcal{L}$ of ω where the cohesive power of $\mathcal{L}$ over any Δ_2 cohesive set has order-type $\omega + \boldsymbol{\sigma}(X)$. Here $\boldsymbol{\sigma}$ denotes the *shuffle* operation. An unexpected byproduct of our work is a new method for constructing infinite Π_1 sets that do not have Δ_2 cohesive subsets. In fact, we construct an infinite Π_1 set that does not have a Δ_2 p-cohesive subset. Infinite Π_1 sets without Δ_2 r-cohesive subsets generalize D. Martin's classic co-infinite c.e. set with no maximal superset and have appeared in the work of Lerman, Shore, and Soare.

1. INTRODUCTION

Ultraproducts and ultrapowers are tools from mathematical logic with a wide variety of applications throughout mathematics [2, 3, 10, 13, 17, 24]. The motivating idea behind the ultrapower construction originates with Skolem's construction of a countable non-standard model of arithmetic. Let $\mathfrak{L} = (0, 1, <, +, \times)$ denote the language of arithmetic. Let $\mathcal{N} = (\mathbb{N}; 0, 1, <, +, \times)$ denote the standard model of arithmetic, which has universe $\mathbb{N}$ and interprets the symbols 0, 1, $<$, $+$, and $\times$ in the usual ways. Skolem's construction produces a countable $\mathfrak{L}$-structure $\mathcal{M}$ that is *elementarily equivalent* to $\mathcal{N}$ (i.e., satisfies the same first-order $\mathfrak{L}$-sentences as $\mathcal{N}$) but is not isomorphic to $\mathcal{N}$. It follows that $\mathcal{N}$ cannot be characterized among the countable $\mathfrak{L}$-structures on the basis of first-order statements alone.

Skolem's construction works as follows. For sets $X, Y \subseteq \mathbb{N}$, let $X \subseteq^* Y$ denote that $X \setminus Y$ is finite, and let $\overline{X} = \mathbb{N} \setminus X$ denote the complement of X. It is not hard to show that for any countable sequence $\vec{A} = (A_n : n \in \mathbb{N})$ of subsets of $\mathbb{N}$, there is an infinite set $C \subseteq^* \mathbb{N}$ such that for every n, either $C \subseteq^* A_n$ or $C \subseteq^* \overline{A}_n$. Such a set C is called *cohesive* for $\vec{A}$, or simply $\vec{A}$-*cohesive*. There are only countably many arithmetically definable sets, so we may fix a set C that is cohesive for this collection. Thus for every arithmetical set A, either $C \subseteq^* A$ or $C \subseteq^* \overline{A}$. Now consider the arithmetically definable functions $f \colon \mathbb{N} \to \mathbb{N}$. Given two such functions f and g, the set $\{n : f(n) = g(n)\}$ is arithmetical, so either $C \subseteq^* \{n : f(n) = g(n)\}$ or $C \subseteq^* \{n : f(n) \neq g(n)\}$. Define $f =_C g$ if $C \subseteq^* \{n : f(n) = g(n)\}$, and notice that $=_C$ is an equivalence relation. Let $[f]$ denote the $=_C$-equivalence class of f, and define an $\mathfrak{L}$-structure $\mathcal{M}$ on the $=_C$-equivalence classes in the following way. Interpret 0 and 1 as the equivalence classes of the constant functions $n \mapsto 0$ and $n \mapsto 1$. Interpret $<$ by letting $[f] < [g]$ if and only if $C \subseteq^* \{n : f(n) < g(n)\}$. Interpret $+$ and $\times$ by letting $[f] + [g] = [f + g]$ and $[f] \times [g] = [f \times g]$, where $f + g$ and $f \times g$ are computed pointwise. One then shows that $\mathcal{M}$ is elementarily equivalent to $\mathcal{N}$. Moreover, $\mathcal{M}$ is countable because there are only countably many arithmetical functions, and $\mathcal{M}$ is not isomorphic to $\mathcal{N}$ because $\mathcal{M}$ contains elements with infinitely many predecessors, such as the element represented by the identity function.

Nowadays we may think of Skolem's construction as effectivizing the ultrapower construction, where $C \subseteq^* A$ plays the role of A being in the ultrafilter. Skolem's construction only considers functions $f \colon \mathbb{N} \to \mathbb{N}$ of arithmetical complexity and only requires the $C \subseteq^* A$ or $C \subseteq^* \overline{A}$ dichotomy for sets A of arithmetical complexity. Restricting the complexity of the sequence $\vec{A}$ also induces a bound on the

---

complexity that suffices to produce an $\vec{A}$-cohesive set C. It becomes natural to ask what happens to Skolem's construction when one restricts to even lower complexities. Call a set C:

- *p-cohesive* if C is cohesive for the collection of primitive recursive sets,
- *r-cohesive* if C is cohesive for the collection of computable (aka recursive) sets, and
- *cohesive* if C is cohesive for the collection of computably enumerable (aka recursively enumerable) sets.

Again let $\mathcal{N}$ denote the standard model of arithmetic. Feferman, Scott, and Tennenbaum [11] consider the effective ultrapowers of $\mathcal{N}$ obtained by restricting to computable functions f and by using r-cohesive sets C to take the powers. They show that the resulting structures are never elementarily equivalent to $\mathcal{N}$. In fact, they show that the resulting structures are never even models of Peano arithmetic. Works such as [14, 15, 20, 26–31, 37] continue studying flavors of effective ultrapowers of the particular structure $\mathcal{N}$, including powers over cohesive sets as discussed here as well as the so-called *recursive ultrapowers* and *r.e. ultrapowers*. There is a particular focus on rigidity. Hirschfeld and Wheeler [15] show that the r.e. ultrapowers of $\mathcal{N}$ are rigid, McLaughlin [27] observes that the same holds of the recursive ultrapowers of $\mathcal{N}$, and later McLaughlin [31] shows that the recursive ultrapowers of $\mathcal{N}$ are totally rigid, meaning that they have no non-trivial isomorphic self-embeddings. This line culminates in recent work of Shavrukov [37], which shows that the r.e. ultrapowers of $\mathcal{N}$ are totally rigid as well. Shavrukov also introduces the *r.e. prime powers* of $\mathcal{N}$ (which he shows are equivalent to the powers of $\mathcal{N}$ over cohesive sets) and shows that they need not always be totally rigid.

Dimitrov [5] generalizes from the particular structure $\mathcal{N}$ to an arbitrary computable structure $\mathcal{A}$, and he calls the effective ultrapower of $\mathcal{A}$ over a cohesive set C the *cohesive power* of $\mathcal{A}$ over C. Cohesive powers of computable structures find applications to the lattice of c.e. subspaces of computable vector spaces in [4, 6, 8]. Rigidity is studied as well. For example, Dimitrov, Harizanov, R. Miller, and Mourad show that cohesive powers of the field of rational numbers over Π_1 cohesive sets are rigid [8].

In the classical case of the ultrapower of a structure $\mathcal{A}$ over an ultrafilter $\mathcal{U}$ (say on $\mathbb{N}$), the resulting structure may depend on the choice of the ultrafilter. Consider, for example, the structure $(\mathbb{N}; <)$ of the natural numbers as a linear order. If the continuum hypothesis fails, then there are $2^{2^{\aleph_0}}$ pairwise non-isomorphic linear orders that arise as ultrapowers of $(\mathbb{N}; <)$ over ultrafilters on $\mathbb{N}$ [18]. In the effective case, Lerman [20] shows that when considering a cohesive power of $\mathcal{N}$ over a Π_1 cohesive set C, the resulting structure depends only on the many-one degree of C. Let C and D be Π_1 cohesive sets. If C and D are many-one equivalent, then the cohesive powers of $\mathcal{N}$ over C and D are isomorphic. If C and D are not many-one equivalent, then the cohesive powers of $\mathcal{N}$ over C and D are not even elementarily equivalent.

Classically, if f is an isomorphism between two structures $\mathcal{A}$ and $\mathcal{B}$ and $\mathcal{U}$ is an ultrafilter, then f induces an isomorphism between the ultrapowers of $\mathcal{A}$ and $\mathcal{B}$ over $\mathcal{U}$. In the effective case, if f is a *computable* isomorphism between two computable structures $\mathcal{A}$ and $\mathcal{B}$ and C is a cohesive set, then f induces an isomorphism between the cohesive powers of $\mathcal{A}$ and $\mathcal{B}$ over C. It is essential that the isomorphism f is computable. In [9] it is shown that for every cohesive set C, there are computable structures $\mathcal{A}$ and $\mathcal{B}$ that are isomorphic (but not computably isomorphic) and are such that the cohesive powers of $\mathcal{A}$ and $\mathcal{B}$ over C are not elementarily equivalent. Indeed, $\mathcal{A}$ and $\mathcal{B}$ may be taken to be isomorphic to $(\mathbb{N}; <)$.

If we assume some level of effectivity of a cohesive set C, then it is possible to gain finer control over cohesive powers over C. This leads to many examples of non-elementarily equivalent cohesive powers of isomorphic computable structures. Let ω, ζ, and η denote the order-types of the natural numbers, the integers, and the rationals, and let $+$ and $\cdot$ denote the usual sum and product of linear orders. It is not hard to show that the cohesive power of the usual presentation $(\mathbb{N}; <)$ of ω over any cohesive set is a linear order of type $\omega + \zeta\eta$. This is the expected outcome because $\omega + \zeta\eta$ is familiar as the order-type of countable non-standard models of Peano arithmetic. On the other hand, in [9] it is shown that for every Π_1 cohesive set C, there is a computable linear order $\mathcal{L}$ of type ω such that the cohesive power of $\mathcal{L}$ over C is a linear order of type $\omega + \eta$. Thus $(\mathbb{N}; <)$ and $\mathcal{L}$ are isomorphic linear orders with non-elementarily equivalent cohesive powers over C. In fact, given any Π_1 cohesive set C, there are countably many computable copies of ω whose cohesive powers over C yield pairwise non-elementarily equivalent linear orders. Let $\boldsymbol{\sigma}(X)$ denote the *shuffle* of a non-empty and at-most-countable set of order-types X (see Definition 3.8), and let ω^* denote the reverse of ω. The most general result of [9] is the following, which we discuss further in Section 3.

Theorem 3.19 ([9, Theorem 6.10]). *Let $X \subseteq \mathbb{N} \setminus \{0\}$ be a Boolean combination of Σ_2 sets, thought of as a set of finite order-types. Let C be a Π_1 cohesive set. Then there is a computable copy $\mathcal{L}$ of ω where the cohesive power of $\mathcal{L}$ over C has order-type $\omega + \boldsymbol{\sigma}(X \cup \{\omega + \zeta\eta + \omega^*\})$. Moreover, if X is finite and non-empty, then there is also a computable copy $\mathcal{L}$ of ω where the cohesive power of $\mathcal{L}$ over C has order-type $\omega + \boldsymbol{\sigma}(X)$.*

In Theorem 3.19, the Π_1 cohesive set C is fixed in advance, and the computable copy $\mathcal{L}$ of ω is built so that the cohesive power of $\mathcal{L}$ over C has a particular order-type. The goal of this work is to improve Theorem 3.19 to a form that is optimal with respect to the complexity of C. First, we generalize from Π_1 cohesive sets to Δ_2 cohesive sets. Second, and most significantly, we swap the quantifiers on $\mathcal{L}$ and C by showing that there is a fixed computable copy $\mathcal{L}$ of ω for which the cohesive power of $\mathcal{L}$ over C has the desired order-type for *every* Δ_2 cohesive set C. Our main result is the following.

Theorem 4.5. *Let $X \subseteq \mathbb{N} \setminus \{0\}$ be a Boolean combination of Σ_2 sets, thought of as a set of finite order-types. Then there is a computable copy $\mathcal{L}$ of ω where the cohesive power of $\mathcal{L}$ over any Δ_2 cohesive set has order-type $\omega + \boldsymbol{\sigma}(X \cup \{\omega + \zeta\eta + \omega^*\})$. Moreover, if X is finite and non-empty, then there is also a computable copy $\mathcal{L}$ of ω where the cohesive power of $\mathcal{L}$ over any Δ_2 cohesive set has order-type $\omega + \boldsymbol{\sigma}(X)$.*

In particular, there is a computable copy $\mathcal{L}$ of ω such that the cohesive power of $\mathcal{L}$ over any Δ_2 cohesive set has order-type $\omega + \eta$. This is the optimal statement of this form because for every computable copy $\mathcal{L}$ of ω, there is a Π_2 cohesive set C such that the cohesive power of $\mathcal{L}$ over C is not isomorphic to $\omega + \eta$ (see Proposition 4.3 below).

An unexpected byproduct of Theorem 4.5 is a new method for constructing infinite Π_1 sets that do not have Δ_2 cohesive subsets. In fact, we construct an infinite Π_1 set that has no Δ_2 p-cohesive subset. A classic construction of D. Martin [25] first produced an infinite Π_1 set with no Π_1 cohesive subset. In connection to questions about the automorphisms of the lattice of c.e. sets, Lerman, Shore, and Soare [22] later produced an infinite Π_1 set with the property of having no Δ_2 r-cohesive subset, though this property was not noted explicitly at the time. Recently, Shavrukov [38] indicated further examples of infinite Π_1 sets without Δ_2 r-cohesive subsets which are connected to the notion of a *single-sky* prime filter of the lattice of c.e. sets modulo finite difference [36]. In terms of the arithmetical hierarchy, an infinite Π_1 set having no Δ_2 p-cohesive/r-cohesive/cohesive subset is optimal because every infinite Π_1 set has a Π_2 cohesive subset. We also observe that there are infinite Π_1 sets that do not have Π_1 r-cohesive subsets but do have Δ_2 cohesive subsets.

This article is organized as follows. In Section 2, we isolate a direct construction of an infinite Π_1 set with no Δ_2 p-cohesive subset. We feel that the construction and its connection to the lattice of c.e. sets is of sufficient independent interest to make a self-contained discussion worthwhile. Section 2 also serves to introduce some of the techniques that are used to prove Theorem 4.5. In Section 3, we introduce cohesive products and cohesive powers, and we summarize the results of [9]. We also improve [9, Theorem 2.18] by showing that the cohesive product of a uniformly computable sequence of structures over a Δ_2 cohesive set is always Σ_1-recursively saturated (Theorem 3.6). Finally, we prove Theorem 4.5 in Section 4.

2. Infinite Π_1 sets without Δ_2 cohesive subsets

Our notation mostly follows that of standard sources, such as [21, 34, 39]. Throughout, $\mathbb{N}$ denotes the natural numbers, especially when used as a domain of computation, and ω denotes the order-type of the linear order $(\mathbb{N}; <)$. Partial computable functions are denoted by φ, ψ, etc. For a partial computable function φ, $\varphi(n)\downarrow$ denotes that φ halts on input n and produces an output, and $\varphi(n)\uparrow$ denotes that φ does not halt on input n. Let $(\varphi_e)_{e \in \mathbb{N}}$ denote the usual effective enumeration of all partial computable functions, and let $\varphi_{e,s}(n)$ denote the result (if any) of running φ_e on input n for s computational steps. Let $W_e = \text{dom}(\varphi_e) = \{n : \varphi_e(n)\downarrow\}$ denote the domain of φ_e for each e. For each $n \geq 1$, $\langle x_0, \ldots, x_{n-1} \rangle \colon \mathbb{N}^n \to \mathbb{N}$ denotes the usual computable bijective tupling function that is increasing in all coordinates. For each $i < n$, π_i denotes the corresponding computable projection function onto coordinate i given by $\pi_i(\langle x_0, \ldots, x_{n-1} \rangle) = x_i$. For a function $g \colon \mathbb{N}^2 \to \mathbb{N}$, say that $\lim_s g(n, s) = y$ if $\exists s_0 \, \forall s \geq s_0 \, (g(s) = y)$; and say that $\lim_s g(n, s)$ *exists* if $\exists y \, (\lim_s g(n, s) = y)$. Say that a function $g \colon \mathbb{N}^2 \to \mathbb{N}$ *approximates* a function $h \colon \mathbb{N} \to \mathbb{N}$ if $\forall n \, (\lim_s g(n, s) = h(n))$. The *limit lemma* (see [39, Lemma III.3.3]) says that for every $f, h \colon \mathbb{N} \to \mathbb{N}$, $h \leq_\text{T} f'$ if and only if there is a $g \colon \mathbb{N}^2 \to \mathbb{N}$ with $g \leq_\text{T} f$ that approximates h.

The lattice of c.e. sets modulo finite difference ordered by $\subseteq^*$ is a major object of study in computability theory, and we refer the reader to [34, 39] for a thorough treatment of the subject. Recall that a c.e. set M is called *maximal* if $M \subsetneq^* \mathbb{N}$ and there is no c.e. set X with $M \subsetneq^* X \subsetneq^* \mathbb{N}$. Friedberg [12] shows that maximal sets exist, and D. Martin [25] shows that there is a c.e. set $A \subsetneq^* \mathbb{N}$ that has no maximal superset. It follows from the definitions that for any set $M \subseteq \mathbb{N}$, M is maximal if and only if $\overline{M}$ is an infinite Π_1 cohesive set. Via this correspondence, Martin's result may be rephrased as stating that there is an infinite Π_1 set that has no Π_1 cohesive subset. There is also an infinite Π_1 set that has no Δ_2 r-cohesive subset, which follows from work of Lerman, Shore, and Soare [22].

Let X and Y be c.e. sets where $Y \subseteq X$ and $X \setminus Y$ is infinite. In this situation,

- Y is *r-maximal* in X if $X \setminus Y$ is r-cohesive;
- Y is a *major* subset of X if $\forall$ c.e. sets W $(\overline{X} \subseteq^* W \to \overline{Y} \subseteq^* W)$;
- Y is an *r-maximal major* subset of X if Y is both r-maximal in X and a major subset of X.

Lerman, Shore, and Soare [22, Theorem 1.2] show that a c.e. set X has an r-maximal major subset if and only if it has a Δ_3 *preference function*, where preference functions are defined as follows.

Definition 2.1 ([22, Definition 1.1])**.** For each e, let $R^{e,1} = \{n : (\forall m \le n \; \varphi_e(m)\downarrow) \; \wedge \; \varphi_e(n) = 1\}$, and let $R^{e,0} = \overline{R^{e,1}}$. Notice that $(R^{e,1})_{e\in\mathbb{N}}$ is a uniformly c.e. sequence of the computable sets. An *inclination function*[1] for a set X is a function $h \colon \mathbb{N} \to \{0,1\}$ such that for all n, the set $X \cap \bigcap_{e<n} R^{e,h(e)}$ is infinite. A *preference function* for a set X is a function $h \colon \mathbb{N} \to \{0,1\}$ that is an inclination function for both X and $\overline{X}$.

Lerman, Shore, and Soare observe that if X is a simple c.e. set (i.e., $\overline{X}$ has no infinite c.e. subset), then every inclination function for $\overline{X}$ is automatically a preference function for X. Thus a simple c.e. X has a preference function if and only if $\overline{X}$ has an inclination function.

It is not hard to show that an infinite Δ_2 set has a Δ_2 r-cohesive subset if and only if it has a Δ_3 inclination function.

Proposition 2.2. *Let $B \subseteq \mathbb{N}$ be an infinite Δ_2 set. Then B has a Δ_2 r-cohesive subset if and only if B has a Δ_3 inclination function.*

Proof. First suppose that $h \le_{\mathrm{T}} 0''$ is an inclination function for B. By the limit lemma, there is a function $g \colon \mathbb{N}^2 \to \{0,1\}$ with $g \le_{\mathrm{T}} 0'$ such that $\lim_s g(e,s) = h(e)$ for every e. We compute an increasing enumeration $c_0 < c_1 < c_2 < \cdots$ of an r-cohesive set $C \subseteq B$ from $0'$ using that $B \le_{\mathrm{T}} 0'$, that $g \le_{\mathrm{T}} 0'$, and the fact the sequence $(R^{e,i} : e \in \omega, i \in \{0,1\})$ is uniformly computable from $0'$. The characteristic function of C may then be computed from its increasing enumeration, so C is the desired Δ_2 r-cohesive subset of B.

Let c_0 be the least element of B. Suppose we have already determined $c_0 < c_1 < \cdots < c_n$. To find c_{n+1}, search for the least pair $\langle s, c \rangle$ with $s > n$, $c > c_n$, and $c \in B \cap \bigcap_{e<n} R^{e,g(e,s)}$. Then let $c_{n+1} = c$. Such a pair $\langle s, c \rangle$ always exists because if s is sufficiently large, then $\forall e < n \; (g(e,s) = h(e))$, in which case $B \cap \bigcap_{e<n} R^{e,g(e,s)}$ is infinite.

To check that C is r-cohesive, let X be a computable set, and let e be such that $X = R^{e,1}$. Let s_0 be large enough so that $g(e,s) = h(e)$ for all $s \ge s_0$. Then c_{n+1} is chosen from $R^{e,h(e)}$ whenever $n \ge s_0$. Thus if $h(e) = 1$, then $C \subseteq^* R^{e,1} = X$; and if $h(e) = 0$, then $C \subseteq^* R^{e,0} = \overline{X}$. So C is r-cohesive.

Conversely, suppose that B has a Δ_2 r-cohesive subset C. The sets $\{e : C \subseteq^* R^{e,0}\}$ and $\{e : C \subseteq^* R^{e,1}\}$ are each Σ_3, and they are complements because C is r-cohesive. Thus the sets are Δ_3. Let h be the characteristic function of $\{e : C \subseteq^* R^{e,1}\}$. Then $C \subseteq^* \bigcap_{e<n} R^{e,h(n)}$ for every n, so $B \cap \bigcap_{e<n} R^{e,h(n)}$ is infinite for every n because $C \subseteq B$. Thus h is a Δ_3 inclination function for B. $\square$

Lerman, Shore, and Soare [22, Theorem 2.7] show that there is a simple (indeed, hyperhypersimple) c.e. set X that has no Δ_3 preference function. Therefore $\overline{X}$ is an infinite Π_1 set that has no Δ_3 inclination function and hence no Δ_2 r-cohesive subset.

We give a direct construction of an infinite Π_1 set with no Δ_2 p-cohesive subset. To organize this, we make use of a *uniform sequence containing all Δ_2-approximations* and a computable *sequence of staggered partitions*. A computable function $g \colon \mathbb{N}^2 \to \{0,1\}$ is called a *Δ_2-approximation* to a set $D \subseteq \mathbb{N}$ if $\forall n \, (\lim_s g(n,s) = D(n))$. A set D is Δ_2 if and only if it has a Δ_2-approximation by the limit lemma.

---

[1]We introduce the term *inclination function* here for expository purposes.

Definition 2.3. A *uniform sequence containing all Δ_2-approximations* is a total computable function $g \colon \mathbb{N}^3 \to \{0,1\}$ such that for every Δ_2 set D, there is an e for which the function $g_e \colon \mathbb{N}^2 \to \{0,1\}$ given by $g_e(n,s) = g(e,n,s)$ is a Δ_2-approximation to D.

A uniform sequence $g \colon \mathbb{N}^3 \to \{0,1\}$ containing all Δ_2-approximations may be computed as follows. On input (e,n,s), search for the greatest $t \leq s$ such that $\varphi_{e,s}(n,t)\!\downarrow$. If $\varphi_{e,s}(n,t) = 1$, then output $g(e,n,s) = 1$. If $\varphi_{e,s}(n,t) \neq 1$ (or if there is no $t \leq s$ such that $\varphi_{e,s}(n,t)\!\downarrow$), then output $g(e,n,s) = 0$. Notice that g is total. Write $g_e(n,s)$ for $g(e,n,s)$ for every e, n, and s. If φ_e is total, $\{0,1\}$-valued, and $\lim_s \varphi_e(n,s)$ exists for every n, then $\lim_s g_e(n,s)$ exists and equals $\lim_s \varphi_e(n,s)$ for every n. Thus if D is a Δ_2 set, then some $\varphi_e(n,s)$ is a Δ_2-approximation to D, in which case g_e is also a Δ_2-approximation to D.

If g is a uniform sequence containing all Δ_2-approximations, then every g_e is total, and every Δ_2 set is approximated by some g_e. However, it is not the case that every g_e is a Δ_2-approximation. There are many e and n for which $\lim_s g_e(n,s)$ does not exist.

For each $n \in \mathbb{N}$, let $\{0,1\}^n$ denote the set of binary sequences of length n.

Definition 2.4. A *sequence of staggered partitions* is a sequence $(A^{i,0}, A^{i,1})_{i\in\mathbb{N}}$ of pairs of subsets of $\mathbb{N}$ such that

- for each i, $A^{i,0}$ and $A^{i,1}$ partition $\mathbb{N}$ into two parts (i.e., $A^{i,1} = \overline{A^{i,0}}$), and

- $\forall n \, \forall \sigma \in \{0,1\}^n \, \big(\bigcap_{i<n} A^{i,\sigma(i)}$ is infinite$\big)$.

An infinite $C \subseteq \mathbb{N}$ is *cohesive* for a sequence of staggered partitions $\vec{A} = (A^{i,0}, A^{i,1})_{i\in\mathbb{N}}$, or *$\vec{A}$-cohesive*, if $\forall i\,(C \subseteq^* A^{i,0} \;\vee\; C \subseteq^* A^{i,1})$.

Of course, C being cohesive for the sequence $(A^{i,0}, A^{i,1})_{i\in\mathbb{N}}$ of staggered partitions means the same thing as C being cohesive for the sequence $(A^{i,0})_{i\in\mathbb{N}}$ (or for the sequence $(A^{i,1})_{i\in\mathbb{N}}$).

We may compute a sequence $(A^{i,0}, A^{i,1})_{i\in\mathbb{N}}$ of staggered partitions as follows. Given i, partition $\mathbb{N}$ into successive pieces of size 2^i, let $A^{i,0}$ consist of every other piece, and let $A^{i,1} = \overline{A^{i,0}}$. Indeed, for $i \in \mathbb{N}$ and $j \in \{0,1\}$, we may take $A^{i,j}$ to be the set of numbers whose binary expansions have bit j in position i. Then each $A^{i,j}$ is primitive recursive. In fact, the sequence $(A^{i,0}, A^{i,1})_{i\in\mathbb{N}}$ is uniformly primitive recursive. We show that for any uniformly computable sequence $\vec{A}$ of staggered partitions, there is an infinite Π_1 set with no Δ_2 $\vec{A}$-cohesive subset. If all the sets of $\vec{A}$ are from a particular class, then it follows that there is an infinite Π_1 set with no Δ_2 subset that is cohesive for that class. There is a uniformly primitive recursive sequence of staggered partitions, so there is an infinite Π_1 set with no Δ_2 p-cohesive subset.

Theorem 2.5. *Let $\vec{A} = (A^{i,0}, A^{i,1})_{i\in\mathbb{N}}$ be a uniformly computable sequence of staggered partitions. Then there is an infinite Π_1 set with no Δ_2 $\vec{A}$-cohesive subset.*

Proof. We implement a movable markers style construction of an infinite Π_1 set B with no Δ_2 $\vec{A}$-cohesive subset. For each $i,s \in \mathbb{N}$, let $m_{i,s}$ denote the position of marker m_i at stage s. Say that a number n is *marked* at stage s if $n = m_{i,s}$ for some i, and say that n is *unmarked* at stage s otherwise. At stage 0, we start with $m_{i,0} = i$ for each i. At stage $s > 0$, for each i, we move marker m_i to the previous position of marker m_j for some $j \geq i$. That is, for each i, we set $m_{i,s} = m_{j,s-1}$ for some $j \geq i$. When we move the markers, we do so in such a way as to maintain that at every stage s, only finitely many numbers are unmarked and $\forall i\,(m_{i,s} < m_{i+1,s})$. If a number is unmarked at stage s, then it is unmarked at all later stages. The set of numbers that are ever unmarked during the course of the construction is Σ_1. Its complement is the desired Π_1 set B, which consists of the final positions of all the markers.

Let $g \colon \mathbb{N}^3 \to \{0,1\}$ be a uniform sequence containing all Δ_2-approximations. As above, write $g_e(n,s)$ for $g(e,n,s)$.

If C is an $\vec{A}$-cohesive set, then for every e, either $C \subseteq^* A^{e,0}$ or $C \subseteq^* A^{e,1}$. The goal of the construction is to arrange, for every Δ_2 $\vec{A}$-cohesive set C, that if $C \subseteq^* A^{e,a}$ for $a \in \{0,1\}$, then almost every marker eventually settles on a member of $A^{e,1-a}$. If this is achieved, then $C \subseteq^* A^{e,a}$ and $B \subseteq^* A^{e,1-a}$, so $B \cap C$ is finite. Thus B has no Δ_2 $\vec{A}$-cohesive subset.

For the purposes of this construction, think of a pair $\langle e, N \rangle$ as coding a guess that g_e is a Δ_2-approximation of an $\vec{A}$-cohesive set C and that N is a threshold by which either $\forall n \geq N\,(n \in C \to n \in A^{e,0})$ or $\forall n \geq N\,(n \in C \to n \in A^{e,1})$.

Say that pair $\langle e, N \rangle$ is *active* at stage s if there is a side $a \in \{0, 1\}$ of the partition $(A^{e,0}, A^{e,1})$ along with a witness w with $N < w < s$ meeting the following conditions.

(a) $g_e(w, s) = 1$.

(b) $\forall n \left((N \leq n \leq w \ \wedge \ g_e(n, s) = 1) \ \rightarrow \ n \in A^{e,a} \right)$.

Notice that if $\langle e, N \rangle$ is active at stage s, then there is a unique $a \in \{0, 1\}$ meeting these conditions. Call this a the *active side*, and call the largest witness w (with $N < w < s$) the *activity witness* for $\langle e, N \rangle$ at stage s. To each pair $q = \langle e, N \rangle$, we also associate a counter $\mathrm{ct}(q)$ that counts the number of times that the active side of q changes. The construction now proceeds as follows.

At stage 0, initialize $m_{i,0} = i$ for each i, and initialize $\mathrm{ct}(q) = 0$ for every pair q.

At stage $s > 0$, consider all pairs $q < s$. First update the counters as follows. For each pair $q < s$, if q is active at stage s and either this is the first stage at which q is active or the active side of q is different than it was at the previous stage at which q was active, then update counter $\mathrm{ct}(q)$ to $\mathrm{ct}(q) + 1$.

Now let $i < s$ be the least number for which there is a pair $q = \langle e, N \rangle < s$ meeting the following conditions.

(1) q is active at stage s with active side a and activity witness w for some a and w.

(2) $q < i$ and $\mathrm{ct}(q) < i$.

(3) $m_{i,s-1} < w$.

(4) $m_{i,s-1} \in A^{e,a}$.

If there is no such i, then put $m_{k,s} = m_{k,s-1}$ for each k and go on to the next stage. If there is such an i, then let q be the least witnessing pair. Let

$$R = \{\pi_0(p) : p \leq q \ \wedge \ \mathrm{ct}(p) < i\}.$$

For each $y \in R$, let v be the greatest activity witness yet achieved by any pair of the form $\langle y, M \rangle \leq q$, let t be the most recent stage at which v was achieved, let M be least such that $\langle y, M \rangle \leq q$ achieved activity witness v at stage t, and let a_y be the corresponding active side. If no pair $\langle y, M \rangle \leq q$ has yet been active, then let $a_y = 0$. Now let $j > i$ be the least number such that

$$m_{j,s-1} \in \bigcap_{y \in R} A^{y, 1-a_y}.$$

Such a j exists because the intersection is infinite and almost every number is marked at stage $s - 1$. Advance the markers by setting $m_{k,s} = m_{k,s-1}$ for all $k < i$ and by setting $m_{k,s} = m_{k+j-i,s-1}$ for all $k \geq i$. In this way we maintain that only finitely many numbers are unmarked at stage s and that $\forall k \, (m_{k,s} < m_{k+1,s})$. We say that pair q has now *moved marker* m_i. This completes the construction.

We show that for each i, marker m_i moves only finitely often. Consider marker m_i, and inductively assume that there is a stage s_0 after which no marker m_k with $k < i$ ever moves. Thus after stage s_0, marker m_i is never moved on account of the movement of a marker m_k with $k < i$. So if marker m_i moves after stage s_0, it is because there is a pair q that meets conditions (1)–(4) for i and hence moves m_i. If $q \geq i$, then q always fails condition (2) and hence never moves marker m_i. Thus it suffices to show that each pair $q < i$ moves marker m_i only finitely often. Fix $q_0 < i$, and inductively assume that there is a stage $s_1 > s_0$ after which no pair $q < q_0$ ever moves marker m_i. There are now several cases.

If q_0 is active only finitely often, then condition (1) eventually always fails and therefore q_0 moves m_i only finitely often.

If q_0 changes its active side at least i many times, then condition (2) eventually always fails on account of $\mathrm{ct}(q_0) \geq i$. Thus q_0 moves m_i only finitely often.

If q_0 does not achieve arbitrarily large activity witnesses, then there is a bound W such that the activity witness of q_0 is below W whenever q_0 is active. When marker m_i moves, it moves to mark a larger number than it did previously. That is, if m_i moves at stage s, then $m_{i,s} > m_{i,s-1}$. Marker m_i can move only finitely often before achieving $m_{i,s} \geq W$. If m_i does achieve $m_{i,s} \geq W$ at some stage s, then condition (3) fails at all later stages. Thus q_0 moves m_i only finitely often.

Finally, suppose that pair $q_0 = \langle e_0, N_0 \rangle$ is active infinitely often, that $\mathrm{ct}(q_0) < i$ at all stages, and that q_0 achieves arbitrarily large activity witnesses. As $\mathrm{ct}(q_0) < i$ at all stages, q_0 changes its active side only finitely often and therefore eventually settles on some active side $a \in \{0, 1\}$. That is, there is a stage $s_2 > s_1$ such that q_0 has active side a whenever it is active at a stage $s > s_2$.

Consider the pairs of the form $\langle e_0, M \rangle < i$. Let $s_3 > s_2$ be large enough so that

- each pair $\langle e_0, M \rangle < i$ that eventually achieves $\operatorname{ct}(\langle e_0, M \rangle) \geq i$ has done so by stage s_3, and

- each pair $\langle e_0, M \rangle < i$ with $\operatorname{ct}(\langle e_0, M \rangle) < i$ at all stages has had $\operatorname{ct}(\langle e_0, M \rangle)$ settle on its final value by stage s_3.

The following Claims 1–3 help us complete the argument that pair $q_0 = \langle e_0, N_0 \rangle$ moves marker m_i only finitely often.

Claim 1. *Consider pairs $\langle e, M_0 \rangle$ and $\langle e, M_1 \rangle$ with $M_0 \leq M_1$, where $\langle e, M_0 \rangle$ is active at stage s with active side $b \in \{0, 1\}$ and activity witness $w > M_1$. Then $\langle e, M_1 \rangle$ is also active at stage s with active side b.*

Proof of Claim. Pair $\langle e, M_0 \rangle$ is active at stage s with active side b and activity witness $w > M_1$. Therefore $g_e(w, s) = 1$; and $n \in A^{e,b}$ whenever $M_0 \leq n \leq w$ and $g_e(n, s) = 1$. Thus b and w also witness that that $\langle e, M_1 \rangle$ is active at stage s with active side b because $M_0 \leq M_1 < w$. $\square$

Claim 2. *If a pair $\langle e_0, M \rangle < i$ is active at stage $s > s_3$ with $\operatorname{ct}(\langle e_0, M \rangle) < i$ and an activity witness $w > i$, then its active side is a.*

Proof of Claim. Suppose that $\langle e_0, M \rangle < i$ is active at stage $s > s_3$ with $\operatorname{ct}(\langle e_0, M \rangle) < i$, active side b, and activity witness $w > i$.

First suppose that $M \leq N_0$, and note that $N_0 \leq \langle e_0, N_0 \rangle < i < w$. Then by Claim 1, pair $q_0 = \langle e_0, N_0 \rangle$ is also active at stage s with active side b. However, q_0 has active side a at stage s because $s > s_3 > s_2$. Thus $b = a$, so $\langle e_0, M \rangle$ has active side a at stage s.

Suppose instead that $M > N_0$. Pair q_0 achieves arbitrarily large activity witnesses, so there is a stage $t > s$ at which q_0 is active with active side a (as $t > s_2$) and activity witness $v > i$. Note that $M \leq \langle e_0, M \rangle < i < v$. By Claim 1, pair $\langle e_0, M \rangle$ is also active at stage t with active side a. However, if $b \neq a$, this means that $\langle e_0, M \rangle$ changes its active side from b to a, thereby incrementing $\operatorname{ct}(\langle e_0, M \rangle)$, at some stage between s and t. This contradicts that $\operatorname{ct}(\langle e_0, M \rangle)$ had already stabilized by stage $s_3 < s$. Therefore $b = a$, and pair $\langle e_0, M \rangle$ must have had active side a at stage s. $\square$

Let $s_4 > s_3$ be the first stage at which pair q_0 achieves an activity witness $w > i$ that is also greater than the maximum activity witness achieved by the pairs of the form $\langle e_0, M \rangle$ with $\langle e_0, M \rangle < i$ at stages $s \leq s_3$. Such an s_4 exists by the assumption that q_0 achieves arbitrarily large activity witnesses.

Claim 3. *If marker m_i moves at a stage $s > s_4$, then m_i moves to mark an element of $A^{e_0, 1-a}$. That is, if $s > s_4$ and $m_{i,s} > m_{i,s-1}$, then $m_{i,s} \in A^{e_0, 1-a}$.*

Proof of Claim. Suppose that m_i moves at stage $s > s_4$. As $s_4 > s_1$, it must be that m_i is moved by a pair q with $q_0 \leq q < i$. As $q_0 = \langle e_0, N_0 \rangle$ and $\operatorname{ct}(q_0) < i$, index e_0 is in the set R used by q to move m_i. We show that $a_{e_0} = a$ at stage s. It then follows that $m_{i,s} \in A^{e_0, 1-a}$ because, at stage s, $m_{i,s}$ is set to $m_{j,s-1}$ for an $m_{j,s-1} \in A^{e_0, 1-a}$.

At stage s, the side a_{e_0} is determined by considering the pairs $\langle e_0, M \rangle \leq q$ with $\operatorname{ct}(\langle e_0, M \rangle) < i$, finding the greatest activity witness v yet achieved by any such pair, and by finding the most recent stage t at which this activity witness was achieved. Pair $q_0 = \langle e_0, N_0 \rangle$ is among the considered pairs, and, at stage s_4, q_0 achieves an activity witness $w > i$ that is greater than any activity witnessed ever achieved by a pair $\langle e_0, M \rangle \leq q < i$ at a stage $r \leq s_3$. Thus it must be that $v > i$, and v must be achieved by some $\langle e_0, M \rangle \leq q < i$ with $\operatorname{ct}(\langle e_0, M \rangle) < i$ at some stage $t > s_3$. By Claim 2, a is the active side for any such pair $\langle e_0, M \rangle$ that is active at a stage $t > s_3$ with activity witness $v > i$. Therefore a_{e_0} is chosen to be a at stage s. $\square$

By Claim 3, whenever marker m_i moves after stage s_4, it moves to mark an element of $A^{e_0, 1-a}$. Thus if m_i moves at stage $s > s_4$, then $\forall t \geq s \ (m_{i,t} \in A^{e_0, 1-a})$. In this case condition (4) fails for q_0 at all stages $t > s$, and therefore q_0 cannot move m_i at any stage $t > s$. Thus pair q_0 moves marker m_i at most once after stage s_4, and therefore q_0 moves m_i only finitely often. This completes the proof that pair q_0 moves marker m_i only finitely often, which completes the proof that marker m_i moves only finitely often. Thus the construction indeed produces an infinite Π_1 set B.

We finish the proof by showing that if C is a Δ_2 $\vec{A}$-cohesive set, then the final position of almost every marker is in $\overline{C}$. This shows that $B \cap C$ is finite and therefore that C is not an $\vec{A}$-cohesive subset of B. It follows that B has no Δ_2 $\vec{A}$-cohesive subset.

Let C be a Δ_2 $\vec{A}$-cohesive set, and let e be such that g_e is a Δ_2-approximation to C. By $\vec{A}$-cohesiveness, either $C \subseteq^* A^{e,0}$ or $C \subseteq^* A^{e,1}$. Let $a \in \{0,1\}$ be such that $C \subseteq^* A^{e,a}$, and let N be least such that $\forall n \geq N\ (n \in C \to n \in A^{e,a})$. Let $q = \langle e, N \rangle$.

Claim 4. *Pair q achieves arbitrarily large activity witnesses and eventually settles on active side a.*

Proof of Claim. Given any number $W > N$, let w be the least number with $w > W$ and $w \in C$. Let $s > w$ be large enough so that $g_e(n,s) = C(n)$ for all $n \leq w$. Then q is active at stage s with active side a and activity witness w or greater. Thus q achieves arbitrarily large activity witnesses.

Let n be the least number with $n > N$ and $n \in C$. Let s_0 be large enough so that $\forall m \leq n\ \forall s \geq s_0\ (g_e(m,s) = C(m))$. Then if q is active at a stage $s > s_0$, it must use an activity witness $w \geq n$, in which case its active side must be a because $g_e(n,s) = 1$ and $n \in A^{e,a}$. That is, q has active side a whenever it is active at a stage later than s_0. Thus q eventually settles on active side a. $\square$

By Claim 4, pair q changes its active side only finitely often, so there is a stage s_0 by which $\mathrm{ct}(q)$ has reached its final value, which we also denote $\mathrm{ct}(q)$. Consider an i with $q < i$ and $\mathrm{ct}(q) < i$. Let $s_1 > s_0$ be a stage by which marker m_i has stopped moving: $\forall s > s_1\ (m_{i,s} = m_{i,s_1})$. Then $m_{i,s_1} \in A^{e,1-a}$. If instead $m_{i,s_1} \in A^{e,a}$, then, by Claim 4, let $s > s_1$ be a stage at which q is active with active side a and activity witness $w > m_{i,s-1}$. Then q meets conditions (1)–(4) at stage s because $m_{i,s-1} = m_{i,s_1} \in A^{e,a}$. Thus marker m_i moves at stage s, either directly by some pair or on account of the movement of a marker m_k with $k < i$. This is a contradiction. We have shown that marker m_i settles on a member of $A^{e,1-a}$ whenever i satisfies $q < i$ and $\mathrm{ct}(q) < i$. We therefore have that $B \subseteq^* A^{e,1-a}$ and that $C \subseteq^* A^{e,a}$. Thus $B \cap C$ is finite, as desired. $\square$

Corollary 2.6. *There is an infinite Π_1 set with no Δ_2 p-cohesive subset.*

Proof. Apply Theorem 2.5 to a uniformly primitive recursive sequence $\vec{A}$ of staggered partitions. Then there is an infinite Π_1 set B with no Δ_2 $\vec{A}$-cohesive subset and hence no p-cohesive subset. $\square$

An infinite Π_1 set B with no Δ_2 p-cohesive subset also has no Σ_2 p-cohesive subset. Every infinite Σ_2 set has an infinite Δ_2 subset by the fact that every infinite c.e. set has an infinite computable subset relativized to $0'$. Also, infinite subsets of p-cohesive sets are p-cohesive. Thus if C were a Σ_2 p-cohesive subset of B, then C would have an infinite Δ_2 subset D, which would be a contradictory Δ_2 p-cohesive subset of B. On the other hand, every infinite Π_1 set has a Π_2 cohesive subset by the fact that every infinite computable set has a Π_1 cohesive subset (see [34, Theorem XI], for example) relativized to $0'$. In fact, relativizing to $0'$ yields that every infinite Δ_2 set has a Π_2 subset that is cohesive for the collection of Σ_2 sets. Thus Corollary 2.6 is optimal in terms of the arithmetical hierarchy: there is an infinite Π_1 set with no Δ_2 p-cohesive subset and therefore no Σ_2 p-cohesive subset, whereas every infinite Π_1 set has a Π_2 cohesive subset.

Lastly, we paste together a few facts from the literature to observe that the collection of infinite Π_1 sets without Δ_2 cohesive subsets does not coincide with the collection of infinite Π_1 sets without Π_1 cohesive subsets. Specifically, we observe that there is an infinite Π_1 set that does not have a Π_1 r-cohesive subset but does have a Δ_2 cohesive subset.

Recall that a set $B \subseteq \mathbb{N}$ is called *semi-low$_2$* if $\{e : W_e \cap B \text{ is infinite}\} \leq_T 0''$. We can use the strategy from Jockusch's proof that every set X with $X' \geq_T 0''$ computes a cohesive set [16, Theorem 4.1] to show that every infinite Δ_2 set that is semi-low$_2$ has a Δ_2 cohesive subset. The proof is also similar to that of Proposition 2.2

Proposition 2.7 (Following [16, Theorem 4.1]). *Let $B \subseteq \mathbb{N}$ be an infinite set that is Δ_2 and semi-low$_2$. Then B has a Δ_2 cohesive subset.*

Proof. Given indices $e_0, \ldots, e_n$, we can effectively produce an index e such that $W_e = \bigcap_{i \leq n} W_{e_i}$. Using this and the fact that B is semi-low$_2$, we can define the following function $f \leq_T 0''$ by recursion.

$$f(0) = \text{the least } e \text{ such that } B \cap W_e \text{ is infinite}$$

$$f(n+1) = \text{the least } e > f(n) \text{ such that } B \cap \bigcap_{i \leq n} W_{f(i)} \cap W_e \text{ is infinite.}$$

By the limit lemma, there is a function $g \colon \mathbb{N}^2 \to \mathbb{N}$ with $g \leq_T 0'$ such that $\lim_s g(n,s) = f(n)$ for every n. We compute an increasing enumeration $c_0 < c_1 < c_2 < \cdots$ of a cohesive set $C \subseteq B$ from $0'$ using that $B \leq_T 0'$, that $g \leq_T 0'$, and the fact that the sets W_e are uniformly computable from $0'$. The

characteristic function of C may then be computed from its increasing enumeration, so C is the desired Δ_2 cohesive subset of B.

Let c_0 be the least element of B. Suppose we have already determined $c_0 < c_1 < \cdots < c_n$. To find c_{n+1}, search for the least pair $\langle s, c \rangle$ with $s > n$, $c > c_n$, and $c \in B \cap \bigcap_{i \leq n} W_{g(i,s)}$. Then let $c_{n+1} = c$. Such a pair $\langle s, c \rangle$ always exists because if s is sufficiently large, then $\forall i \leq n\ (g(i,s) = f(i))$, in which case $B \cap \bigcap_{i \leq n} W_{g(i,s)}$ is infinite.

We check that C is cohesive. Suppose that $e \in \mathrm{ran}(f)$. Let i be such that $f(i) = e$, and let s_0 be large enough so that $g(i,s) = f(i) = e$ for all $s \geq s_0$. Then c_{n+1} is chosen from $W_{f(i)} = W_e$ whenever $n \geq \max\{i, s_0\}$. Thus $C \subseteq^* W_e$. Now suppose that $e \notin \mathrm{ran}(f)$, and let n be least such that $f(n) > e$. Then $B \cap \bigcap_{i<n} W_{f(i)} \cap W_e$ must be finite because otherwise we would have $f(n) \leq e$. We just showed that $C \subseteq^* B \cap \bigcap_{i<n} W_{f(i)}$, so $C \cap W_e$ must be finite as well. Therefore C is cohesive. $\square$

Proposition 2.8. *There is an infinite Π_1 set that has no Π_1 r-cohesive subset but does have a Δ_2 cohesive subset.*

Proof. Let A be Lachlan's hyperhypersimple set with no maximal superset from [19]. Recall that a co-infinite c.e. set is called r-maximal if its complement is r-cohesive. Then in fact A has no r-maximal superset because hyperhypersimiplicity is $\subseteq$-upwards closed in the co-infinite c.e. sets and because r-maximality and hyperhypersimiplicity together imply maximality by [39, Proposition 4.5]. So if A had an r-maximal superset X, then X would be a contradictory maximal superset of A. Thus $\overline{A}$ is an infinite Π_1 set with no Π_1 r-cohesive subset. Maass [23] observes that $\overline{A}$ is semi-low$_2$ (see also [39, Section XVI.1]), so $\overline{A}$ has a Δ_2 cohesive subset by Proposition 2.7. Thus $B = \overline{A}$ is an infinite Π_1 set that has no Π_1 r-cohesive subset but does have a Δ_2 cohesive subset. $\square$

One may of course wonder if 'no Π_1 r-cohesive subset' can be improved to 'no Π_1 p-cohesive subset' in Proposition 2.8. We did not attempt to determine this.

3. Cohesive products, cohesive powers, and computable linear orders

We present the necessary background material concerning cohesive products and powers of computable structures, with emphasis on computable linear orders. We refer the reader to [1, 32] as general resources for computable structure theory and to [35] as a general resource for linear orders.

Computable structures, cohesive products, and cohesive powers. Fix a computable language $\mathfrak{L}$. A *computable $\mathfrak{L}$-structure* $\mathcal{A}$ consists of a non-empty computable domain $A \subseteq \mathbb{N}$ and uniformly computable interpretations of the relation, function, and constant symbols of $\mathfrak{L}$. Often the domain of a structure $\mathcal{A}$ (computable or not) is denoted by $|\mathcal{A}|$. Likewise, a *uniformly computable sequence of $\mathfrak{L}$-structures* $(\mathcal{A}_i : i \in \mathbb{N})$ consists of a uniformly computable sequence $(A_i : i \in \mathbb{N})$ of non-empty domains, where $A_i \subseteq \mathbb{N}$ for each i, along with uniformly computable interpretations all the symbols of $\mathfrak{L}$ in the structures $(\mathcal{A}_i : i \in \mathbb{N})$. Equivalently, an $\mathfrak{L}$-structure $\mathcal{A}$ is computable if its atomic diagram is computable, and a sequence of $\mathfrak{L}$-structures $(\mathcal{A}_i : i \in \mathbb{N})$ is uniformly computable if the corresponding sequence of atomic diagrams is uniformly computable. More generally, a computable $\mathfrak{L}$-structure is *decidable* (*n-decidable*) if its elementary diagram (Σ_n-elementary diagram) is computable, and a uniformly computable sequence of $\mathfrak{L}$-structures is *uniformly decidable* (*uniformly n-decidable*) if the corresponding sequence of elementary diagrams (Σ_n-elementary diagrams) is uniformly computable. Note that a 0-decidable $\mathfrak{L}$-structure is the same thing as a computable $\mathfrak{L}$-structure, and a uniformly 0-decidable sequence of $\mathfrak{L}$-structures is the same thing as a uniformly computable sequence of $\mathfrak{L}$-structures.

We present the definition of *cohesive products* and *cohesive powers* as in [9]. See also [5, 7].

Definition 3.1. Let $\mathfrak{L}$ be a computable language. Let $(\mathcal{A}_n : n \in \mathbb{N})$ be a uniformly computable sequence of $\mathfrak{L}$-structures with corresponding uniformly computable sequence of non-empty domains $(|\mathcal{A}_n| : n \in \mathbb{N})$. Let $C \subseteq \mathbb{N}$ be a cohesive set. The *cohesive product of $(\mathcal{A}_n : n \in \mathbb{N})$ over C* is the $\mathfrak{L}$-structure $\prod_C \mathcal{A}_n$ defined as follows.

- Let D be the set of partial computable functions φ such that $\forall n\, (\varphi(n)\downarrow \to \varphi(n) \in |\mathcal{A}_n|)$ and $C \subseteq^* \mathrm{dom}(\varphi)$.
- For $\varphi, \psi \in D$, let $\varphi =_C \psi$ denote $C \subseteq^* \{n : \varphi(n)\downarrow = \psi(n)\downarrow\}$. The relation $=_C$ is an equivalence relation on D. Let $[\varphi]$ denote the equivalence class of $\varphi \in D$ with respect to $=_C$.

- The domain of $\prod_C \mathcal{A}_n$ is the set $|\prod_C \mathcal{A}_n| = \{[\varphi] : \varphi \in D\}$.

- Let R be an m-ary relation symbol of $\mathfrak{L}$. For each $[\varphi_0], \ldots, [\varphi_{m-1}] \in |\prod_C \mathcal{A}_n|$, define $R^{\prod_C \mathcal{A}_n}([\varphi_0], \ldots, [\varphi_{m-1}])$ by

$$R^{\prod_C \mathcal{A}_n}([\varphi_0], \ldots, [\varphi_{m-1}]) \iff C \subseteq^* \{n : R^{\mathcal{A}_n}(\varphi_0(n), \ldots, \varphi_{m-1}(n))\}.$$

  Here, $R^{\mathcal{A}_n}(\varphi_0(n), \ldots, \varphi_{m-1}(n))$ includes the condition that $\varphi_i(n)\!\downarrow$ for each $i < m$.

- Let f be an m-ary function symbol of $\mathfrak{L}$. For each $[\varphi_0], \ldots, [\varphi_{m-1}] \in |\prod_C \mathcal{A}_n|$, let ψ be the partial computable function defined by

$$\psi(n) \simeq f^{\mathcal{A}_n}(\varphi_0(n), \ldots, \varphi_{m-1}(n)),$$

  and notice that $C \subseteq^* \operatorname{dom}(\psi)$ because $C \subseteq^* \operatorname{dom}(\varphi_i)$ for each $i < m$. Define $f^{\prod_C \mathcal{A}_n}$ by

$$f^{\prod_C \mathcal{A}_n}([\varphi_0], \ldots, [\varphi_{m-1}]) = [\psi].$$

- Let c be a constant symbol of $\mathfrak{L}$. Let ψ be the total computable function defined by $\psi(n) = c^{\mathcal{A}_n}$, and define $c^{\prod_C \mathcal{A}_n} = [\psi]$.

In the case where $\mathcal{A}_n$ is the same fixed computable structure $\mathcal{A}$ for every n, the cohesive product $\prod_C \mathcal{A}_n$ is called the *cohesive power of $\mathcal{A}$ over C* and is denoted $\prod_C \mathcal{A}$.

As in the classical case, a computable structure $\mathcal{A}$ always naturally embeds into its cohesive powers. For $a \in |\mathcal{A}|$, let f_a be the total computable function with constant value a. Then for any cohesive set C, the so-called *canonical embedding* $a \mapsto [f_a]$ embeds $\mathcal{A}$ into $\prod_C \mathcal{A}$.

The extent to which analogs of Łoś's theorem hold for cohesive products and powers depends on what formulas are decidable in the structures. We make the following definitions as in [9]. Say that a computable sequence of formulas $(\Phi_i : i \in \mathbb{N})$ is *uniformly decidable* in a uniformly computable sequence of structures $(\mathcal{A}_n : n \in \mathbb{N})$ if there is an algorithm that, given an n, an i, a subformula $\Psi(\vec{y})$ of Φ_i, and a sequence of parameters $\vec{a}$ from $|\mathcal{A}_n|$ of appropriate length, determines whether $\mathcal{A}_n \models \Psi(\vec{a})$. That is, $(\Phi_i : i \in \mathbb{N})$ is uniformly decidable in $(\mathcal{A}_n : n \in \mathbb{N})$ if the set

$$\{\langle n, i, \Psi(\vec{y}), \vec{a}\rangle : \Psi \text{ is a subformula of } \Phi_i \ \wedge \ |\vec{a}| = |\vec{y}| \ \wedge \ \mathcal{A}_n \models \Psi(\vec{a})\}$$

is computable. In the case of a single formula Φ (and its subformulas), we say that Φ is *uniformly decidable* in $(\mathcal{A}_n : n \in \mathbb{N})$.

Lemma 3.2 ([9, Lemma 2.5]). *Let $\mathfrak{L}$ be a computable language, let $(\mathcal{A}_n : n \in \mathbb{N})$ be a uniformly computable sequence of $\mathfrak{L}$-structures, and let C be a cohesive set. Let $\Phi(\vec{x}, \vec{y}, v_0, \ldots, v_{m-1})$ be a formula that is uniformly decidable in $(\mathcal{A}_n : n \in \mathbb{N})$.*

(1) For any $[\varphi_0], \ldots, [\varphi_{m-1}] \in |\prod_C \mathcal{A}_n|$,

$$\prod_C \mathcal{A}_n \models \exists \vec{x} \forall \vec{y} \, \Phi(\vec{x}, \vec{y}, [\varphi_0], \ldots, [\varphi_{m-1}]) \ \Rightarrow \ C \subseteq^* \{n : \mathcal{A}_n \models \exists \vec{x} \forall \vec{y} \, \Phi(\vec{x}, \vec{y}, \varphi_0(n), \ldots, \varphi_{m-1}(n))\}.$$

(2) For any $[\varphi_0], \ldots, [\varphi_{m-1}] \in |\prod_C \mathcal{A}_n|$,

$$C \subseteq^* \{n : \mathcal{A}_n \models \forall \vec{x} \exists \vec{y} \, \Phi(\vec{x}, \vec{y}, \varphi_0(n), \ldots, \varphi_{m-1}(n))\} \ \Rightarrow \ \prod_C \mathcal{A}_n \models \forall \vec{x} \exists \vec{y} \, \Phi(\vec{x}, \vec{y}, [\varphi_0], \ldots, [\varphi_{m-1}]).$$

We obtain the following analogs of Łoś's theorem from Lemma 3.2. We abuse the terminology somewhat by saying that a formula is Δ_n if it is logically equivalent to both a Σ_n formula and a Π_n formula.

Theorem 3.3 ([9, Theorem 2.7]). *Let $\mathfrak{L}$ be a computable language, let $(\mathcal{A}_i : i \in \mathbb{N})$ be a uniformly n-decidable sequence of $\mathfrak{L}$-structures, and let C be a cohesive set.*

(1) Let $\Phi(v_0, \ldots, v_{m-1})$ be a Σ_{n+2} formula. Then for any $[\varphi_0], \ldots, [\varphi_{m-1}] \in |\prod_C \mathcal{A}_i|$,

$$\prod_C \mathcal{A}_i \models \Phi([\varphi_0], \ldots, [\varphi_{m-1}]) \ \Rightarrow \ C \subseteq^* \{i : \mathcal{A}_i \models \Phi(\varphi_0(i), \ldots, \varphi_{m-1}(i))\}.$$

(2) Let $\Phi(v_0, \ldots, v_{m-1})$ be a Π_{n+2} formula. Then for any $[\varphi_0], \ldots, [\varphi_{m-1}] \in |\prod_C \mathcal{A}_i|$,

$$C \subseteq^* \{i : \mathcal{A}_i \models \Phi(\varphi_0(i), \ldots, \varphi_{m-1}(i))\} \ \Rightarrow \ \prod_C \mathcal{A}_i \models \Phi([\varphi_0], \ldots, [\varphi_{m-1}]).$$

(3) *Let* $\Phi(v_0, \ldots, v_{m-1})$ *be a* Δ_{n+2} *formula. Then for any* $[\varphi_0], \ldots, [\varphi_{m-1}] \in |\prod_C \mathcal{A}_i|$,

$$\prod_C \mathcal{A}_i \models \Phi([\varphi_0], \ldots, [\varphi_{m-1})]) \iff C \subseteq^* \{i : \mathcal{A}_i \models \Phi(\varphi_0(i), \ldots, \varphi_{m-1}(i))\}.$$

The analog of Łoś's theorem achieves an extra quantifier in the case of sentences and cohesive powers.

Theorem 3.4 ([9, Theorem 2.9])**.** *Let* $\mathfrak{L}$ *be a computable language, let* $\mathcal{A}$ *be an* n-*decidable* $\mathfrak{L}$-*structure, and let* C *be a cohesive set.*

   (1) *Let* Φ *be a* Δ_{n+3} *sentence. Then* $\mathcal{A} \models \Phi$ *if and only if* $\prod_C \mathcal{A} \models \Phi$.

   (2) *Let* Φ *be a* Σ_{n+3} *sentence. If* $\mathcal{A} \models \Phi$, *then* $\prod_C \mathcal{A} \models \Phi$.

It follows that the full analogs of Łoś's theorem hold when the structures are uniformly decidable. If $(\mathcal{A}_i : i \in \mathbb{N})$ is a uniformly decidable sequence of structures, then the conclusion of Theorem 3.3 item (3) holds for every first-order formula Φ. Similarly, if $\mathcal{A}$ is a decidable structure, then the conclusion of Theorem 3.4 item (1) holds for every first-order sentence Φ.

In [9] it is shown (following [33]) that cohesive products of uniformly decidable sequences of structures are recursively saturated and that, for $n > 0$, cohesive products of uniformly n-decidable sequences of structures are Σ_n-recursively saturated. More interestingly, it is shown that we obtain an extra level of saturation as well as the $n = 0$ case when the cohesive set is assumed to be Π_1: cohesive products of uniformly n-decidable sequences of structures over Π_1 cohesive sets are Σ_{n+1}-recursively saturated [9, Theorem 2.18]. We now show that the same holds for cohesive products over Δ_2 cohesive sets: cohesive products of uniformly n-decidable sequences of structures over Δ_2 cohesive sets are Σ_{n+1}-recursively saturated.

Recall the definitions pertaining to saturation. Let $\mathfrak{L}$ be a language, let $\mathcal{A}$ be an $\mathfrak{L}$-structure, and let $D \subseteq |\mathcal{A}|$ be a collection of parameters from $|\mathcal{A}|$. Let $\mathfrak{L}_D = \mathfrak{L} \cup D$ be the language obtained by augmenting $\mathfrak{L}$ with fresh constant symbols identified with the members of D. A *type (of $\mathcal{A}$) over D* is a set of $\mathfrak{L}_D$-formulas $p(\vec{x}) = p(x_0, \ldots, x_{m-1})$ in m fixed variables $x_0, \ldots, x_{m-1}$ that is finitely satisfied in $\mathcal{A}$: for every $\Phi_0(\vec{x}), \ldots, \Phi_{k-1}(\vec{x}) \in p(\vec{x})$, $\mathcal{A} \models \exists \vec{x} \bigwedge_{i<k} \Phi_i(\vec{x})$. A type $p(\vec{x})$ of $\mathcal{A}$ over D is *realized* if there are $a_0, \ldots, a_{m-1} \in |\mathcal{A}|$ such that for all $\Phi(\vec{x}) \in p(\vec{x})$, $\mathcal{A} \models \Phi(\vec{a})$. A type is a Σ_n-*type* if every formula in the type is Σ_n. Now let $\mathcal{L}$ be a computable language. An $\mathfrak{L}$-structure $\mathcal{A}$ is *recursively saturated* if it realizes every computable type over a finite set of parameters, and it is Σ_n-*recursively saturated* if it realizes every computable Σ_n-type over a finite set of parameters.

Let $p(\vec{x})$ be a type of some structure $\mathcal{A}$ over parameters D. For us, D is always finite or countable, and we enumerate it as a sequence $\vec{c}$ of appropriate length. We write $p(\vec{x}; \vec{c})$ for $p(\vec{x})$ and $\Phi(\vec{x}; \vec{c})$ for a formula of $p(\vec{x}; \vec{c})$ when we want to highlight the parameters. Here, $\Phi(\vec{x}; \vec{c})$ is shorthand for $\Phi(\vec{x}; \vec{c} \restriction k)$, where $\vec{c} \restriction k$ is the shortest initial segment of $\vec{c}$ containing all the parameters appearing in Φ. We also write $\Phi(\vec{x}; \vec{y})$ for the $\mathcal{L}$-formula corresponding to $\Phi(\vec{x}; \vec{c})$, with fresh variables $\vec{y}$ in place of the parameters $\vec{c}$.

The following lemma extends [9, Lemma 2.17] by allowing Δ_2 cohesive sets and by allowing infinite sequences of uniformly partial computable parameters. Say that a sequence $([\psi_\ell] : \ell \in \mathbb{N})$ of elements of some cohesive product is *uniformly partial computable* if the sequence $(\psi_\ell : \ell \in \mathbb{N})$ of representatives is uniformly partial computable.

Lemma 3.5 (Extending [9, Lemma 2.17])**.** *Let* $\mathfrak{L}$ *be a computable language, let* $(\mathcal{A}_n : n \in \mathbb{N})$ *be a uniformly computable sequence of* $\mathfrak{L}$-*structures, and let* C *be a* Δ_2 *cohesive set. Let* $p = p(\vec{x}; \overrightarrow{[\psi]})$ *be a computable type of* $\prod_C \mathcal{A}_n$ *over a uniformly partial computable sequence of parameters* $\overrightarrow{[\psi]} = ([\psi_\ell] : \ell \in \mathbb{N})$. *Assume that* p *consists of formulas of the form* $\exists \vec{z}\, \Phi(\vec{x}, \vec{z}; \overrightarrow{[\psi]})$ *with computable enumeration* $(\exists \vec{z}_i\, \Phi_i(\vec{x}, \vec{z}_i; \overrightarrow{[\psi]}) : i \in \mathbb{N})$. *Further assume that the formulas* $(\Phi_i(\vec{x}, \vec{z}_i; \vec{y}) : i \in \mathbb{N})$ *are uniformly decidable in the structures* $(\mathcal{A}_n : n \in \mathbb{N})$. *Then* $\prod_C \mathcal{A}_n$ *realizes* p.

Proof. As $p(\vec{x}; \overrightarrow{[\psi]})$ is a type,

$$\prod_C \mathcal{A}_n \models \exists \vec{x} \bigwedge_{i<k} \exists \vec{z}_i\, \Phi_i(\vec{x}, \vec{z}_i; \overrightarrow{[\psi]})$$

for each k. To streamline the notation, let $\psi \colon \mathbb{N}^2 \to \mathbb{N}^{<\mathbb{N}}$ be the partial computable function given by $\psi(i, n) \simeq \langle \psi_0(n), \ldots, \psi_{\ell_i - 1}(n) \rangle$, where ℓ_i is least such that $\overrightarrow{[\psi]} \restriction \ell_i$ contains all the parameters of $\overrightarrow{[\psi]}$ appearing in Φ_i. Notice that $C \subseteq^* \{n : \psi(i, n)\downarrow\}$ for every i.

Our goal is to partially compute a function $\theta\colon \mathbb{N} \to \mathbb{N}^m$ so that $C \subseteq^* \mathrm{dom}(\theta)$ and

$$\exists^\infty n \in C \left(\mathcal{A}_n \models \exists \vec{z}_i\, \Phi_i(\theta(n), \vec{z}_i; \psi(i,n))\right) \tag{$*$}$$

for each i. The set

$$\left\{n : \mathcal{A}_n \models \exists \vec{z}_i\, \Phi_i(\theta(n), \vec{z}_i; \psi(i,n))\right\}$$

is c.e. for each i because Φ_i is uniformly decidable in $(\mathcal{A}_n : n \in \mathbb{N})$. Thus $(*)$ implies that

$$\forall^\infty n \in C \left(\mathcal{A}_n \models \exists \vec{z}_i\, \Phi_i(\theta(n), \vec{z}_i; \psi(i,n))\right)$$

for each i by cohesiveness. Once θ has been defined, we let $\varphi_j = \pi_j \circ \theta$ for each $j < m$. We then have that $[\varphi_0], \ldots, [\varphi_{m-1}] \in |\prod_C \mathcal{A}_n|$ and that

$$\prod\nolimits_C \mathcal{A}_n \models \exists \vec{z}_i\, \Phi_i \left(\overrightarrow{[\varphi]}, \vec{z}_i; \overrightarrow{[\psi]}\right)$$

for each i by Lemma 3.2 item (2). Thus $[\varphi], \ldots, [\varphi_{m-1}]$ realize $p(\vec{x}; \overrightarrow{[\psi]})$ in $\prod_C \mathcal{A}_n$.

Let $f\colon \mathbb{N}^2 \to \{0,1\}$ be a Δ_2-approximation to the cohesive set C. Let $(U_k : k \in \mathbb{N})$ be the uniformly c.e. sequence of sets given by

$$U_k = \left\{\langle \vec{a}, n\rangle \in \mathbb{N}^m \times \mathbb{N} : \mathcal{A}_n \models \bigwedge_{i<k} \exists \vec{z}_i\, \Phi_i(\vec{a}, \vec{z}_i; \psi(i,n))\right\}$$

with uniformly computable $\subseteq$-increasing enumerations $(U_{k,s})_{s\in\mathbb{N}}$ for each k. The sequence $(U_k : k \in \mathbb{N})$ is uniformly c.e. because the formulas $(\Phi_i : i \in \mathbb{N})$ are uniformly decidable in $(\mathcal{A}_n : n \in \mathbb{N})$. Notice that if $k \leq j$, then $U_j \subseteq U_k$. We can therefore arrange the enumerations so that if $k \leq j$, then $\forall s\, (U_{j,s} \subseteq U_{k,s})$.

We partially compute θ by computing an increasing sequence $\theta_0 \subseteq \theta_1 \subseteq \theta_2 \subseteq \cdots$ of finite approximations to θ. Start at stage 0 with $\theta_0 = \emptyset$. At stage s, we have θ_s, and we define θ_{s+1}.

Say that n *covers* k *at stage* s if the following conditions hold.

(a) $n > k$.

(b) $f(n, s) = 1$.

(c) $\theta_s(n)\!\downarrow$.

(d) $\langle \theta_s(n), n\rangle \in U_{k,s}$.

If there is an n that covers k at stage s, then we say that k is *covered* at stage s. Let k_s^0 be the least number that is not covered at the start of stage s. If $s > 0$, then let k_s^1 be the least number (if it exists) for which there is an n with $f(n, s) = 0$ that covered k_s^1 at stage $s - 1$, but no $\widehat{n} \leq n$ covers k_s^1 at stage s. If k_s^1 is defined, then let $k_s = \min\{k_s^0, k_s^1\}$. Otherwise, let $k_s = k_s^0$. Now check if there is an $n < s$ meeting the following conditions.

(i) $n > k_s$.

(ii) $f(n, s) = 1$.

(iii) $\theta_s(n)\!\uparrow$.

(iv) There is an $\vec{a}$ with $\langle \vec{a}, n\rangle \in U_{k_s, s}$.

If there is such an n, choose the least such n, choose the first corresponding $\vec{a}$ as in item (iv), and extend θ_s to θ_{s+1} by setting $\theta_{s+1}(n) = \vec{a}$. If there is no such n, then set $\theta_{s+1} = \theta_s$. Now go to stage $s + 1$. This completes the definition of θ.

Suppose that n covers k at some stage s_0. Then the only way that n could fail to cover k at some stage $s > s_0$ is by the failure of condition (b). If, however, $n \in C$, then there is a stage $s_1 > s_0$ such that $\forall s \geq s_1\, (f(n, s) = 1)$. Then n covers k at all stages $s \geq s_1$. In this situation, we say that k is *covered by an* $n \in C$. We show by induction on k that every k is eventually covered by an $n \in C$.

Claim. *Every k is eventually covered by an $n \in C$.*

Proof of Claim. Proceed by induction on k. Let s_0 be a stage by which all $\widehat{k} < k$ have been covered by members of C. Let c be the greatest member of C covering a $\widehat{k} < k$ at stage s_0. Let $s_1 > s_0$ be a stage by which f has settled to its final value on all $n \leq c$: $\forall n \leq c \, \forall s \geq s_1 \, (f(n, s) = C(n))$. Then $k_s \geq k$ at all stages $s > s_1$. By assumption,

$$\prod_C \mathcal{A}_n \models \exists \vec{x} \bigwedge_{i < k} \exists \vec{z}_i \, \Phi_i \left(\vec{x}, \vec{z}_i; \overrightarrow{[\psi]} \right),$$

and therefore

$$C \subseteq^* \left\{ n : \mathcal{A}_n \models \exists \vec{x} \bigwedge_{i < k} \exists \vec{z}_i \, \Phi_i \left(\vec{x}, \vec{z}_i; \psi(i, n) \right) \right\}$$

by Lemma 3.2 item (1). The lemma applies because prenexing the formula $\exists \vec{x} \bigwedge_{i < k} \exists \vec{z}_i \, \Phi_i(\vec{x}, \vec{z}_i; \vec{y})$ yields a formula of the form $\exists \vec{w} \, \Psi(\vec{w}; \vec{y})$, where Ψ is uniformly decidable in $(\mathcal{A}_n : n \in \mathbb{N})$. Let n_0 be least with $n_0 > k$, $n_0 \in C$, $\theta_{s_1}(n_0)\!\!\uparrow$, and $\exists \vec{a} \, (\langle \vec{a}, n_0 \rangle \in U_k)$. If $\theta_s(n_0)$ is defined for the first time during a stage $s > s_1$, then it is to cover a j with $k \leq j < n_0$. That is, if the value of $\theta_s(n_0)$ is determined at stage $s > s_1$, then it is chosen so that $\langle \theta_s(n_0), n_0 \rangle \in U_{j,s} \subseteq U_{k,s}$ for a $j \geq k$. Therefore n_0 covers k at any stage $s > s_1$ at which $\theta_s(n_0)\!\!\downarrow$ and $f(n_0, s) = 1$.

Let $s_2 > \max\{n_0, s_1\}$ be large enough so that $\forall n \leq n_0 \, \forall s \geq s_2 \, (f(n, s) = C(n))$ and that $\exists \vec{a} \, (\langle \vec{a}, n_0 \rangle \in U_{k, s_2})$. Consider stage s_2. If k is not covered at stage s_2, then it must be that $k_{s_2} = k$ and that $\theta_{s_2}(n_0)\!\!\uparrow$. Furthermore, by choice of n_0 and s_2, $n_0 < s_2$ is the least number meeting conditions (i)–(iv) at stage s_2. Therefore $\theta_{s_2+1}(n_0)$ is defined to cover k at stage s_2, so k is covered by an element of C.

Suppose instead that k is covered at stage s_2. Let n be the least number for which there is a stage $s_3 \geq s_2$ at which n covers k. If $n \in C$, then k is covered by an element of C, as desired. If $n \notin C$, then there is a least stage $s > s_3$ with $f(n, s) = 0$. The number n covers k at stage $s - 1$, but by the choice of n, no $\widehat{n} \leq n$ covers k at stage s. Thus $k_s^1 = k$, so $k_s = k$. If $\theta_s(n_0)\!\!\downarrow$, then n_0 must already cover k as observed above. If $\theta_s(n_0)\!\!\uparrow$, then $n_0 < s$ is the least number meeting conditions (i)–(iv) at stage s. Therefore $\theta_{s+1}(n_0)$ is defined to cover k at stage s, so k is covered by an element of C. This completes the proof of the claim. $\qquad\square$

To complete the proof, consider the formula $\exists \vec{z}_i \, \Phi_i$. By the claim, every k is eventually covered by an $n \in C$. Thus for every $k > i$, there is an $n > k$ with $n \in C$, $\theta(n)\!\!\downarrow$, and $\langle \theta(n), n \rangle \in U_k$. Thus $C \subseteq^* \mathrm{dom}(\theta)$ by cohesiveness, and

$$\exists^\infty n \in C \left(\mathcal{A}_n \models \exists \vec{z}_i \, \Phi_i(\theta(n), \vec{z}_i; \psi(i, n)) \right)$$

as desired. $\qquad\square$

Theorem 3.6 (Extending [9, Theorem 2.18])**.** *Let $\mathfrak{L}$ be a computable language, and let C be a Δ_2 cohesive set.*

    (1) *Let $(\mathcal{A}_i : i \in \mathbb{N})$ be a uniformly n-decidable sequence of $\mathfrak{L}$-structures. Then $\prod_C \mathcal{A}_i$ realizes every computable Σ_{n+1}-type over a uniformly partial computable sequence of parameters. In particular, $\prod_C \mathcal{A}_i$ is Σ_{n+1}-recursively saturated.*

    (2) *Let $\mathcal{A}$ be an n-decidable $\mathfrak{L}$-structure. Then $\prod_C \mathcal{A}$ realizes every computable Σ_{n+1}-type over a uniformly partial computable sequence of parameters. In particular, $\prod_C \mathcal{A}$ is Σ_{n+1}-recursively saturated.*

Proof. Item (1) follows from Lemma 3.5. A computable Σ_{n+1}-type can be computably enumerated as $(\exists \vec{z}_j \, \Phi_j : j \in \mathbb{N})$, where Φ_j is Π_n for every j. The formulas $(\Phi_j : j \in \mathbb{N})$ are then uniformly decidable in the uniformly n-decidable sequence of structures $(\mathcal{A}_i : i \in \mathbb{N})$. Item (2) is the special case of item (1) where $\mathcal{A}_i$ is $\mathcal{A}$ for each i. $\qquad\square$

Cohesive products and powers of computable linear orders. A *linear order* $\mathcal{L} = (L; \prec)$ consists of a non-empty set L equipped with a binary relation $\prec$ satisfying the following axioms.

- $\forall x \, (x \not\prec x)$.
- $\forall x \, \forall y \, \forall z \, \big((x \prec y \wedge y \prec z) \rightarrow x \prec z \big)$.
- $\forall x \, \forall y \, (x \prec y \vee x = y \vee y \prec x)$.

Furthermore, a linear order $\mathcal{L}$ is *dense* if $\forall x \, \forall y \, \exists z \, (x \prec y \;\rightarrow\; x \prec z \prec y)$ and *has no endpoints* if $\forall x \, \exists y \, \exists z \, (y \prec x \prec z)$. A computable linear order $\mathcal{L} = (L; \prec_{\mathcal{L}})$ therefore consists of a non-empty computable set $L \subseteq \mathbb{N}$ and a computable relation $\prec_{\mathcal{L}} \subseteq L \times L$ that defines a linear order on L.

Let $\mathcal{L} = (L; \prec_{\mathcal{L}})$ be a linear order, and let $a, b \in L$. Let $\min_{\prec_{\mathcal{L}}}\{a, b\}$ and $\max_{\prec_{\mathcal{L}}}\{a, b\}$ denote the minimum and maximum of a and b with respect to $\prec_{\mathcal{L}}$; let $(a, b)_{\mathcal{L}} = \{x \in L : a \prec_{\mathcal{L}} x \prec_{\mathcal{L}} b\}$ and $[a, b]_{\mathcal{L}} = \{x \in L : a \preceq_{\mathcal{L}} x \preceq_{\mathcal{L}} b\}$ denote the open and closed intervals defined by a and b; and let $|(a, b)_{\mathcal{L}}|$ and $|[a, b]_{\mathcal{L}}|$ denote the cardinalities of the respective intervals. It is convenient to allow $b \prec_{\mathcal{L}} a$ in the interval notation, in which case $(a, b)_{\mathcal{L}} = [a, b]_{\mathcal{L}} = \emptyset$. Let $a \ll_{\mathcal{L}} b$ denote that the interval $(a, b)_{\mathcal{L}}$ is infinite.

Let ω denote the order-type of the natural numbers $(\mathbb{N}; <)$, let ζ denote the order-type of the integers $(\mathbb{Z}; <)$, and let η denote the order-type of the rationals $(\mathbb{Q}; <)$, all with their usual orders. We refer to $(\mathbb{N}; <)$, $(\mathbb{Z}; <)$, and $(\mathbb{Q}; <)$ as the *usual presentations* of ω, ζ, and η. For each $n \geq 1$, let $\boldsymbol{n}$ denote the order-type of the finite linear order $(\{0, 1, \ldots, n-1\}; <)$. For any order-type α, a computable linear order of type α is called a *computable copy* of α. A computable copy of ω is not necessarily isomorphic to the usual presentation of ω via a computable isomorphism. Every countable dense linear order without endpoints has order-type η, and every computable copy of η is computably isomorphic to the usual presentation of η (see [35, Theorem 2.8 and Exercise 16.4]).

Let $\mathcal{L}_0 + \mathcal{L}_1$ and $\mathcal{L}_0 \mathcal{L}_1$ denote the usual sum and product of linear orders $\mathcal{L}_0$ and $\mathcal{L}_1$. Let $\mathcal{L}^*$ denote the reverse of the linear order $\mathcal{L}$. Furthermore, recall the *generalized sum* of a sequence $(\mathcal{M}_\ell : \ell \in |\mathcal{L}|)$ of linear orders indexed by the elements of a linear order $\mathcal{L}$.

Definition 3.7 (see [35, Definition 1.38]). Let $\mathcal{L}$ be a linear order, and let $(\mathcal{M}_\ell : \ell \in |\mathcal{L}|)$ be a sequence of linear orders indexed by $|\mathcal{L}|$. The *generalized sum* $\sum_{\ell \in |\mathcal{L}|} \mathcal{M}_\ell$ of $(\mathcal{M}_\ell : \ell \in |\mathcal{L}|)$ over $\mathcal{L}$ is the linear order $\mathcal{S} = (S; \prec_{\mathcal{S}})$ defined as follows. Write $\mathcal{L} = (L; \prec_{\mathcal{L}})$, and write $\mathcal{M}_\ell = (M_\ell; \prec_{\mathcal{M}_\ell})$ for each $\ell \in L$. Define $S = \{(\ell, m) : \ell \in L \land m \in M_\ell\}$, and define

$$(\ell_0, m_0) \prec_{\mathcal{S}} (\ell_1, m_1) \quad \text{if and only if} \quad (\ell_0 \prec_{\mathcal{L}} \ell_1) \lor (\ell_0 = \ell_1 \land m_0 \prec_{\mathcal{M}_{\ell_0}} m_1).$$

Generalized sums may be used to define the *shuffle* of a non-empty and at-most-countable collection of linear orders.

Definition 3.8 (see [35, Definition 7.14]). Let X be a non-empty collection of linear orders with $|X| \leq \aleph_0$. Let $f : \mathbb{Q} \to X$ be a function such that $f^{-1}(\mathcal{M})$ is dense in $\mathbb{Q}$ for each linear order $\mathcal{M} \in X$. Let $\mathcal{S} = \sum_{q \in \mathbb{Q}} f(q)$ be the generalized sum of the sequence $(f(q) : q \in \mathbb{Q})$ over $\mathbb{Q}$. By density, the order-type of $\mathcal{S}$ does not depend on the particular choice of f. Therefore $\mathcal{S}$ is called the *shuffle* of X and is denoted $\boldsymbol{\sigma}(X)$.

In Definition 3.8, it is helpful to identify each linear order in X with a unique color and think of the function $f : \mathbb{Q} \to X$ as a coloring of $\mathbb{Q}$ in which every color occurs densely. The shuffle $\boldsymbol{\sigma}(X)$ is then obtained by replacing each element of $\mathbb{Q}$ by the linear order with which it is colored. Also, we usually think of the X in a shuffle $\boldsymbol{\sigma}(X)$ as a collection of order-types instead of as a collection of concrete linear orders.

If $\mathcal{L}$ is a computable linear order and $(\mathcal{M}_\ell : \ell \in |\mathcal{L}|)$ is a uniformly computable sequence of linear orders indexed by $|\mathcal{L}|$, then the pairing function may be used to compute a copy of $\sum_{\ell \in |\mathcal{L}|} \mathcal{M}_\ell$. Furthermore, if $(\mathcal{M}_n : n \in \mathbb{N})$ is a uniformly computable sequence of linear orders, then a computable dense coloring $f : \mathbb{Q} \to \mathbb{N}$ of $\mathbb{Q}$ may be used to compute a copy of $\boldsymbol{\sigma}(\{\mathcal{M}_n : n \in \mathbb{N}\})$.

Recall now the *condensations* and in particular the *finite condensation* of a linear order. In general, a condensation of a linear order $\mathcal{L}$ is obtained by partitioning $\mathcal{L}$ into non-empty intervals and then by collapsing each interval to a point.

Definition 3.9. Let $\mathcal{L} = (L; \prec_{\mathcal{L}})$ be a linear order. A *condensation* of $\mathcal{L}$ is any linear order $\mathcal{M} = (M; \prec_{\mathcal{M}})$ obtained by partitioning L into a collection M of non-empty intervals and, for intervals $I, J \in M$, defining $I \prec_{\mathcal{M}} J$ if and only if $\forall a \in I \, \forall b \in J \, (a \prec_{\mathcal{L}} b)$.

Definition 3.10. Let $\mathcal{L} = (L; \prec_{\mathcal{L}})$ be a linear order. For $x \in L$, let $\mathbf{c}_{\mathrm{F}}(x)$ denote the set of $y \in L$ for which there are only finitely many elements between x and y:

$$\mathbf{c}_{\mathrm{F}}(x) = \left\{ y \in L : \text{the interval } \left[\min_{\prec_{\mathcal{L}}}\{x, y\}, \max_{\prec_{\mathcal{L}}}\{x, y\} \right]_{\mathcal{L}} \text{ is finite} \right\}.$$

The set $\mathbf{c}_{\mathrm{F}}(x)$ is always a non-empty interval, as $x \in \mathbf{c}_{\mathrm{F}}(x)$. The *finite condensation* $\mathbf{c}_{\mathrm{F}}(\mathcal{L})$ of $\mathcal{L}$ is the condensation obtained from the partition $\{\mathbf{c}_{\mathrm{F}}(x) : x \in L\}$.

When a linear order $\mathcal{L} = (L; \prec_{\mathcal{L}})$ is partitioned into non-empty intervals as in Definition 3.9, the intervals of the partition are called the *blocks* of $\mathcal{L}$. For the finite condensation of $\mathcal{L}$, the blocks are the sets of the form $\mathbf{c}_F(x)$ for $x \in L$, each of which has order-type either ω, ω^*, ζ, or $\boldsymbol{n}$ for some $n \geq 1$. For $x, y \in L$, we also have that $\mathbf{c}_F(x) \prec_{\mathbf{c}_F(\mathcal{L})} \mathbf{c}_F(y)$ if and only if $x \ll_{\mathcal{L}} y$.

Linear orders are axiomatized by Π_1 sentences, so it follows from Theorem 3.3 item (2) that a cohesive product of a uniformly computable sequence of linear orders is again a linear order. The properties of a linear order being dense and having no endpoints can each be expressed by Π_2 sentences, so also a cohesive product of a uniformly computable sequence of dense linear orders is a dense linear order, and a cohesive product of a uniformly computable sequence of linear orders without endpoints is a linear order without endpoints.

In [9] it is shown that cohesive powers commute with the sum, product, and reverse operations on computable linear orders.

Theorem 3.11 ([9, Theorem 3.6]). *Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be computable linear orders, and let C be a cohesive set. Then*

(1) $\prod_C(\mathcal{L}_0 + \mathcal{L}_1) \cong \prod_C \mathcal{L}_0 + \prod_C \mathcal{L}_1$,

(2) $\prod_C(\mathcal{L}_0 \mathcal{L}_1) \cong \left(\prod_C \mathcal{L}_0\right)\left(\prod_C \mathcal{L}_1\right)$, *and*

(3) $\prod_C(\mathcal{L}_0^*) \cong \left(\prod_C \mathcal{L}_0\right)^*$.

In [9], it is also shown that the finite condensation of the cohesive product of a uniformly computable sequence of linear orders over a Π_1 cohesive set is always dense. Furthermore, it is shown that the finite condensation of the cohesive power of a computable copy of ω over a Π_1 cohesive set always has order-type $\boldsymbol{1} + \eta$. We extend both results to Δ_2 cohesive sets and recall a few helpful lemmas from [9].

Lemma 3.12 ([9, Lemma 3.7]). *Let $(\mathcal{L}_n : n \in \mathbb{N})$ be a uniformly computable sequence of linear orders, let C be a cohesive set, and let $[\psi]$ and $[\varphi]$ be elements of $\prod_C \mathcal{L}_n$. Then the following are equivalent.*

(1) $[\varphi]$ *is the* $\prec_{\prod_C \mathcal{L}_n}$*-immediate successor of* $[\psi]$.

(2) $\forall^\infty n \in C \, \big(\varphi(n) \text{ is the } \prec_{\mathcal{L}_n}\text{-immediate successor of } \psi(n)\big)$.

(3) $\exists^\infty n \in C \, \big(\varphi(n) \text{ is the } \prec_{\mathcal{L}_n}\text{-immediate successor of } \psi(n)\big)$.

Lemma 3.13 ([9, Lemma 3.8]). *Let $(\mathcal{L}_n : n \in \mathbb{N})$ be a uniformly computable sequence of linear orders, let C be a cohesive set, and let $[\psi]$ and $[\varphi]$ be elements of $\prod_C \mathcal{L}_n$. Then the following are equivalent.*

(1) $[\psi] \ll_{\prod_C \mathcal{L}_n} [\varphi]$.

(2) $\lim_{n \in C} |(\psi(n), \varphi(n))_{\mathcal{L}_n}| = \infty$.

(3) $\limsup_{n \in C} |(\psi(n), \varphi(n))_{\mathcal{L}_n}| = \infty$.

Theorem 3.14 (Extending [9, Theorem 3.9]). *Let $(\mathcal{L}_n : n \in \mathbb{N})$ be a uniformly computable sequence of linear orders, and let C be a Δ_2 cohesive set. Then $\mathbf{c}_F(\prod_C \mathcal{L}_n)$ is dense.*

Proof. The cohesive product $\prod_C \mathcal{L}_n$ is Σ_1-recursively saturated by Theorem 3.6, so it suffices to show that the finite condensation of a Σ_1-recursively saturated linear order is dense.

Let $\mathcal{M} = (M; \prec_{\mathcal{M}})$ be a Σ_1-recursively saturated linear order, and let $a, b \in M$ be such that $a \ll_{\mathcal{M}} b$. We need to find a $c \in M$ with $a \ll_{\mathcal{M}} c \ll_{\mathcal{M}} b$. For each $k \in \mathbb{N}$, let $\Phi_k(x; a, b)$ be the following formula (with parameters a and b) expressing that the intervals $(a, x)_{\mathcal{M}}$ and $(x, b)_{\mathcal{M}}$ have at least k elements each:

$$\Phi_k(x; a, b) \quad \equiv \quad \exists w_0, \dots, w_{k-1}, z_0, \dots, z_{k-1} \, (a \prec w_0 \prec \cdots \prec w_{k-1} \prec x \prec z_0 \prec \cdots \prec z_{k-1} \prec b).$$

The set $p(x; a, b) = \{\Phi_k(x; a, b) : k \in \mathbb{N}\}$ is a computable set of Σ_1 formulas, and it is a type over $\{a, b\}$ because the interval $(a, b)_{\mathcal{M}}$ is infinite. Thus $p(x; a, b)$ is realized by some $c \in M$ by Σ_1-recursive saturation. The intervals $(a, c)_{\mathcal{M}}$ and $(c, b)_{\mathcal{M}}$ are therefore both infinite, so $a \ll_{\mathcal{M}} c \ll_{\mathcal{M}} b$ as desired. $\square$

If $\mathcal{L}$ is a computable copy of ω and C is a cohesive set, then the canonical embedding embeds $\mathcal{L}$ into $\prod_C \mathcal{L}$ as an initial segment.

Lemma 3.15 ([9, Lemma 4.1]). *Let $\mathcal{L}$ be a computable copy of ω, and let C be a cohesive set. Then the image of the canonical embedding of $\mathcal{L}$ into $\prod_C \mathcal{L}$ is an initial segment of $\prod_C \mathcal{L}$ of order-type ω.*

Let $\mathcal{L} = (L; \prec_{\mathcal{L}})$ be a computable copy of ω, and let C be a cohesive set. It is straightforward to check that if $\varphi \colon \mathbb{N} \to L$ is a total computable injection, then $[\varphi]$ is not in the image of the canonical embedding. Therefore the cohesive power $\prod_C \mathcal{L}$ has the form $\omega + \mathcal{M}$ for some non-empty linear order $\mathcal{M}$. Call an element of $\prod_C \mathcal{L}$ *standard* if it is in the image of the canonical embedding, and call the element *non-standard* otherwise.

Lemma 3.16 ([9, Lemma 4.2]). *Let $\mathcal{L}$ be a computable copy of ω, let C be a cohesive set, and let $[\varphi]$ be an element of $\prod_C \mathcal{L}$. Then $[\varphi]$ is non-standard if and only if $\liminf_{n \in C} \varphi(n) = \infty$.*

In Lemma 3.16, the condition $\liminf_{n \in C} \varphi(n) = \infty$ may be replaced by $\lim_{n \in C} \varphi(n) = \infty$ and by $\limsup_{n \in C} \varphi(n) = \infty$.

Lemma 3.17 ([9, Lemma 4.3]). *Let $\mathcal{L}$ be a computable copy of ω, let C be a cohesive set, and let $[\varphi]$ be a non-standard element of $\prod_C \mathcal{L}$. Then there are non-standard elements $[\psi^-]$ and $[\psi^+]$ of $\prod_C \mathcal{L}$ with $[\psi^-] \ll_{\prod_C \mathcal{L}} [\varphi] \ll_{\prod_C \mathcal{L}} [\psi^+]$.*

Theorem 3.18 (Extending [9, Theorem 4.4]). *Let $\mathcal{L}$ be a computable copy of ω, and let C be a Δ_2 cohesive set. Then $\mathbf{c}_F(\prod_C \mathcal{L})$ has order-type $\mathbf{1} + \eta$.*

Proof. The standard elements of $\prod_C \mathcal{L}$ form a single initial block by Lemma 3.15. The blocks of $\prod_C \mathcal{L}$ containing the non-standard elements form a countable linear order that is dense by Theorem 3.14 and has no endpoints by Lemma 3.17. Thus $\mathbf{c}_F(\prod_C \mathcal{L}) \cong \mathbf{1} + \eta$. $\square$

The main result of [9] is that for a variety of countable linear orders $\mathcal{M}$ with $\mathbf{c}_F(\mathcal{M}) \cong \mathbf{1} + \eta$ and for a given Π_1 cohesive set C, it is possible to design a computable copy $\mathcal{L}$ of ω that achieves $\prod_C \mathcal{L} \cong \mathcal{M}$.

Theorem 3.19 ([9, Theorem 6.10]). *Let $X \subseteq \mathbb{N} \setminus \{0\}$ be a Boolean combination of Σ_2 sets, thought of as a set of finite order-types. Let C be a Π_1 cohesive set. Then there is a computable copy $\mathcal{L}$ of ω where the cohesive power $\prod_C \mathcal{L}$ has order-type $\omega + \boldsymbol{\sigma}(X \cup \{\omega + \zeta\eta + \omega^*\})$. Moreover, if X is finite and non-empty, then there is also a computable copy $\mathcal{L}$ of ω where the cohesive power $\prod_C \mathcal{L}$ has order-type $\omega + \boldsymbol{\sigma}(X)$.*

The proof of Theorem 3.19 involves a coloring apparatus whereby a computable copy of ω is colored in such a way as to induce a coloring on its cohesive power with certain density properties. We end this section by introducing the coloring apparatus.

Definition 3.20 ([9, Definition 5.1]). A *colored linear order* is a structure $\mathcal{O} = (L, \mathbb{N}; \prec_{\mathcal{L}}, F)$, where $\mathcal{L} = (L; \prec_{\mathcal{L}})$ is a linear order and F is (the graph of) a function $F \colon L \to \mathbb{N}$, thought of as a coloring of L. The colored linear order $\mathcal{O}$ is a *colored copy of ω* if $\mathcal{L} \cong \omega$.

A colored linear order is a two-sorted structure, where one sort is the domain L of the linear order and the other sort is the set $\mathbb{N}$ of colors. The language consists of unary relation symbols for L and $\mathbb{N}$, a binary relation symbol for F, and the binary relation symbol $\prec$. Though we must technically formalize F as a relation, we use the function notation $F(\ell) = d$ instead of the relation notation $F(\ell, d)$.

We typically follow the notational convention of letting $\mathcal{O} = (L, \mathbb{N}; \prec_{\mathcal{L}}, F)$ denote a colored linear order with the additional coloring structure and letting $\mathcal{L} = (L; \prec_{\mathcal{L}})$ denote the underlying linear order. So let $\mathcal{O}$ be a computable colored linear order, let $\mathcal{L}$ be the underlying linear order, and let C be a cohesive set. The cohesive power $\prod_C \mathcal{O}$ consists of a linear order $\prod_C \mathcal{L}$, a collection of colors $\prod_C \mathbb{N}$, and a (graph of a) function $F^{\prod_C \mathcal{O}} \colon |\prod_C \mathcal{L}| \to |\prod_C \mathbb{N}|$ thought of as a coloring of $\prod_C \mathcal{L}$. If $[\varphi]$ is in the domain of $\prod_C \mathcal{O}$, then, by cohesiveness, either $\varphi(n)$ is an element of the linear order (i.e., is of the first sort) for almost every $n \in C$, or $\varphi(n)$ is a color (i.e., is of the second sort) for almost every $n \in C$. Therefore $[\varphi]$ is either an element of the linear order $\prod_C \mathcal{L}$ or an element of the collection of colors $\prod_C \mathbb{N}$. See also the discussion of reducts and substructures of cohesive powers in [9, Section 2]. That $\prec_{\mathcal{L}}$ linear orders L is expressible by a Π_1 sentence, so $\prec_{\prod_C \mathcal{L}}$ indeed linearly orders $|\prod_C \mathcal{L}|$ by Theorem 3.4. That F is the graph of a function with domain L and codomain $\mathbb{N}$ is expressible by a Π_2 sentence, so $F^{\prod_C \mathcal{O}}$ is indeed the graph of a function with domain $|\prod_C \mathcal{L}|$ and codomain $|\prod_C \mathbb{N}|$ by Theorem 3.4 as well.

Again let $\mathcal{O} = (L, \mathbb{N}; \prec_{\mathcal{L}}, F)$ be a computable colored linear order, let $\mathcal{L} = (L; \prec_{\mathcal{L}})$, and let C be a cohesive set. To disambiguate between the elements of the linear order $\prod_C \mathcal{L}$ and the colors of $\prod_C \mathbb{N}$ inside the cohesive power $\prod_C \mathcal{O}$, we write $[\varphi]$ for the elements of $\prod_C \mathcal{L}$, and we write $[\![\delta]\!]$ for the elements of $\prod_C \mathbb{N}$. Call a color $[\![\delta]\!] \in |\prod_C \mathbb{N}|$ a *solid color* if δ is eventually constant on C: $\exists d \in \mathbb{N} \, \forall^\infty n \in C \, (\delta(n) = d)$. Otherwise, call $[\![\delta]\!]$ a *striped color*.

We want to color a computable copy of ω so that in the cohesive power by a given cohesive set, between any two distinct non-standard elements there are elements of every solid color plus at least one element of a striped color. In this situation, we say that the cohesive power is *colorful*.

Definition 3.21 ([9, Definition 5.2]). Let $\mathcal{O} = (L, \mathbb{N}; \prec_\mathcal{L}, F)$ be a computable colored copy of ω, and let $\mathcal{L}$ denote $(L; \prec_\mathcal{L})$. Let C be a cohesive set. The cohesive power $\prod_C \mathcal{O}$ is *colorful* if the following items hold.

(1) For every pair of non-standard elements $[\varphi], [\psi] \in |\prod_C \mathcal{L}|$ with $[\psi] \prec_{\prod_C \mathcal{L}} [\varphi]$ and every solid color $[\![\delta]\!] \in |\prod_C \mathbb{N}|$, there is a $[\theta] \in |\prod_C \mathcal{L}|$ with $[\psi] \prec_{\prod_C \mathcal{L}} [\theta] \prec_{\prod_C \mathcal{L}} [\varphi]$ and $F^{\prod_C \mathcal{O}}([\theta]) = [\![\delta]\!]$.

(2) For every pair of non-standard elements $[\varphi], [\psi] \in |\prod_C \mathcal{L}|$ with $[\psi] \prec_{\prod_C \mathcal{L}} [\varphi]$, there is a $[\theta] \in |\prod_C \mathcal{L}|$ with $[\psi] \prec_{\prod_C \mathcal{L}} [\theta] \prec_{\prod_C \mathcal{L}} [\varphi]$ where $F^{\prod_C \mathcal{O}}([\theta])$ is a striped color.

In Definition 3.21, item (1) implies item (2) when the cohesive set is Δ_2. Let $\mathcal{O} = (L, \mathbb{N}; \prec_\mathcal{L}, F)$ be a computable colored copy of ω, let C be a Δ_2 cohesive set, and suppose that $\prod_C \mathcal{O}$ satisfies Definition 3.21 item (1). Let $[\varphi]$ and $[\psi]$ be non-standard elements of $\prod_C \mathcal{L}$ with $[\psi] \prec_{\prod_C \mathcal{L}} [\varphi]$. Let $\delta_i : \mathbb{N} \to \mathbb{N}$ be the constant function with value i for each i, so that $([\![\delta_i]\!] : i \in \mathbb{N})$ is a uniformly computable sequence of all the solid colors of $\prod_C \mathbb{N}$. Now consider the computable Σ_1-type $p(x; [\varphi], [\psi], [\![\delta_0]\!], [\![\delta_1]\!], \dots)$ consisting of the formula expressing $[\psi] \prec x \prec [\varphi]$ and, for each i, the formula expressing $F(x) \neq [\![\delta_i]\!]$. The set of formulas p is indeed a type because every solid color occurs between $[\varphi]$ and $[\psi]$. Thus p is realized by some $[\theta] \in |\prod_C \mathcal{L}|$ by Theorem 3.6 item (2). Then $[\psi] \prec_{\prod_C \mathcal{L}} [\theta] \prec_{\prod_C \mathcal{L}} [\varphi]$, and $F^{\prod_C \mathcal{O}}([\theta])$ is a striped color because it is not equal to any solid color. Thus $\prod_C \mathcal{O}$ also satisfies Definition 3.21 item (2).

Given a Π_1 cohesive set C, we can compute a colored copy $\mathcal{O}$ of ω such that $\prod_C \mathcal{O}$ is colorful by [9, Theorem 5.3]. In Section 4, we compute a single colored copy $\mathcal{O}$ of ω such that $\prod_C \mathcal{O}$ is colorful for *every* Δ_2 cohesive set C.

If $\mathcal{O} = (R, \mathbb{N}; \prec_\mathcal{R}, F)$ is a computable colored copy of ω, $\mathcal{R}$ denotes $(R; \prec_\mathcal{R})$, and C is a cohesive set for which $\prod_C \mathcal{O}$ is colorful, then $\prod_C \mathcal{R} \cong \omega + \eta$. This is because the standard elements of $\prod_C \mathcal{R}$ form an initial segment of order-type ω by Lemma 3.15, the non-standard elements have no endpoints by Lemma 3.17, and the colorfulness of $\prod_C \mathcal{O}$ implies that the non-standard elements of $\prod_C \mathcal{R}$ are dense. In [9, Section 6], order-types other than $\omega + \eta$ are achieved by starting with $\mathcal{R}$ and computing another copy $\mathcal{L}$ of ω by replacing each element of $\mathcal{R}$ by some finite linear order depending on the element's color and the desired order-type. The technique for producing $\mathcal{L}$ from $\mathcal{O}$ does not depend on the cohesive set C: so long as $\prod_C \mathcal{O}$ is colorful, the cohesive power $\prod_C \mathcal{L}$ has the desired order-type.

Lemma 3.22 ([9, Lemmas 6.5 and 6.9]). *Let $X \subseteq \mathbb{N} \setminus \{0\}$ be a Boolean combination of Σ_2 sets, thought of as a set of finite order-types. Let $\mathcal{O}$ be a computable colored copy of ω.*

- *There is a computable copy $\mathcal{L}$ of ω (constructed from $\mathcal{O}$) such that for every cohesive set C, if $\prod_C \mathcal{O}$ is colorful, then $\prod_C \mathcal{L}$ has order-type $\omega + \boldsymbol{\sigma}(X \cup \{\omega + \zeta\eta + \omega^*\})$.*

- *Moreover, if X is finite and non-empty, then there is also a computable copy $\mathcal{L}$ of ω (constructed from $\mathcal{O}$) such that for every cohesive set C, if $\prod_C \mathcal{O}$ is colorful, then $\prod_C \mathcal{L}$ has order-type $\omega + \boldsymbol{\sigma}(X)$.*

We emphasize that the linear order $\mathcal{L}$ in the conclusion of Lemma 3.22 is not the underlying linear order of $\mathcal{O}$, but it is obtained from the underlying linear order of $\mathcal{O}$ by replacing its elements by finite linear orders. Theorem 3.19 is achieved by starting with a Π_1 cohesive set C, computing a colored copy $\mathcal{O}$ of ω such that $\prod_C \mathcal{O}$ is colorful as in [9, Theorem 5.3], and then applying Lemma 3.22.

4. Cohesive powers of ω over Δ_2 cohesive sets

The goal of this section is to compute a single computable colored copy $\mathcal{O}$ of ω such that $\prod_C \mathcal{O}$ is colorful for every Δ_2 cohesive set C. As a corollary, we obtain an infinite Π_1 set with no Δ_2 cohesive subset, which is a weaker version of Corollary 2.6. We then apply Lemma 3.22 to show that there are computable copies of ω whose cohesive powers over Δ_2 cohesive sets have order-types of the form ω + various shuffles.

Theorem 4.1. *There is a computable colored copy $\mathcal{O}$ of ω such that $\prod_C \mathcal{O}$ is colorful whenever C is a Δ_2 cohesive set.*

Proof. We compute a colored copy $\mathcal{O} = (L, \mathbb{N}; \prec_{\mathcal{L}}, F)$ of ω so that $\prod_C \mathcal{O}$ is colorful whenever C is a Δ_2 cohesive set. We take the domain of the underlying linear order $\mathcal{L} = (L; \prec_{\mathcal{L}})$ to be $L = \mathbb{N}$. Given a cohesive set C, recall that an element $[\varphi]$ of $\prod_C \mathcal{L}$ is non-standard if and only if $\lim_{n \in C} \varphi(n) = \infty$ by Lemma 3.16.

The goal is to arrange that

$$\forall^\infty n \in C \left(\psi(n){\downarrow} \prec_{\mathcal{L}} \varphi(n){\downarrow} \right.$$

$$\left. \Rightarrow \forall d \leq \max_{<} \{\varphi(n), \psi(n)\} \, \exists k \left((\psi(n) \prec_{\mathcal{L}} k \prec_{\mathcal{L}} \varphi(n)) \wedge (F(k) = d) \right) \right) \qquad (*)$$

whenever C is a Δ_2 cohesive set and φ and ψ are partial computable functions with $C \subseteq^* \mathrm{dom}(\varphi)$, $C \subseteq^* \mathrm{dom}(\psi)$, and $\lim_{n \in C} \varphi(n) = \lim_{n \in C} \psi(n) = \infty$.

Suppose we have arranged that $\mathcal{L} \cong \omega$, suppose that C is a Δ_2 cohesive set, and suppose that $[\varphi]$ and $[\psi]$ are non-standard elements of $\prod_C \mathcal{L}$ with $[\psi] \prec_{\prod_C \mathcal{L}} [\varphi]$. Then $C \subseteq^* \mathrm{dom}(\varphi)$, $C \subseteq^* \mathrm{dom}(\psi)$, and $\lim_{n \in C} \varphi(n) = \lim_{n \in C} \psi(n) = \infty$. Further suppose that we have achieved $(*)$ for C, φ, and ψ. Fix any color d, and let δ be the constant function with value d. Partially compute a function $\theta(n)$ by searching for a k with $\psi(n) \prec_{\mathcal{L}} k \prec_{\mathcal{L}} \varphi(n)$ and $F(k) = d$. If there is such a k, let $\theta(n)$ be the first such k found. Property $(*)$ and the fact that $\lim_{n \in C} \varphi(n) = \lim_{n \in C} \psi(n) = \infty$ ensure that there is such a k for almost every $n \in C$. Therefore $C \subseteq^* \mathrm{dom}(\theta)$, $[\psi] \prec_{\prod_C \mathcal{L}} [\theta] \prec_{\prod_C \mathcal{L}} [\varphi]$, and $F^{\prod_C \mathcal{O}}([\theta]) = [\![\delta]\!]$. This shows that for every solid color $[\![\delta]\!]$, there is an element $[\theta]$ of $\prod_C \mathcal{L}$ between $[\psi]$ and $[\varphi]$ with color $[\![\delta]\!]$. Thus $\prod_C \mathcal{O}$ is colorful because it satisfies Definition 3.21 item (1) and C is Δ_2. Therefore, achieving $(*)$ suffices to prove the theorem, again provided we also arrange that $\mathcal{L} \cong \omega$.

Let $g \colon \mathbb{N}^3 \to \{0, 1\}$ be a uniform sequence containing all Δ_2-approximations as in Definition 2.3 and the discussion following it. Write $g_e(n, s)$ in place of $g(e, n, s)$ for all e, n, and s. Let $(A^{i,0}, A^{i,1})_{i \in \mathbb{N}}$ be a computable sequence of staggered partitions as in Definition 2.4 and the discussion following it.

The construction acts when quadruples $q = \langle \ell, r, e, N \rangle$ meet certain conditions. Think of $\langle \ell, r, e, N \rangle$ as coding a pair $(\varphi_\ell, \varphi_r)$ of partial computable functions along with a guess that g_e approximates a Δ_2 cohesive set and, moreover, that N is a threshold by which certain cohesive behavior begins. For notational convenience, we often write $\langle x, N \rangle$ for $\langle \ell, r, e, N \rangle$, where $x = \langle \ell, r, e \rangle$. In this notation, we still call $\langle x, N \rangle$ a 'quadruple' because x codes a triple.

To each triple $x = \langle \ell, r, e \rangle$, assign partitions $(A^{2x,0}, A^{2x,1})$ and $(A^{2x+1,0}, A^{2x+1,1})$. Notice that if C is a cohesive set with $C \subseteq^* \mathrm{dom}(\varphi_\ell)$ and $C \subseteq^* \mathrm{dom}(\varphi_r)$, then there is a pair of sides $(a, b) \in \{0, 1\} \times \{0, 1\}$ of the partitions $(A^{2x,0}, A^{2x,1})$ and $(A^{2x+1,0}, A^{2x+1,1})$ such that $\forall^\infty n \in C \, (\varphi_\ell(n) \in A^{2x,a})$ and $\forall^\infty n \in C \, (\varphi_r(n) \in A^{2x+1,b})$.

The goal of a quadruple of the form $\langle x, N \rangle = \langle \ell, r, e, N \rangle$ is to attempt to satisfy $(*)$ for $\varphi = \varphi_r$, $\psi = \varphi_\ell$, and C the Δ_2 cohesive set approximated by g_e (if g_e indeed approximates a Δ_2 cohesive set) if also there is a pair $(a, b) \in \{0, 1\} \times \{0, 1\}$ such that $\varphi_\ell(n) \in A^{2x,a}$ and $\varphi_r(n) \in A^{2x+1,b}$ whenever $n \geq N$ and $n \in C$. If ever it looks like $\varphi_\ell(n) \prec_{\mathcal{L}} \varphi_r(n)$ for an $n \geq N$ with $n \in C$ but there are not elements of every color $d \leq \max_{<} \{\varphi_\ell(n), \varphi_r(n)\}$ in the interval $(\varphi_\ell(n), \varphi_r(n))_{\mathcal{L}}$, then $\langle x, N \rangle$ needs to add elements of the missing colors to the interval. The difficulty is that we must produce a linear order of type ω, and therefore we can only place finitely many elements $\prec_{\mathcal{L}}$-below any given element. Priority prevents certain quadruples from adding elements $\prec_{\mathcal{L}}$-below certain other elements. Even so, it may still be that a quadruple q_0 adds an element k_0, which induces a quadruple $q_1 \leq q_0$ to add an element $k_1 \prec_{\mathcal{L}} k_0$, which induces a quadruple $q_2 \leq q_1$ to add another element $k_2 \prec_{\mathcal{L}} k_0$, and so on. It may even be that $q_0 = q_1 = q_2 = \cdots$.

To avoid the sort of behavior indicated above, when a quadruple q wants to add elements to $\mathcal{L}$, it looks at the initial triples y of higher-or-equal priority requirements $\langle y, M \rangle = \langle \ell, r, e, M \rangle \leq q$, supposes that g_e approximates a Δ_2 cohesive set C_e, and tries to choose the elements that it adds to avoid each $\varphi_\ell(C_e)$ and $\varphi_r(C_e)$. To do this, for each such $y = \langle \ell, r, e \rangle$, q looks at the most recent guess of sides (a_y, b_y) such that $\varphi_\ell(C_e) \subseteq^* A^{2y,a_y}$ and $\varphi_r(C_e) \subseteq^* A^{2y+1,b_y}$ made by any $\langle y, M \rangle \leq q$ for this y. Then q chooses the elements it adds to $\mathcal{L}$ from the set

$$\bigcap_{\langle y, M \rangle \leq q} A^{2y, 1-a_y} \cap A^{2y+1, 1-b_y}.$$

By choosing elements from the opposite sides of the partitions, q attempts to avoid adding elements from the sets $\varphi_\ell(C_e)$ and $\varphi_r(C_e)$ corresponding to higher-or-equal priority quadruples. The staggering of the partitions ensures that there are infinitely many elements for q to choose among.

If for $x = \langle \ell, r, e \rangle$, g_e really does approximate a Δ_2 cohesive set C with $C \subseteq^* \mathrm{dom}(\varphi_\ell)$ and $C \subseteq^* \mathrm{dom}(\varphi_r)$, then any quadruple $\langle x, N \rangle$ with sufficiently large N eventually settles on the correct guess of sides (a, b) such that $\varphi_\ell(C) \subseteq^* A^{2x,a}$ and $\varphi_r(C) \subseteq^* A^{2x+1,b}$. However, many g_e do not approximate Δ_2 sets at all, let alone Δ_2 cohesive sets. If g_e is particularly ill-behaved, then a quadruple $\langle x, N \rangle = \langle \ell, r, e, N \rangle$ with this e may change its guess (a, b) concerning the sides of the partitions $(A^{2x,0}, A^{2x,1})$ and $(A^{2x+1,0}, A^{2x+1,1})$ infinitely often. This makes it impossible for lower priority quadruples to predict which elements to avoid adding to $\mathcal{L}$ in order to avoid inciting a reaction from quadruple $\langle x, N \rangle$. To combat this problem, we count how many times $\langle x, N \rangle$ changes its guess. The more $\langle x, N \rangle$ changes its guess, the more elements of $\mathcal{L}$ we prevent $\langle x, N \rangle$ from adding elements $\prec_{\mathcal{L}}$-below. This makes it safe for lower priority quadruples to add elements where $\langle x, N \rangle$ cannot.

We now describe the construction in full detail. Define $\prec_{\mathcal{L}}$ and F in stages. By the end of stage s, $\prec_{\mathcal{L}}$ will have been defined on $L_s \times L_s$ and F will have been defined on L_s for some finite $L_s \supseteq \{0, 1, \ldots, s\}$.

Say that quadruple $q = \langle x, N \rangle = \langle \ell, r, e, N \rangle$ is *active* at stage s if there is a pair of sides $(a, b) \in \{0, 1\} \times \{0, 1\}$ of the partitions $(A^{2x,0}, A^{2x,1})$ and $(A^{2x+1,0}, A^{2x+1,1})$ along with a witness w with $N < w < s$ meeting the following conditions.

(a) $g_e(w, s) = 1$.

(b) For all m with $N \leq m \leq w$ and $g_e(m, s) = 1$:
 - $\varphi_{\ell,s}(m){\downarrow} \in A^{2x,a}$, and
 - $\varphi_{r,s}(m){\downarrow} \in A^{2x+1,b}$.

Notice that if q is active at stage s, then there is a unique pair (a, b) for which there is a w meeting these conditions. Call this pair (a, b) the *active sides*, and call the largest witness w (with $N < w < s$) the *activity witness* for q at stage s. To each quadruple q, we associate a counter $\mathrm{ct}(q)$ that counts the number of times that the active sides of q change.

At stage 0, set $L_0 = \{0\}$ with $0 \not\prec_{\mathcal{L}} 0$ and $F(0) = 0$. Initialize $\mathrm{ct}(q) = 0$ for every quadruple q.

At stage $s > 0$, initially set $L_s = L_{s-1}$. If $s \notin L_s$, then add s to L_s, define it to be the $\prec_{\mathcal{L}}$-maximum element of L_s, and define $F(s) = 0$. Then consider the active quadruples $q < s$. If this is the first stage at which q is active or if the active sides of q are different than they were at the previous stage at which q was active, then update the counter $\mathrm{ct}(q)$ to $\mathrm{ct}(q) + 1$.

The quadruple $q = \langle \ell, r, e, N \rangle$ *demands action* at stage s if it is active with active sides (a, b) and activity witness w and there is a least *action input* n with $N \leq n \leq w$ meeting the following conditions.

(1) $g_e(n, s) = 1$.

(2) $\varphi_\ell(n), \varphi_r(n) \in L_s$ and $\varphi_\ell(n) \prec_{\mathcal{L}} \varphi_r(n)$, but there is a $d \leq \max_<\{\varphi_\ell(n), \varphi_r(n)\}$ for which there is no $k \in L_s$ with $\varphi_\ell(n) \prec_{\mathcal{L}} k \prec_{\mathcal{L}} \varphi_r(n)$ and $F(k) = d$.

(3) $\varphi_\ell(n)$ is $\preceq_{\mathcal{L}}$-above all of $0, 1, \ldots, \max_<\{q, \mathrm{ct}(q)\}$.

If $q = \langle \ell, r, e, N \rangle$ demands action with action input n, then let

$$S = \big\{ p \leq q : \varphi_\ell(n) \text{ is } \preceq_{\mathcal{L}}\text{-above all of } 0, 1, \ldots, \max_<\{p, \mathrm{ct}(p)\} \big\}$$
$$R = \big\{ y : \exists M (\langle y, M \rangle \in S) \big\}.$$

The idea is that S consists of the higher-or-equal priority quadruples $p \leq q$ that are currently permitted to add elements immediately $\prec_{\mathcal{L}}$-above $\varphi_\ell(n)$, and that R consists of the initial triples of each quadruple in S. For each $y \in R$, let w be the greatest activity witness yet achieved by any $\langle y, M \rangle \in S$, let t be the most recent stage at which w was achieved, let M be least such that $\langle y, M \rangle \in S$ achieved activity witness w at stage t, and let (a_y, b_y) be the corresponding active sides. If no $\langle y, M \rangle \in S$ has yet been active, then let $(a_y, b_y) = (0, 0)$. Let $c = \max_<\{\varphi_\ell(n), \varphi_r(n)\}$, and let $k_0 < k_1 < \cdots < k_c$ be the $c + 1$ least members of

$$\bigcap_{y \in R} \left(A^{2y,1-a_y} \cap A^{2y+1,1-b_y} \right) \setminus L_s, \qquad (\star)$$

which exist because the intersection is infinite and L_s is finite. Add $k_0, \ldots, k_c$ to L_s. Let $v \in L_s$ be the current $\prec_{\mathcal{L}}$-greatest element of the interval $(\varphi_\ell(n), \varphi_r(n))_{\mathcal{L}}$ (or $v = \varphi_\ell(n)$ if the interval is empty), and set

$$\varphi_\ell(n) \preceq_{\mathcal{L}} v \prec_{\mathcal{L}} k_0 \prec_{\mathcal{L}} \cdots \prec_{\mathcal{L}} k_c \prec_{\mathcal{L}} \varphi_r(n).$$

Also set $F(k_i) = i$ for each $i \leq c$, and say that q *has acted and added k's*. This completes the construction.

The constructed $\mathcal{L}$ is a computable linear order. We show that $\mathcal{L} \cong \omega$ by showing that for each u, there are only finitely many elements $\prec_{\mathcal{L}}$-below u.

Fix u, and note that u appears in L_s at stage $s = u$ at the latest. Consider the evolution of the construction at stages $s > u$.

Claim 1. *Suppose that quadruple q acts and adds elements to L_s at stage $s > u$ and either $q \geq u$ or $\mathrm{ct}(q) \geq u$ at stage s. Then the elements added by the action of q are $\prec_{\mathcal{L}}$-above u.*

Proof of Claim. If q acts at stage $s > u$ with action input n and either $q \geq u$ or $\mathrm{ct}(q) \geq u$ at stage s, then $u \leq \max_{<}\{q, \mathrm{ct}(q)\}$ at stage s. Thus it must be that $u \prec_{\mathcal{L}} \varphi_\ell(n) \prec_{\mathcal{L}} \varphi_r(n)$ by condition (3). In this case, the action adds elements to L_s and places them in the interval $(\varphi_\ell(n), \varphi_r(n))_{\mathcal{L}}$ and hence places them $\prec_{\mathcal{L}}$-above u. $\square$

It follows from Claim 1 that no quadruple $q \geq u$ acts to add elements $\prec_{\mathcal{L}}$-below u at stages $s > u$. Thus to show that there are only finitely many elements $\prec_{\mathcal{L}}$-below u, it suffices to show that each quadruple $q < u$ only ever acts to add finitely many elements $\prec_{\mathcal{L}}$-below u. The following Claims 2–4 aid this analysis.

Claim 2. *Let $x = \langle \ell, r, e \rangle$, and consider quadruples $\langle x, M_0 \rangle$ and $\langle x, M_1 \rangle$ with $M_0 < M_1$. Suppose that for each $i \in \{0, 1\}$, $\langle x, M_i \rangle$ achieves arbitrarily large activity witnesses and eventually settles on a pair of active sides (a_i, b_i). Then $(a_0, b_0) = (a_1, b_1)$.*

Proof of Claim. We show that at infinitely many stages t, both $\langle x, M_0 \rangle$ and $\langle x, M_1 \rangle$ are active with active sides (a_0, b_0). It follows that $(a_0, b_0) = (a_1, b_1)$ because $\langle x, M_1 \rangle$ changes its active sides only finitely often. To this end, given any s, let $t > s$ be a stage at which $\langle x, M_0 \rangle$ is active with active sides (a_0, b_0) and activity witness $w > M_1$. Such a stage t exists because $\langle x, M_0 \rangle$ achieves arbitrarily large activity witnesses. Then $g_e(w, t) = 1$, and also $\varphi_{\ell,t}(m)\downarrow \in A^{2x, a_0}$ and $\varphi_{r,t}(m)\downarrow \in A^{2x+1, b_0}$ for all m with $M_0 \leq m \leq w$ and $g_e(m, t) = 1$. Therefore (a_0, b_0) and w also witness that $\langle x, M_1 \rangle$ is active at stage t with active sides (a_0, b_0) because $M_0 < M_1 < w$. $\square$

Claim 3. *For a given quadruple q, each number n can be the action input for q at most once.*

Proof of Claim. Suppose that $q = \langle \ell, r, e, N \rangle$ demands action with action input n at some stage s. Then q adds elements of every color $i \leq \max_{<}\{\varphi_\ell(n), \varphi_r(n)\}$ to L_s and places them $\prec_{\mathcal{L}}$-between $\varphi_\ell(n)$ and $\varphi_r(n)$. Thus condition (2) is never again satisfied for q with action input n at any stage $t > s$. $\square$

Let $q_0 < u$, and assume inductively that there is a stage $s_0 > u$ such that no quadruple $p < q_0$ acts to add elements $k \prec_{\mathcal{L}} u$ after stage s_0. In order for q_0 to act and add elements at some stage s, it must be that q_0 is active at stage s with some activity witness w, and there must be an action input n with $n \leq w$. Therefore, if the activity witnesses for q_0 are always $\leq W$ for some fixed W, then q_0 only acts to add elements finitely often. This is because when q_0 acts to add elements, it must use an action input $n \leq W$, and, by Claim 3, q_0 can use each $n \leq W$ as an action input at most once. Furthermore, if quadruple q_0 changes its active sides at least u many times, then $\mathrm{ct}(q_0)$ is incremented at least u many times, so there is a stage s such that $\mathrm{ct}(q_0) \geq u$ at all stages $t \geq s$. Thus if q_0 acts at a stage $t \geq s$, then the elements it adds are $\prec_{\mathcal{L}}$-above u by Claim 1. Thus q_0 only ever adds finitely elements $\prec_{\mathcal{L}}$-below u in this case as well.

For the remaining case, suppose that our quadruple $q_0 = \langle x_0, N_0 \rangle = \langle \ell_0, r_0, e_0, N_0 \rangle$ achieves arbitrarily large activity witnesses and changes its active sides fewer than u many times. Let $\mathrm{ct}_0 < u$ denote the final value of $\mathrm{ct}(q_0)$, and let (a_0, b_0) be the final active sides of q_0. Let $v = \max_{\prec_{\mathcal{L}}}\{0, 1, \ldots, \max_{<}\{q_0, \mathrm{ct}_0\}\}$. We claim that eventually every element k added to L_s with $v \prec_{\mathcal{L}} k \prec_{\mathcal{L}} u$ is in $A^{2x_0, 1-a_0} \cap A^{2x_0+1, 1-b_0}$.

Claim 4. *There is a stage $s_1 \geq s_0$ such that whenever an element k is added to L_s and $v \prec_{\mathcal{L}} k \prec_{\mathcal{L}} u$ is defined at some stage $s \geq s_1$, we have that $k \in A^{2x_0, 1-a_0} \cap A^{2x_0+1, 1-b_0}$.*

Proof of Claim. By the choice of s_0, the fact that $s_0 > u$, and Claim 1, we already know that no quadruple p with either $p < q_0$ or $p \geq u$ acts to add elements $\prec_{\mathcal{L}}$-below u after stage s_0. Thus we need only consider the behavior of quadruples p with $q_0 \leq p < u$.

Recall that our quadruple q_0 is $q_0 = \langle x_0, N_0 \rangle$. Consider for a moment all quadruples $\langle x_0, M \rangle < u$ with this same x_0, including quadruple q_0. Let t_0 and s_1 with $s_1 > t_0 > s_0$ be large enough so that the following hold.

- Each $\langle x_0, M \rangle < u$ that eventually achieves $\mathrm{ct}(\langle x_0, M \rangle) \geq u$ has done so by stage t_0.

- Each $\langle x_0, M \rangle < u$ that changes its active sides only finitely often has settled on its final active sides and on its final value of $\mathrm{ct}(\langle x_0, M \rangle)$ by stage t_0.

- By stage t_0, q_0 has achieved an activity witness larger than all activity witnesses ever achieved by the $\langle x_0, M \rangle < u$ that do not achieve arbitrarily large activity witnesses.

- By stage s_1, q_0 has achieved an activity witness larger than all the activity witnesses achieved by all $\langle x_0, M \rangle < u$ at stages $t \leq t_0$.

Such t_0 and s_1 exist because q_0 achieves arbitrarily large activity witnesses.

Suppose that quadruple $q = \langle \ell, r, e, N \rangle$ with $q_0 \leq q < u$ acts and adds k's in the interval $(v, u)_{\mathcal{L}}$ at a stage $s \geq s_1$. Then it must be that $v \preceq_{\mathcal{L}} \varphi_\ell(n) \prec_{\mathcal{L}} \varphi_r(n) \preceq_{\mathcal{L}} u$, where n is the action input for q at stage s. In particular, at stage s, $\varphi_\ell(n)$ is $\preceq_{\mathcal{L}}$-above all of $0, 1, \ldots, \max_<\{q_0, \mathrm{ct}(q_0)\}$. This means that $q_0 \in S$ and $x_0 \in R$, where S and R are the sets used by q when it acts at stage s. Thus when q acts at stage s, it chooses active sides (a_{x_0}, b_{x_0}) corresponding to x_0. To do this, the action of q finds the greatest activity witness w yet achieved by any $\langle x_0, M \rangle \in S$, the most recent stage t at which w was achieved, the least M such that $\langle x_0, M \rangle \in S$ achieved activity witness w at stage t, and then takes (a_{x_0}, b_{x_0}) to be the active sides of $\langle x_0, M \rangle$ at stage t. We show that $(a_{x_0}, b_{x_0}) = (a_0, b_0)$.

Consider the quadruple $\langle x_0, M \rangle$ used to choose the active sides (a_{x_0}, b_{x_0}) corresponding to x_0 during the action of q at stage s as described above. Note that $\langle x_0, M \rangle \leq q < u$. Quadruple $\langle x_0, M \rangle$ changes its active sides only finitely often. If $\langle x_0, M \rangle$ changes its active sides infinitely often, then eventually $\langle x_0, M \rangle$ achieves $\mathrm{ct}(\langle x_0, M \rangle) \geq u$ and hence has done so by stage $s > t_0$. Therefore, at stage s, $\varphi_\ell(n)$ is not $\preceq_{\mathcal{L}}$-above all of $0, 1, \ldots, \max_<\{\langle x_0, M \rangle, \mathrm{ct}(\langle x_0, M \rangle)\}$ because $\varphi_\ell(n) \prec_{\mathcal{L}} u$ but $\mathrm{ct}(\langle x_0, M \rangle) \geq u$. This implies that $\langle x_0, M \rangle \notin S$, which is a contradiction.

Quadruple $\langle x_0, M \rangle$ achieves arbitrarily large activity witnesses. If not, then by stage $s > t_0$ quadruple $q_0 = \langle x_0, N_0 \rangle$ has already achieved an activity witness greater than any activity witness ever achieved by $\langle x_0, M \rangle$. Quadruple q_0 is in S, so the greatest activity witness achieved by a quadruple of the form $\langle x_0, \widehat{M} \rangle$ in S by stage s was not achieved by $\langle x_0, M \rangle$. This contradicts that q uses $\langle x_0, M \rangle$ to choose (a_{x_0}, b_{x_0}) at stage S.

The most recent stage $t \leq s$ at which quadruple $\langle x_0, M \rangle$ achieves activity witness w must satisfy $t > t_0$. If $t \leq t_0$, then, by choice of s_1, quadruple $q_0 = \langle x_0, N_0 \rangle$ has already achieved an activity witness greater than w by stage $s > s_1$. This contradicts that w is the greatest activity witness achieved by a quadruple of the form $\langle x_0, \widehat{M} \rangle$ in S by stage s.

It now follows that (a_{x_0}, b_{x_0}) are the final active sides of $\langle x_0, M \rangle$. Quadruple $\langle x_0, M \rangle$ changes its active sides finitely often, so it settles on its final active sides by stage t_0 by choice of t_0. The pair (a_{x_0}, b_{x_0}) is the active sides of $\langle x_0, M \rangle$ at the most recent stage $t \leq s$ at which $\langle x_0, M \rangle$ achieved activity witness w. We showed that $t > t_0$, so (a_{x_0}, b_{x_0}) must be the final active sides of $\langle x_0, M \rangle$.

Recall that (a_0, b_0) are the final active sides of q_0. Quadruples $\langle x_0, M \rangle$ and $q_0 = \langle x_0, N_0 \rangle$ both achieve arbitrarily large activity witnesses and eventually settle on their active sides. Therefore $\langle x_0, M \rangle$ and q_0 settle on the same active sides by Claim 2. Thus $(a_{x_0}, b_{x_0}) = (a_0, b_0)$. Therefore, the k's that the action of q chooses to add to L_s are in $A^{2x_0, 1-a_0} \cap A^{2x_0+1, 1-b_0}$.

Thus we have found an $s_1 \geq s_0$ such that whenever an element k is added to L_s and $v \prec_{\mathcal{L}} k \prec_{\mathcal{L}} u$ is defined at a stage $s \geq s_1$, it is on account of a quadruple q with $q_0 \leq q < u$ and we have that $k \in A^{2x_0, 1-a_0} \cap A^{2x_0+1, 1-b_0}$. $\qquad\square$

We may now show that our quadruple $q_0 = \langle x_0, N_0 \rangle = \langle \ell_0, r_0, e_0, N_0 \rangle$ adds only finitely many elements $k \prec_{\mathcal{L}} u$. Let s_1 be the stage from Claim 4, and additionally assume that q_0 has settled on its final active sides (a_0, b_0) and that $\mathrm{ct}(q_0)$ has reached its final value ct_0 by stage s_1. Recall that $v = \max_{\prec_{\mathcal{L}}}\{0, 1, \ldots, \max_<\{q_0, \mathrm{ct}_0\}\}$. By condition (3), every k that q_0 adds to L_s at a stage $s \geq s_1$ satisfies $v \prec_{\mathcal{L}} k$. So suppose that q_0 acts at some stage $s \geq s_1$, adds an element k to L_s, and defines $k \prec_{\mathcal{L}} u$, in which case it also defines $v \prec_{\mathcal{L}} k$. Then at stage s, q_0 is active with active sides (a_0, b_0), and it acts with action input n, where $\varphi_{\ell_0}(n) = i$ for some $i \in A^{2x_0, a_0}$, $\varphi_{r_0}(n) = j$ for some $j \in A^{2x_0+1, b_0}$, and $v \preceq_{\mathcal{L}} i \prec_{\mathcal{L}} j \preceq_{\mathcal{L}} u$. The action then places k's of each color $d \leq \max_<\{i, j\}$ in the interval $(i, j)_{\mathcal{L}}$. If q_0 acts again at some later stage $t > s$ with some action input m, then again $\varphi_{\ell_0}(m) \in A^{2x_0, a_0}$ and $\varphi_{r_0}(m) \in A^{2x_0+1, b}$. However, it cannot again be that $\varphi_{\ell_0}(m) = i$ and $\varphi_{r_0}(m) = j$ because condition (2) would fail in this situation. Thus when adding a number $k \prec_{\mathcal{L}} u$, the action input n used by q_0 specifies a pair $(i, j) = (\varphi_{\ell_0}(n), \varphi_{r_0}(n)) \in A^{2x_0, a_0} \times A^{2x_0+1, b_0}$ with $v \preceq_{\mathcal{L}} i \prec_{\mathcal{L}} j \preceq_{\mathcal{L}} u$, and each such pair can be

specified by q_0 at most once. By Claim 4, every element added to the interval $(v, u)_{\mathcal{L}}$ after stage s_1 is in $A^{2x_0,1-a_0} \cap A^{2x_0+1,1-b_0}$. Therefore, there are only finitely many pairs $(i, j) \in A^{2x_0,a_0} \times A^{2x_0+1,b_0}$ with $v \preceq_{\mathcal{L}} i \prec_{\mathcal{L}} j \preceq_{\mathcal{L}} u$, and therefore quadruple q_0 can only add finitely many elements $k \prec_{\mathcal{L}} u$. This completes the proof that $\mathcal{L} \cong \omega$.

To complete the proof of the theorem, we must show that $(*)$ is satisfied whenever C is a Δ_2 cohesive set and φ and ψ are partial computable functions with $C \subseteq^* \operatorname{dom}(\varphi)$, $C \subseteq^* \operatorname{dom}(\psi)$, and $\lim_{n \in C} \varphi(n) = \lim_{n \in C} \psi(n) = \infty$.

Claim 5. *Suppose that quadruple $q = \langle x, N \rangle = \langle \ell, r, e, N \rangle$ and pair (a, b) are such that*

- *g_e is a Δ_2-approximation to an infinite Δ_2 set C,*
- *$\forall m \geq N \ (m \in C \ \rightarrow \ \varphi_\ell(m)\downarrow \in A^{2x,a})$, and*
- *$\forall m \geq N \ (m \in C \ \rightarrow \ \varphi_r(m)\downarrow \in A^{2x+1,b})$.*

Then q achieves arbitrarily large activity witnesses, and q eventually settles on active sides (a, b).

Proof of Claim. Given any number W, let w be the least number with $w > W$ and $w \in C$. Let $s > w$ be large enough so that $g_e(m, s) = C(m)$ for all $m \leq w$ and so that $\varphi_{\ell,s}(m)\downarrow \in A^{2x,a}$ and $\varphi_{r,s}(m)\downarrow \in A^{2x+1,b}$ for all $m \in C$ with $N \leq m \leq w$. Then q is active at stage s with activity witness w or greater. Thus q achieves arbitrarily large activity witnesses.

Let n be the least number with $n > N$ and $n \in C$. Let s_0 be large enough so that $\forall m \leq n \ \forall s \geq s_0 \ (g_e(m, s) = C(m))$, $\varphi_{\ell,s_0}(n)\downarrow \in A^{2x,a}$, and $\varphi_{r,s_0}(n)\downarrow \in A^{2x+1,b}$. Then if q is active at a stage $s > s_0$, it must use an activity witness $w \geq n$, in which case its active sides must be (a, b) because $g_e(n, s) = 1$, $\varphi_{\ell,s}(n)\downarrow \in A^{2x,a}$, and $\varphi_{r,s}(n)\downarrow \in A^{2x+1,b}$. That is, q has active sides (a, b) whenever it is active at a stage later than s_0. Thus q eventually settles on active sides (a, b). $\square$

Let C be a Δ_2 cohesive set, and let φ and ψ be partial computable functions with $C \subseteq^* \operatorname{dom}(\varphi)$, $C \subseteq^* \operatorname{dom}(\psi)$, and $\lim_{n \in C} \varphi(n) = \lim_{n \in C} \psi(n) = \infty$. We show that $(*)$ holds for C, φ, and ψ. Assume that $\forall^\infty n \in C \ (\psi(n) \prec_{\mathcal{L}} \varphi(n))$, for otherwise $(*)$ vacuously holds. Let ℓ and r be such that $\varphi_\ell = \psi$ and $\varphi_r = \varphi$. Let e be such that g_e is a Δ_2-approximation to C. Let $x = \langle \ell, r, e \rangle$. By cohesiveness, let (a, b) and N be such that, for all $n \in C$ with $n \geq N$, $\varphi_\ell(n) \in A^{2x,a}$ and $\varphi_r(n) \in A^{2x+1,b}$. Let q be the quadruple $q = \langle x, N \rangle = \langle \ell, r, e, N \rangle$. Quadruple q and pair (a, b) satisfy the hypotheses of Claim 5, so the active sides of q eventually settle on (a, b) and therefore $\operatorname{ct}(q)$ also reaches a final value ct. Let $v = \max_{\prec_{\mathcal{L}}}\{0, 1, \dots, \max_<\{q, \operatorname{ct}\}\}$. Let $n_0 \geq N$ be large enough so that $v \preceq_{\mathcal{L}} \varphi_\ell(n)$ whenever $n \geq n_0$ and $n \in C$. Such an n_0 exists because $\lim_{n \in C} \varphi_\ell(n) = \infty$, but there are only finitely many elements $\prec_{\mathcal{L}}$-below v because $\mathcal{L} \cong \omega$.

Suppose that $n \in C$ and $n \geq n_0$, and furthermore suppose for a contradiction that there is a $d \leq \max_<\{\varphi_\ell(n), \varphi_r(n)\}$ such that there is no k with $\varphi_\ell(n) \prec_{\mathcal{L}} k \prec_{\mathcal{L}} \varphi_r(n)$ and $F(k) = d$. Then conditions (1)–(3) hold for n at all sufficiently large stages s, with (1) holding because $n \in C$, (2) holding by assumption, and (3) holding by the choice of n_0. By Claim 5, quadruple q achieves arbitrarily large activity witnesses. Thus infinitely often q is active with an activity witness $w > n$. By Claim 3, each $m < n$ can be the action input for q at most once. Thus at some stage, q eventually demands action with action input n. The action of q defines $\varphi_\ell(n) \prec_{\mathcal{L}} k \prec_{\mathcal{L}} \varphi_r(n)$ and $F(k) = d$ for some k, which contradicts that there is no such k. This shows that $(*)$ holds for C, $\varphi = \varphi_r$ and $\psi = \varphi_\ell$, which completes the proof. $\square$

Theorem 4.1 immediately provides a computable copy $\mathcal{L}$ of ω for which $\prod_C \mathcal{L} \cong \omega + \eta$ whenever C is a Δ_2 cohesive set.

Corollary 4.2. *There is a computable copy $\mathcal{L}$ of ω such that for every Δ_2 cohesive set C, $\prod_C \mathcal{L} \cong \omega + \eta$.*

Proof. Let $\mathcal{O} = (L, \mathbb{N}; \prec_{\mathcal{L}}, F)$ be the computable colored copy of ω from Theorem 4.1, and let $\mathcal{L} = (L; \prec_{\mathcal{L}})$ be the underlying computable linear order of type ω. Let C be a Δ_2 cohesive set. Then $\prod_C \mathcal{O}$ is colorful, so $\prod_C \mathcal{L} \cong \omega + \eta$ as explained in the discussion following Definition 3.21. $\square$

Corollary 4.2 is as good as possible, in the sense that Δ_2 cannot be improved to Π_2.

Proposition 4.3. *For every computable copy $\mathcal{L}$ of ω, there is a Π_2 cohesive set C such that $\prod_C \mathcal{L} \not\cong \omega + \eta$.*

Proof. Let $\mathcal{L} = (L; \prec_{\mathcal{L}})$ be a computable copy of ω. Fix an $\ell \in L$. Define total computable functions ψ and φ by

$$\psi(\langle m, n \rangle) = \begin{cases} m & \text{if } m \in L \\ \ell & \text{if } m \notin L \end{cases}$$

$$\varphi(\langle m, n \rangle) = \begin{cases} n & \text{if } n \in L \\ \ell & \text{if } n \notin L. \end{cases}$$

Let

$$X = \{\langle m, n \rangle : m, n \in L \text{ and } n \text{ is the } \prec_{\mathcal{L}}\text{-immediate successor of } m\}.$$

Then X is an infinite Π_1 set because $\mathcal{L}$ is a computable linear order of type ω. Therefore X has a Π_2 cohesive subset C as explained in the discussion following Corollary 2.6.

Consider the elements $[\psi]$ and $[\varphi]$ of $\prod_C \mathcal{L}$. Every element of $\mathcal{L}$ has exactly one $\prec_{\mathcal{L}}$-immediate successor, so for every m there is at most one n with $\langle m, n \rangle \in C$, and for every n there is at most one m with $\langle m, n \rangle \in C$. It follows that ψ and φ are injective when restricted to C and therefore that $\lim_{\langle m,n \rangle \in C} \psi(\langle m, n \rangle) = \lim_{\langle m,n \rangle \in C} \varphi(\langle m, n \rangle) = \infty$. Thus $[\psi]$ and $[\varphi]$ are non-standard elements of $\prod_C \mathcal{L}$ by Lemma 3.16. By the choice of X and C, $\varphi(\langle m, n \rangle) = n$ is the $\prec_{\mathcal{L}}$-immediate successor of $\psi(\langle m, n \rangle) = m$ for every $\langle m, n \rangle \in C$. Therefore $[\varphi]$ is the $\prec_{\prod_C \mathcal{L}}$-immediate successor of $[\psi]$ in $\prod_C \mathcal{L}$ by Lemma 3.12. Thus the non-standard elements of $\prod_C \mathcal{L}$ are not dense, so $\prod_C \mathcal{L} \not\cong \omega + \eta$. $\square$

Remark 4.4. We obtain an infinite Π_1 set with no Δ_2 cohesive subset as a consequence of Corollary 4.2. Let $\mathcal{L} = (L; \prec_{\mathcal{L}})$ be a computable copy of ω such that $\prod_C \mathcal{L} \cong \omega + \eta$ for every Δ_2 cohesive set C. Let

$$X = \{\langle m, n \rangle : m, n \in L \text{ and } n \text{ is the } \prec_{\mathcal{L}}\text{-immediate successor of } m\}.$$

as in the proof of Proposition 4.3. Then X is an infinite Π_1 set. The proof of Proposition 4.3 shows that if C is a cohesive subset of X, then there is a non-standard $\prec_{\prod_C \mathcal{L}}$-immediate successor pair $[\psi] \prec_{\prod_C \mathcal{L}} [\varphi]$ in $\prod_C \mathcal{L}$. Therefore, if C is a cohesive subset of X, then $\prod_C \mathcal{L} \not\cong \omega + \eta$. As $\prod_C \mathcal{L} \cong \omega + \eta$ for every Δ_2 cohesive set C, it follows that X cannot have a Δ_2 cohesive subset. So X is an infinite Π_1 set with no Δ_2 cohesive subset.

Finally, we reach the main result by combining Theorem 4.1 with Lemma 3.22.

Theorem 4.5. *Let $X \subseteq \mathbb{N} \setminus \{0\}$ be a Boolean combination of Σ_2 sets, thought of as a set of finite order-types. Then there is a computable copy $\mathcal{L}$ of ω such that for every Δ_2 cohesive set C, the cohesive power $\prod_C \mathcal{L}$ has order-type $\omega + \boldsymbol{\sigma}(X \cup \{\omega + \zeta\eta + \omega^*\})$. Moreover, if X is finite and non-empty, then there is also a computable copy $\mathcal{L}$ of ω such that for every Δ_2 cohesive set C, the cohesive power $\prod_C \mathcal{L}$ has order-type $\omega + \boldsymbol{\sigma}(X)$.*

Proof. By Theorem 4.1, let $\mathcal{O}$ be a computable colored copy of ω such that for every Δ_2 cohesive set C, the cohesive power $\prod_C \mathcal{O}$ is colorful. Then apply Lemma 3.22 to $\mathcal{O}$ to get the desired computable copy $\mathcal{L}$ of ω in either case. $\square$

Similar to [9, Example 5.5], we can define a countable collection of pairwise isomorphic (but not computably isomorphic) linear orders whose cohesive powers over Δ_2 cohesive sets are pairwise non-elementarily equivalent.

Example 4.6. There are computable copies $\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3, \ldots$ of ω such that the cohesive powers $\prod_C \mathcal{L}^k$ and $\prod_D \mathcal{L}^m$ are not elementarily equivalent whenever $1 \le k < m$ and C and D are Δ_2 cohesive sets. (We put the index in the superscript to emphasize that we mean the cohesive powers of the individual structures, not the cohesive product of the sequence.) For each $k \ge 1$, apply Theorem 4.5 to the set $X = \{k\}$ to get a computable copy $\mathcal{L}^k$ of ω such that for every Δ_2 cohesive set C, $\prod_C \mathcal{L}^k \cong \omega + \boldsymbol{\sigma}(\{\boldsymbol{k}\}) \cong \omega + \boldsymbol{k}\eta$. If $1 \le k < m$ and C and D are Δ_2 cohesive sets, then $\prod_C \mathcal{L}^k \cong \omega + \boldsymbol{k}\eta$ and $\prod_D \mathcal{L}^m \cong \omega + \boldsymbol{m}\eta$. The order-types $\omega + \boldsymbol{k}\eta$ and $\omega + \boldsymbol{m}\eta$ are not elementarily equivalent because they disagree on the Σ_3 sentence expressing that there is a maximal block (in the sense of the finite condensation) of size k.

In fact, Theorem 4.5 is not necessary to obtain this example. Instead, let $\mathcal{L}$ be the computable copy of ω from Corollary 4.2, and let $\mathcal{L}^k = \boldsymbol{k}\mathcal{L}$ for each $k \geq 1$. Then

$$\prod_C \mathcal{L}^k \cong \prod_C (\boldsymbol{k}\mathcal{L}) \cong \left(\prod_C \boldsymbol{k}\right)\left(\prod_C \mathcal{L}\right) \cong \boldsymbol{k}(\omega + \eta) \cong \omega + \boldsymbol{k}\eta$$

for every $k \geq 1$ and every Δ_2 cohesive set C. The second isomorphism is by Theorem 3.11 item (2), and $\prod_C \boldsymbol{k} \cong \boldsymbol{k}$ in the third isomorphism because $\boldsymbol{k}$ is finite. If $\mathcal{A}$ is a finite computable structure and C is cohesive, then, by cohesiveness, every element of $\prod_C \mathcal{A}$ is in the range of the canonical embedding of $\mathcal{A}$ into $\prod_C \mathcal{A}$. Therefore $\prod_C \mathcal{A} \cong \mathcal{A}$. Here it is apparent that the sequence $\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3, \ldots$ may be taken to be uniformly computable. The sequence may be taken to be uniformly computable in the first situation too because Lemma 3.22 is uniform.

Acknowledgments

References

[1] Christopher J. Ash and Julia Knight, *Computable Structures and the Hyperarithmetical Hierarchy*, Studies in Logic and the Foundations of Mathematics, vol. 144, North-Holland Publishing Co., Amsterdam, 2000.

[2] Paul Bankston, *A survey of ultraproduct constructions in general topology*, Topology Atlas Invited Contributions **8** (2003), no. 2, 1–32.

[3] Mauro Di Nasso, Isaac Goldbring, and Martino Lupini, *Nonstandard Methods in Ramsey Theory and Combinatorial Number Theory*, Lecture Notes in Mathematics, vol. 2239, Springer, Cham, 2019.

[4] Rumen Dimitrov, *A class of Σ_3^0 modular lattices embeddable as principal filters in $\mathcal{L}^*(V_\infty)$*, Archive for Mathematical Logic **47** (2008), no. 2, 111–132.

[5] ———, *Cohesive powers of computable structures*, Godishnik na Sofiĭskiya Universitet "Sv. Kliment Ohridski". Fakultet po Matematika i Informatika. Annuaire de l'Université de Sofia "St. Kliment Ohridski". Faculté de Mathématiques et Informatique **99** (2009), 193–201.

[6] Rumen Dimitrov and Valentina Harizanov, *Orbits of maximal vector spaces*, Algebra and Logic **54** (2016), no. 6, 440–477. (English translation).

[7] ———, *Effective ultrapowers and applications*, Aspects of Computation and Automata Theory with Applications, Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, 2023.

[8] Rumen Dimitrov, Valentina Harizanov, Russell Miller, and K. J. Mourad, *Isomorphisms on non-standard fields and Ash's conjecture*, Lecture Notes in Computer Science 8493: Language, Life, Limits, Springer, Cham, 2014, pp. 143–152.

[9] Rumen Dimitrov, Valentina Harizanov, Andrey Morozov, Paul Shafer, Alexandra A. Soskova, and Stefan V. Vatev, *On cohesive powers of linear orders*, The Journal of Symbolic Logic (2023), 1–58.

[10] Paul C. Eklof, *Ultraproducts for algebraists*, Handbook of Mathematical Logic, 1977, pp. 105–137.

[11] Solomon Feferman, Dana S. Scott, and Stanley Tennenbaum, *Models of arithmetic through function rings*, Notices of the American Mathematical Society **6** (1959), no. 2, 173–174. Abstract 556-31.

[12] Richard M. Friedberg, *Three theorems on recursive enumeration. I. Decomposition. II. Maximal set. III. Enumeration without duplication*, The Journal of Symbolic Logic **23** (1958), 309–316.

[13] C. Ward Henson and José Iovino, *Ultraproducts in analysis*, Analysis and Logic (Mons, 1997), 2002, pp. 1–110.

[14] Yoram Hirschfeld, *Models of arithmetic and recursive functions*, Israel Journal of Mathematics **20** (1975), no. 2, 111–126.

[15] Yoram Hirschfeld and William H. Wheeler, *Forcing, Arithmetic, Division Rings*, Lecture Notes in Mathematics, vol. 454, Springer-Verlag, Berlin–New York, 1975.

[16] Carl G. Jockusch Jr., *The degrees of hyperhyperimmune sets*, The Journal of Symbolic Logic **34** (1969), 489–493.

[17] H. Jerome Keisler, *The ultraproduct construction*, Ultrafilters Across Mathematics, 2010, pp. 163–179.

[18] Linus Kramer, Saharon Shelah, Katrin Tent, and Simon Thomas, *Asymptotic cones of finitely presented groups*, Advances in Mathematics **193** (2005), no. 1, 142–173.

[19] Alistair H. Lachlan, *On the lattice of recursively enumerable sets*, Transactions of the American Mathematical Society **130** (1968), 1–37.

[20] Manuel Lerman, *Recursive functions modulo co-r-maximal sets*, Transactions of the American Mathematical Society **148** (1970), 429–444.

[21] ———, *Degrees of Unsolvability: Local and Global Theory*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1983.

[22] Manuel Lerman, Richard A. Shore, and Robert I. Soare, *r-maximal major subsets*, Israel Journal of Mathematics **31** (1978), no. 1, 1–18.

[23] Wolfgang Maass, *On the orbits of hyperhypersimple sets*, The Journal of Symbolic Logic **49** (1984), no. 1, 51–62.

[24] Maryanthe Malliaris, *Model theory and ultraproducts*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures, 2018, pp. 83–97.

[25] Donald A. Martin, *A theorem on hyperhypersimple sets*, The Journal of Symbolic Logic **28** (1963), 273–278.

[26] Thomas G. McLaughlin, *Embeddings of and into Nerode semirings*, Israel Journal of Mathematics **60** (1987), no. 1, 65–88.

[27] ———, *Some extension and rearrangement theorems for Nerode semirings*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik **35** (1989), no. 3, 197–209.

[28] ———, *Sub-arithmetical ultrapowers: a survey*, Annals of Pure and Applied Logic **49** (1990), no. 2, 143–191.

[29] ———, *Recursive ultrapowers, simple models, and cofinal extensions*, Archive for Mathematical Logic **31** (1992), no. 4, 287–296.

[30] ———, *A note on effective ultrapowers: uniform failure of bounded collection*, Mathematical Logic Quarterly **39** (1993), no. 4, 431–435.

[31] ———, Δ_1 *ultrapowers are totally rigid*, Archive for Mathematical Logic **46** (2007), no. 5-6, 379–384.

[32] Antonio Montalbán, *Computable Structure Theory—Within the Arithmetic*, Perspectives in Logic, Cambridge University Press, Cambridge; Association for Symbolic Logic, Ithaca, NY, 2021.

[33] George C. Nelson, *Constructive ultraproducts and isomorphisms of recursively saturated ultrapowers*, Notre Dame Journal of Formal Logic **33** (1992), no. 3, 433–441.

[34] Hartley Rogers Jr., *Theory of Recursive Functions and Effective Computability*, Second edition, MIT Press, Cambridge, MA, 1987.

[35] Joseph G. Rosenstein, *Linear Orderings*, Pure and Applied Mathematics, vol. 98, Academic Press, New York–London, 1982.

[36] V. Yu. Shavrukov, *Duality, non-standard elements, and dynamic properties of r.e. sets*, Annals of Pure and Applied Logic **167** (2016), no. 10, 939–981.

[37] ———, *R.e. prime powers and total rigidity*, Advances in Mathematics **360** (2020), 106884, 50.

[38] ———, *Personal communication*, 2022.

[39] Robert I. Soare, *Recursively Enumerable Sets and Degrees*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1987.

School of Mathematics, University of Leeds, Leeds, LS2 9JT, United Kingdom

Email address: p.e.shafer@leeds.ac.uk

URL: http://www1.maths.leeds.ac.uk/~matpsh/