CHARACTERIZING THE JOIN-IRREDUCIBLE MEDVEDEV DEGREES

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Abstract. We characterize the join-irreducible Medvedev degrees as the degrees of complements of Turing ideals, thereby solving a problem posed by Sorbi. We use this characterization to prove that there are Medvedev degrees above the second-least degree that do not bound any join-irreducible degrees above this second-least degree. This solves a problem posed by Sorbi and Terwijn. Finally, we prove that the filter generated by the degrees of closed sets is not prime. This solves a problem posed by Bianchini and Sorbi.

1. Introduction

We present solutions to three problems concerning the Medvedev degrees. A *mass problem* is a set $\mathcal{A} \subseteq \omega^\omega$. For mass problems $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ *Medvedev reduces* to $\mathcal{B}$ ($\mathcal{A} \leq_\mathrm{M} \mathcal{B}$) if there is a Turing functional Φ such that $\Phi(\mathcal{B}) \subseteq \mathcal{A}$. That is, $\Phi(f) \in \mathcal{A}$ for all $f \in \mathcal{B}$. We say that $\mathcal{A}$ and $\mathcal{B}$ are *Medvedev equivalent* ($\mathcal{A} \equiv_\mathrm{M} \mathcal{B}$) if $\mathcal{A} \leq_\mathrm{M} \mathcal{B}$ and $\mathcal{B} \leq_\mathrm{M} \mathcal{A}$. The equivalence class $[\mathcal{A}]$ is called the *Medvedev degree* of $\mathcal{A}$, and the structure $\mathfrak{M} = (2^{\omega^\omega}/\equiv_\mathrm{M}, \leq_\mathrm{M})$ is called the *Medvedev degrees*. See Sorbi [15] for a good introduction to the theory of the Medvedev degrees.

For $f, g \in \omega^\omega$, let $f \oplus g$ be the function $(f \oplus g)(2n) = f(n)$ and $(f \oplus g)(2n+1) = g(n)$. For $m \in \omega$ and $f \in \omega^\omega$, let $m^\frown f$ be the function $(m^\frown f)(0) = m$ and $(m^\frown f)(n+1) = f(n)$. In general, '$^\frown$' denotes string concatenation. Functions $f \in \omega^\omega$ are interpreted as ω-length strings when appropriate. For a mass problem $\mathcal{A}$, let $m^\frown\mathcal{A} = \{m^\frown f \mid f \in \mathcal{A}\}$. Given mass problems $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{A} + \mathcal{B} = \{f \oplus g \mid f \in \mathcal{A} \wedge g \in \mathcal{B}\}$ and let $\mathcal{A} \times \mathcal{B} = 0^\frown\mathcal{A} \cup 1^\frown\mathcal{B}$. Then $[\mathcal{A}] + [\mathcal{B}] = [\mathcal{A} + \mathcal{B}]$ is the *join* (i.e., $\leq_\mathrm{M}$-least upper bound) of $[\mathcal{A}]$ and $[\mathcal{B}]$, while $[\mathcal{A}] \times [\mathcal{B}] = [\mathcal{A} \times \mathcal{B}]$ is the *meet* (i.e., $\leq_\mathrm{M}$-greatest lower bound) of $[\mathcal{A}]$ and $[\mathcal{B}]$. Hence $\mathfrak{M}$ is a lattice. In fact, $\mathfrak{M}$ is a distributive lattice, meaning that join and meet distribute over each other: $\mathbf{a} + (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{c})$ and $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$. Notation for join and meet appears in the literature variously as $+$, $\times$, as $\vee$, $\wedge$, and confusingly as $\wedge$, $\vee$. We choose the $+$, $\times$ notation to avoid conflict with the logical notation and to match Sorbi and Terwijn [16].

$\mathfrak{M}$ has a least element $\mathbf{0} = [\omega^\omega]$ (and any $\mathcal{A}$ containing a recursive function has this degree), a second-least element $\mathbf{0}' = [\{f \mid f >_\mathrm{T} 0\}]$, and a greatest element $\mathbf{1} = [\emptyset]$. (The Medvedev degree $\mathbf{0}'$ has little to do with $0'$, the Turing jump of the 0 function. Here $\mathbf{0}'$ always refers to the second-least Medvedev degree.)

In any lattice, an element $\mathbf{a}$ is called *join-reducible* if there are $\mathbf{x}, \mathbf{y} < \mathbf{a}$ such that $\mathbf{a} = \mathbf{x} + \mathbf{y}$. Otherwise $\mathbf{a}$ is called *join-irreducible*. Dually, $\mathbf{a}$ is called *meet-reducible* if there are $\mathbf{x}, \mathbf{y} > \mathbf{a}$ such that $\mathbf{a} = \mathbf{x} \times \mathbf{y}$. Otherwise $\mathbf{a}$ is called *meet-irreducible*. Dyment [3] characterized the meet-reducible Medvedev degrees in the following theorem. Its corollary helps identify meet-irreducible Medvedev degrees.

Theorem 1.1 ([3]). *A Medvedev degree* $\mathbf{a}$ *is meet-reducible if and only if* $\mathbf{a} = [\mathcal{A}]$ *for a mass problem* $\mathcal{A}$ *for which there are r.e. sets* $V_0, V_1 \subseteq \omega^{<\omega}$ *such that*

- $(\forall f \in \mathcal{A})(\exists \sigma \in V_0 \cup V_1)(\sigma \subset f)$,
- *The following mass problems are* $\leq_\mathrm{M}$*-incomparable:*

$$\{f \in \mathcal{A} \mid (\exists \sigma \in V_0)(\sigma \subset f)\} \text{ and } \{f \in \mathcal{A} \mid (\exists \sigma \in V_1)(\sigma \subset f)\}$$

Corollary 1.2 ([3])**.** *If $\mathcal{A}$ is a mass problem such that $\sigma^\frown\mathcal{A} \subseteq \mathcal{A}$ for all $\sigma \in \omega^{<\omega}$, then $[\mathcal{A}]$ is meet-irreducible.*

In particular, $\mathbf{0}'$ is meet-irreducible because $\sigma^\frown f >_{\mathrm{T}} 0$ whenever $\sigma \in \omega^{<\omega}$ and $f >_{\mathrm{T}} 0$.

The problem of characterizing the join-irreducible Medvedev degrees was posed in [15]. In Section 2, we prove that $\mathbf{a} \in \mathfrak{M}$ is join-irreducible if and only if $\mathbf{a} = [\omega^\omega - \mathcal{I}]$ for some Turing ideal $\mathcal{I}$.

We have seen that $\mathfrak{M}$ is a distributive lattice with $\mathbf{0}$ and $\mathbf{1}$. In fact, $\mathfrak{M}$ is a Brouwer algebra. A *Brouwer algebra* is a distributive lattice with $\mathbf{0}$ and $\mathbf{1}$ such that for every $\mathbf{a}$ and $\mathbf{b}$ there is a least $\mathbf{c}$ such that $\mathbf{a} + \mathbf{c} \geq \mathbf{b}$. This least $\mathbf{c}$ is denoted by $\mathbf{a} \rightarrow \mathbf{b}$. For mass problems $\mathcal{A}$ and $\mathcal{B}$, define $\mathcal{A} \rightarrow \mathcal{B} = \{e^\frown g \mid (\forall f \in \mathcal{A})(\Phi_e(f \oplus g) \in \mathcal{B})\}$. Then $[\mathcal{A}] \rightarrow [\mathcal{B}] = [\mathcal{A} \rightarrow \mathcal{B}]$. A Brouwer algebra is dual to a Heyting algebra, but $\mathfrak{M}$ is proved not to be a Heyting algebra in Sorbi [12].

Brouwer algebras give semantics for propositional logic. For any Brouwer algebra $\mathfrak{B}$, a *valuation* is a function ν: propositional variables $\rightarrow \mathfrak{B}$. A valuation ν extends to all propositional formulas φ by defining

$$\begin{aligned}
\nu(\varphi \wedge \psi) &= \nu(\varphi) + \nu(\psi), \\
\nu(\varphi \vee \psi) &= \nu(\varphi) \times \nu(\psi), \\
\nu(\varphi \rightarrow \psi) &= \nu(\varphi) \rightarrow \nu(\psi), \text{ and} \\
\nu(\neg\varphi) &= \nu(\varphi) \rightarrow \mathbf{1}.
\end{aligned}$$

A propositional formula φ is called *valid* in $\mathfrak{B}$ if $\nu(\varphi) = \mathbf{0}$ for every valuation ν. Let $\mathrm{Th}(\mathfrak{B})$ denote the set of propositional formulas valid in $\mathfrak{B}$. The axioms of intuitionistic logic are valid in every Brouwer algebra $\mathfrak{B}$, so $\mathrm{IPC} \subseteq \mathrm{Th}(\mathfrak{B}) \subseteq \mathrm{CPC}$ for every Brouwer algebra $\mathfrak{B}$. Here IPC denotes intuitionistic logic and CPC denotes classical logic. Logics L for which $\mathrm{IPC} \subseteq L \subseteq \mathrm{CPC}$ are called *intermediate logics*.

Providing semantics for propositional logic was one of Medvedev's main motivations behind introducing $\mathfrak{M}$, and he proved $\mathrm{Th}(\mathfrak{M}) = \mathrm{JAN}$ in Medvedev [8]. JAN denotes the logic $\mathrm{IPC} + \neg p \vee \neg\neg p$ named after Jankov who studied it in Jankov [5]. In any Brouwer algebra $\mathfrak{B}$, the quotient of $\mathfrak{B}$ by the principal filter generated by $\mathbf{a} \in \mathfrak{B}$ is denoted by $\mathfrak{B}/\mathbf{a}$. The quotient $\mathfrak{B}/\mathbf{a}$ is isomorphic to the interval $[\mathbf{0}, \mathbf{a}]$ which is a Brouwer algebra under the operations inherited from $\mathfrak{B}$. Logics of the form $\mathrm{Th}(\mathfrak{M}/\mathbf{a})$ have been studied in Skvortsova [10], Sorbi [14], and Sorbi and Terwijn [16]. (Skvortsova and Dyment are the same person. Dyment got married and became Skvortsova.) The results in Section 3 and Section 4 are motivated by the following question which remains open:

Question 1.3 ([16])**.** Is $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}$ for all $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$?

Sorbi and Terwijn's study of Question 1.3 in [16] lead them to ask whether every degree $>_{\mathrm{M}} \mathbf{0}'$ bounds a join-irreducible degree $>_{\mathrm{M}} \mathbf{0}'$ because a "yes" answer to this question implies a "yes" answer to Question 1.3. However, Sorbi and Terwijn conjectured that there is a degree $>_{\mathrm{M}} \mathbf{0}'$ that bounds no join-irreducible degree $>_{\mathrm{M}} \mathbf{0}'$, and we prove that this is correct in Section 3. In Section 4 we provide slight extensions to some of the results in [14], thereby widening the class of degrees $\mathbf{a}$ for which $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}$ is known.

Lastly, in Section 5 we use techniques similar to those used to characterize the join-irreducible degrees to prove that the filter generated by the degrees of mass problems closed in ω^ω is not prime. This problem was posed in Bianchini and Sorbi [2] and in Sorbi [15].

2. Characterizing the join-irreducible Medvedev degrees

A *Turing ideal* is a set $\mathcal{I} \subseteq \omega^\omega$ that is closed downward under $\leq_{\mathrm{T}}$ (i.e., $f \in \mathcal{I} \wedge g \leq_{\mathrm{T}} f \rightarrow g \in \mathcal{I}$) and closed under $\oplus$ (i.e., $f, g \in \mathcal{I} \rightarrow f \oplus g \in \mathcal{I}$). We prove that $\mathbf{a} \in \mathfrak{M}$ is join-irreducible if and only if $\mathbf{a} = [\omega^\omega - \mathcal{I}]$ for some Turing ideal $\mathcal{I}$. We frequently use the following well-known lemma without mention:

Lemma 2.1 (see [1] Section III.2)**.** *In a distributive lattice,* $\mathbf{a}$ *is join-irreducible if and only if for all* $\mathbf{x}$ *and* $\mathbf{y}$*,* $\mathbf{a} \leq \mathbf{x} + \mathbf{y}$ *implies* $\mathbf{a} \leq \mathbf{x}$ *or* $\mathbf{a} \leq \mathbf{y}$*. Dually,* $\mathbf{a}$ *is meet-irreducible if and only if for all* $\mathbf{x}$ *and* $\mathbf{y}$*,* $\mathbf{a} \geq \mathbf{x} \times \mathbf{y}$ *implies* $\mathbf{a} \geq \mathbf{x}$ *or* $\mathbf{a} \geq \mathbf{y}$*.*

Proof. Suppose $\mathbf{a}$ is join-irreducible and $\mathbf{a} \leq \mathbf{x} + \mathbf{y}$. Then

$$\mathbf{a} = \mathbf{a} \times (\mathbf{x} + \mathbf{y}) = (\mathbf{a} \times \mathbf{x}) + (\mathbf{a} \times \mathbf{y}).$$

Thus $\mathbf{a} = \mathbf{a} \times \mathbf{x}$ or $\mathbf{a} = \mathbf{a} \times \mathbf{y}$ which means $\mathbf{a} \leq \mathbf{x}$ or $\mathbf{a} \leq \mathbf{y}$. Conversely, if $\mathbf{a}$ is join-reducible, then by definition there are $\mathbf{x}, \mathbf{y} < \mathbf{a}$ with $\mathbf{a} = \mathbf{x} + \mathbf{y}$. The proof for the meet-irreducible case is obtained by dualizing the proof for the join-irreducible case. □

For a mass problem $\mathcal{A}$, let $C(\mathcal{A})$ denote the *Turing upward-closure* of $\mathcal{A}$: $C(\mathcal{A}) = \{f \mid (\exists g \in \mathcal{A})(f \geq_{\mathrm{T}} g)\}$. A mass problem $\mathcal{A}$ is called *Turing upward-closed* if $\mathcal{A} = C(\mathcal{A})$. The identity functional witnesses $C(\mathcal{A}) \leq_{\mathrm{M}} \mathcal{A}$ for any mass problem $\mathcal{A}$, and if $\mathcal{A}$ and $\mathcal{B}$ are mass problems such that $\mathcal{A}$ is Turing upward-closed, then $\mathcal{A} \leq_{\mathrm{M}} \mathcal{B}$ if and only if $\mathcal{B} \subseteq \mathcal{A}$. Our starting point is the following observation:

Lemma 2.2 ([15])**.** *If* $\mathcal{A}$ *is a mass problem such that* $[\mathcal{A}]$ *is join-irreducible, then* $\omega^\omega - C(\mathcal{A})$ *is a Turing ideal.*

Proof. We prove the contrapositive. If $\omega^\omega - C(\mathcal{A})$ is not a Turing ideal, then there are $f, g \notin C(\mathcal{A})$ with $f \oplus g \in C(\mathcal{A})$. This means that $\{f\}, \{g\} \not\geq_{\mathrm{M}} \mathcal{A}$ but $\{f\} + \{g\} \geq_{\mathrm{M}} \mathcal{A}$. Thus $[\mathcal{A}]$ is join-reducible. □

The next lemma is the main step in our characterization.

Lemma 2.3. *If* $\mathcal{A}$ *is a mass problem such that* $[\mathcal{A}]$ *is join-irreducible, then* $\mathcal{A} \equiv_{\mathrm{M}} C(\mathcal{A})$

Proof. We prove the contrapositive. Suppose $\mathcal{A} \not\equiv_{\mathrm{M}} C(\mathcal{A})$. Then it must be that $\mathcal{A} \not\leq_{\mathrm{M}} C(\mathcal{A})$. We find mass problems $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{X}, \mathcal{Y} \not\geq_{\mathrm{M}} \mathcal{A}$ but $\mathcal{X} + \mathcal{Y} \geq_{\mathrm{M}} \mathcal{A}$. Thus $[\mathcal{A}]$ is join-reducible.

To find $\mathcal{X}$ and $\mathcal{Y}$, first find a sequence $(h_n \mid n \in \omega)$ of functions and a sequence $(e_n \mid n \in \omega)$ of indices such that

(i) $\Phi_{e_n}(h_n) \in \mathcal{A}$ for all $n \in \omega$,
(ii) $\Phi_n(h_{2n}) \notin \mathcal{A}$ and $\Phi_n(h_{2n+1}) \notin \mathcal{A}$ for all $n \in \omega$, and
(iii) $h_n(0) = \langle n, e_0, e_1, \ldots, e_{n-1} \rangle$ for all $n \in \omega$.

We find the desired sequences by iterating the following claim:

Claim. *If* $\mathcal{A} \not\leq_{\mathrm{M}} C(\mathcal{A})$*, then for every* $e, m \in \omega$ *there is an* $h \in C(\mathcal{A})$ *such that* $h(0) = m$ *and* $\Phi_e(h) \notin \mathcal{A}$*.*

Proof of claim. Suppose not. Then there are $e, m \in \omega$ such that $h(0) = m$ implies $\Phi_e(h) \in \mathcal{A}$ for all $h \in C(\mathcal{A})$. Thus $h \mapsto \Phi_e(m^\frown h)$ is a reduction witnessing $\mathcal{A} \leq_{\mathrm{M}} C(\mathcal{A})$, a contradiction. □

Suppose we have h_i and e_i for all $i < n$. To find h_n and e_n, let $e = \lfloor n/2 \rfloor$ and let $m = \langle n, e_0, e_1, \ldots, e_{n-1} \rangle$. By the claim, there is an $h_n \in C(\mathcal{A})$ such that $h_n(0) = m$ and $\Phi_e(h_n) \notin \mathcal{A}$. The fact that $h_n \in C(\mathcal{A})$ means that there is an e_n such that $\Phi_{e_n}(h_n) \in \mathcal{A}$.

Now set $\mathcal{X} = \{h_{2n} \mid n \in \omega\}$ and $\mathcal{Y} = \{h_{2n+1} \mid n \in \omega\}$. Then $\Phi_e(\mathcal{X}) \not\subseteq \mathcal{A}$ and $\Phi_e(\mathcal{Y}) \not\subseteq \mathcal{A}$ for each e by item (ii). Hence $\mathcal{X}, \mathcal{Y} \not\geq_{\mathrm{M}} \mathcal{A}$. The following reduction witnesses $\mathcal{X} + \mathcal{Y} \geq_{\mathrm{M}} \mathcal{A}$.

Given h, decompose h as $h = f \oplus g$ and decode $f(0)$ and $g(0)$ as $f(0) = \langle 2n, x_0, x_1, \ldots, x_{2n-1} \rangle$ and $g(0) = \langle 2m+1, y_0, y_1, \ldots, y_{2m} \rangle$. If either $f(0)$ or $g(0)$ is not of the required form, then output the 0 function (as such an h cannot be in $\mathcal{X} + \mathcal{Y}$). Otherwise output $\Phi_{x_{2m+1}}(g)$ if $2n > 2m+1$ and output $\Phi_{y_{2n}}(f)$ if $2m+1 > 2n$.

Suppose this reduction is applied to some $h = h_{2n} \oplus h_{2m+1} \in \mathcal{X} + \mathcal{Y}$. In this case $f = h_{2n}$, $g = h_{2m+1}$, and $f(0)$ and $g(0)$ are of the required form by item (iii). So if $2n > 2m+1$ we output $\Phi_{e_{2m+1}}(h_{2m+1})$ and if $2m+1 > 2n$ we output $\Phi_{e_{2n}}(h_{2n})$. Both alternatives are in $\mathcal{A}$ by item (i). Thus $\mathcal{X} + \mathcal{Y} \geq_{\mathrm{M}} \mathcal{A}$. □

Theorem 2.4. *A Medvedev degree* $\mathbf{a}$ *is join-irreducible if and only if* $\mathbf{a} = [\omega^\omega - \mathcal{I}]$ *for some Turing ideal* $\mathcal{I}$.

Proof. Suppose $\mathbf{a}$ is join-irreducible, and let $\mathcal{A}$ be a mass problem such that $\mathbf{a} = [\mathcal{A}]$. Then $\mathcal{I} = \omega^\omega - C(\mathcal{A})$ is a Turing ideal by Lemma 2.2, $\mathcal{A} \equiv_{\mathrm{M}} C(\mathcal{A})$ by Lemma 2.3, and therefore $\mathcal{A} \equiv_{\mathrm{M}} C(\mathcal{A}) = \omega^\omega - \mathcal{I}$. Hence $\mathbf{a} = [\omega^\omega - \mathcal{I}]$ for the Turing ideal $\mathcal{I}$.

Conversely, suppose $\mathcal{I}$ is a Turing ideal and let $\mathcal{X}$ and $\mathcal{Y}$ be mass problems such that $\mathcal{X}, \mathcal{Y} \ngeq_{\mathrm{M}} \omega^\omega - \mathcal{I}$. We show that $\mathcal{X} + \mathcal{Y} \ngeq_{\mathrm{M}} \omega^\omega - \mathcal{I}$. Observe $\mathcal{X}, \mathcal{Y} \nsubseteq \omega^\omega - \mathcal{I}$ for otherwise the identity functional would witness $\mathcal{X}, \mathcal{Y} \geq_{\mathrm{M}} \omega^\omega - \mathcal{I}$. Let $f \in \mathcal{X} \cap \mathcal{I}$ and let $g \in \mathcal{Y} \cap \mathcal{I}$, thereby making $f \oplus g \in (\mathcal{X} + \mathcal{Y}) \cap \mathcal{I}$. The function $f \oplus g$ is in $\mathcal{X} + \mathcal{Y}$, but it does not compute any member of $\omega^\omega - \mathcal{I}$. Therefore $\mathcal{X} + \mathcal{Y} \ngeq_{\mathrm{M}} \omega^\omega - \mathcal{I}$. Hence $[\omega^\omega - \mathcal{I}]$ is join-irreducible. □

Theorem 2.4 is also valid for the *Muchnik degrees* $\mathfrak{M}_{\mathrm{w}}$ in place of $\mathfrak{M}$, a fact first noticed by Terwijn [17]. $\mathfrak{M}_{\mathrm{w}}$ is defined just as $\mathfrak{M}$, but with *Muchnik reducibility* (also called *weak reducibility*) $\leq_{\mathrm{w}}$ in place of $\leq_{\mathrm{M}}$: $\mathcal{A} \leq_{\mathrm{w}} \mathcal{B}$ if for every $f \in \mathcal{B}$ there is a $g \in \mathcal{A}$ such that $f \geq_{\mathrm{T}} g$. $\mathfrak{M}_{\mathrm{w}}$ is a Brouwer algebra with $+$, $\times$, and $\to$ defined by $[\mathcal{A}]_{\mathrm{w}} + [\mathcal{B}]_{\mathrm{w}} = [\mathcal{A} + \mathcal{B}]_{\mathrm{w}}$, $[\mathcal{A}]_{\mathrm{w}} \times [\mathcal{B}]_{\mathrm{w}} = [\mathcal{A} \times \mathcal{B}]_{\mathrm{w}}$, and $[\mathcal{A}]_{\mathrm{w}} \to [\mathcal{B}]_{\mathrm{w}} = [\{g \mid (\forall f \in \mathcal{A})(\exists h \in \mathcal{B})(h \leq_{\mathrm{T}} f \oplus g)\}]_{\mathrm{w}}$. The proof of Lemma 2.2 also works for $\mathfrak{M}_{\mathrm{w}}$, and it is easy to check that $\mathcal{A} \equiv_{\mathrm{w}} C(\mathcal{A})$ for any mass problem $\mathcal{A}$ (i.e., the $\mathfrak{M}_{\mathrm{w}}$ analogue of Lemma 2.3 is trivial). This gives the forward direction of Theorem 2.4 for $\mathfrak{M}_{\mathrm{w}}$. The proof of the reverse direction of Theorem 2.4 also works for $\mathfrak{M}_{\mathrm{w}}$.

3. Degrees that bound no join-irreducible degrees $>_{\mathrm{M}} \mathbf{0}'$

Recall that JAN is the intermediate logic IPC $+ \neg p \vee \neg\neg p$. The results of this section and the next are motivated by Question 1.3: is $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq$ JAN for every $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$?

$\mathrm{Th}(\mathfrak{M}/\mathbf{0}') =$ CPC because $\mathfrak{M}/\mathbf{0}' \cong [\mathbf{0}, \mathbf{0}'] = \{\mathbf{0}, \mathbf{0}'\}$. In fact, $\mathbf{0}'$ is the only degree for which $\mathrm{Th}(\mathfrak{M}/\mathbf{0}') =$ CPC. This is because if $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$, then $\mathbf{0}' \to \mathbf{a} = \mathbf{a}$, hence $\mathbf{0}' \times (\mathbf{0}' \to \mathbf{a}) = \mathbf{0}'$. Thus let $p = \mathbf{0}'$ to see that the formula $p \vee \neg p$ is not valid in $\mathrm{Th}(\mathfrak{M}/\mathbf{a})$.

Furthermore, if $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$, then we cannot have $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \supsetneq$ JAN. It is an easy check that in any Brouwer algebra $\mathfrak{B}$ with meet-irreducible $\mathbf{0}$ (such as the algebras $\mathfrak{M}/\mathbf{a}$), $\neg p \vee \neg\neg p \in \mathrm{Th}(\mathfrak{B})$ if and only if $\mathbf{1}$ is join-irreducible. However, if $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ is join-irreducible, then $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) =$ JAN [14]. Thus if $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ and $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \supseteq$ JAN, then $\neg p \vee \neg\neg p \in \mathrm{Th}(\mathfrak{M}/\mathbf{a})$ which implies that $\mathbf{a}$ is join-irreducible which implies that $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) =$ JAN. Thus a "no" answer to Question 1.3 must yield a degree $\mathbf{a}$ such that $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \nsubseteq$ JAN and JAN $\nsubseteq \mathrm{Th}(\mathfrak{M}/\mathbf{a})$.

The following theorem shows that to give a "yes" answer to Question 1.3 it suffices to show that every $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ bounds a finite meet of join-irreducible degrees $>_{\mathrm{M}} \mathbf{0}'$.

Theorem 3.1 ([14]). *If* $\mathbf{a}$ *is a degree such that* $\mathbf{a} \geq_{\mathrm{M}} \prod_{i=0}^{n} \mathbf{d}_i$ *for join-irreducible degrees* $\mathbf{d}_i >_{\mathrm{M}} \mathbf{0}'$, $i \leq n$, *then* $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq$ JAN.

(The above theorem is stated more generally in [14]. Each degree $\mathbf{d}_i$ for $i \leq n$ is allowed to be either join-irreducible or $\mathfrak{De}$-irreducible. See the parenthetical discussion following Theorem 4.1 for the definition of $\mathfrak{De}$-irreducible and an explanation of why we do not consider such degrees here. Theorem 4.1 is a restatement of [14] Theorem 2.11 which is the main tool used to prove Theorem 3.1.)

The degrees of the mass problems $\mathcal{B}_f = \{g \mid g \nleq_{\mathrm{T}} f\}$ play an important role in the study of Question 1.3. The following lemma from Sorbi [13] encapsulates the properties of the $[\mathcal{B}_f]$'s that we need in this section and the next.

Lemma 3.2 ([13]).

(i) *Every* $[\mathcal{B}_f]$ *is join-irreducible.*
(ii) *Every* $\sum_{i=1}^{n} [\mathcal{B}_{f_i}]$ *is meet-irreducible.*

(iii) *Let V and J be finite sets and let U_v and I_j be finite sets for each $v \in V$ and $j \in J$. Let $\mathbf{x}_u^v$ and $\mathbf{y}_i^j$ be degrees of the form $[\mathcal{B}_f]$ for every $v \in V$, $u \in U_v$, $j \in J$, and $i \in I_j$. Let $\mathbf{a} = \sum_{v \in V} \prod_{u \in U_v} \mathbf{x}_u^v$ and $\mathbf{b} = \sum_{j \in J} \prod_{i \in I_j} \mathbf{y}_i^j$. Then $\mathbf{a} \leq_{\mathrm{M}} \mathbf{b}$ if and only if*

$$\big(\forall v \in V\big)\big(\exists j \in J\big)\big(\forall i \in I_j\big)\big(\exists u \in U_v\big)\big(\mathbf{x}_u^v \leq_{\mathrm{M}} \mathbf{y}_i^j\big).$$

(iv) *In the notation of item (iii),*

$$\mathbf{a} \to \mathbf{b} = \sum \Big\{ \prod_{i \in I_j} \mathbf{y}_i^j \;\Big|\; \Big(\forall v \in V\Big)\Big(\prod_{i \in I_j} \mathbf{y}_i^j \nleq_{\mathrm{M}} \prod_{u \in U_v} \mathbf{x}_u^v\Big)\Big\}$$

(where the empty join is $\mathbf{0}$).

Proof. Item (i) is by Theorem 2.4 and item (ii) is by Corollary 1.2. Item (iv) is proved in [13]. Item (iii) follows from item (iv) because $\mathbf{a} \leq_{\mathrm{M}} \mathbf{b}$ if and only if $\mathbf{b} \to \mathbf{a} = \mathbf{0}$. □

In [16] it is asked if every degree $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ bounds a join-irreducible degree $>_{\mathrm{M}} \mathbf{0}'$, and it is conjectured that this is not the case based on the evidence provided by the following theorem.

Theorem 3.3 ([16])**.** *There is a degree $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ such that $\mathbf{a} \ngeq_{\mathrm{M}} [\mathcal{B}_f]$ for every $f >_{\mathrm{T}} 0$.*

Our characterization of the join-irreducible degrees implies that every join-irreducible degree $>_{\mathrm{M}} \mathbf{0}'$ bounds some degree $[\mathcal{B}_f]$ with $f >_{\mathrm{T}} 0$. Thus the conjecture is correct.

Corollary 3.4 (to Theorem 2.4)**.** *If $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ is join-irreducible, then $\mathbf{a} \geq_{\mathrm{M}} [\mathcal{B}_f]$ for some $f >_{\mathrm{T}} 0$.*

Proof. If $\mathbf{a}$ is join-irreducible, then, by Theorem 2.4, $\mathbf{a} = [\omega^\omega - \mathcal{I}]$ for some Turing ideal $\mathcal{I}$. If $[\omega^\omega - \mathcal{I}] >_{\mathrm{M}} \mathbf{0}'$, then $\mathcal{I}$ contains some function $f >_{\mathrm{T}} 0$. Thus $\omega^\omega - \mathcal{I} \subseteq \mathcal{B}_f$. Hence $\mathbf{a} = [\omega^\omega - \mathcal{I}] \geq_{\mathrm{M}} [\mathcal{B}_f]$. □

Theorem 3.5. *There is a degree $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ such that every degree $\mathbf{x}$ with $\mathbf{0}' <_{\mathrm{M}} \mathbf{x} \leq_{\mathrm{M}} \mathbf{a}$ is join-reducible.*

Proof. By Theorem 3.3, let $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ be such that $\mathbf{a} \ngeq_{\mathrm{M}} [\mathcal{B}_f]$ for every $f >_{\mathrm{T}} 0$. This $\mathbf{a}$ is the desired degree because, by Corollary 3.4, if $\mathbf{a} \geq_{\mathrm{M}} \mathbf{x}$ for some join-irreducible $\mathbf{x} >_{\mathrm{M}} \mathbf{0}'$, then $\mathbf{a} \geq_{\mathrm{M}} [\mathcal{B}_f]$ for some $f >_{\mathrm{T}} 0$. □

The degree $\mathbf{a}$ satisfying Theorem 3.3 was constructed by diagonalization in [16]. We can give somewhat more concrete examples of degrees satisfying Theorem 3.3 and Theorem 3.5. Recall the following definitions. Functions $f, g >_{\mathrm{T}} 0$ are a *Turing minimal pair* if, for all h, $h \leq_{\mathrm{T}} f, g$ implies $h \leq_{\mathrm{T}} 0$. A function f has *minimal Turing degree* if, for all h, $h <_{\mathrm{T}} f$ implies $h \leq_{\mathrm{T}} 0$. Minimal pairs and minimal degrees exist. In fact, there are continuum many distinct minimal Turing degrees. See Lerman [6] Section II.4 and Section V.2.

Theorem 3.6. *If f and g are a minimal pair, then the degree $\mathbf{a} = [\mathcal{B}_f] \times [\mathcal{B}_g]$ witnesses Theorem 3.5.*

Proof. Let f and g be a minimal pair. Then $[\mathcal{B}_f], [\mathcal{B}_g] >_{\mathrm{M}} \mathbf{0}'$ because $f, g >_{\mathrm{T}} 0$. Thus $[\mathcal{B}_f] \times [\mathcal{B}_g] >_{\mathrm{M}} \mathbf{0}'$ because $\mathbf{0}'$ is meet-irreducible by Corollary 1.2. To show that $[\mathcal{B}_f] \times [\mathcal{B}_g]$ bounds no join-irreducible degree $>_{\mathrm{M}} \mathbf{0}'$, it suffices by Corollary 3.4 to show that $[\mathcal{B}_f] \times [\mathcal{B}_g]$ bounds no $[\mathcal{B}_h]$ for $h >_{\mathrm{T}} 0$. This is true because f, g is a minimal pair, so for any $h >_{\mathrm{T}} 0$, either $h \nleq_{\mathrm{T}} f$ or $h \nleq g$. Thus either $h \in \mathcal{B}_f$ or $h \in \mathcal{B}_g$ which means $\mathcal{B}_f \times \mathcal{B}_g$ contains a function $\equiv_{\mathrm{T}} h$. $\mathcal{B}_h$ contains no function $\leq_{\mathrm{T}} h$, therefore $\mathcal{B}_f \times \mathcal{B}_g \ngeq_{\mathrm{M}} \mathcal{B}_h$. □

We can extend the idea behind Theorem 3.6 to find a degree $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ that does not bound any finite meet of join-irreducible degrees $>_{\mathrm{M}} \mathbf{0}'$. Several of our examples in this section and the next are of the form $\big[\bigcup_{i \in \omega} i^\frown \mathcal{D}_i\big]$ for mass problems $\mathcal{D}_i$, $i \in \omega$.

Lemma 3.7. *Let $\mathbf{d} = \big[\bigcup_{i \in \omega} i^\frown \mathcal{D}_i\big]$ where $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$ for each $i \in \omega$. Then $\mathbf{d} >_{\mathrm{M}} \mathbf{0}'$.*

Proof. Suppose for a contradiction that Φ is a reduction witnessing $\mathbf{d} \leq_{\mathrm{M}} \mathbf{0}'$ (i.e., $\Phi(f) \in \bigcup_{i\in\omega} i^\frown\mathcal{D}_i$ for all $f >_{\mathrm{T}} 0$). Let $\sigma \in \omega^{<\omega}$ be such that $\Phi(\sigma)(0)\downarrow$ and let $i = \Phi(\sigma)(0)$. Then $f \mapsto \Phi(\sigma^\frown f)$ is a reduction witnessing $\mathbf{0}' \geq_{\mathrm{M}} [\mathcal{D}_i]$, contradicting $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$. □

Theorem 3.8. *There is a degree $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ such that no degree $\mathbf{x}$ with $\mathbf{0}' <_{\mathrm{M}} \mathbf{x} \leq_{\mathrm{M}} \mathbf{a}$ is of the form $\prod_{i=0}^n \mathbf{d}_i$ for join-irreducible degrees $\mathbf{d}_i >_{\mathrm{M}} \mathbf{0}'$, $i \leq n$.*

Proof. By Corollary 3.4, it suffices to find a degree $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ which is not above any degree of the form $\prod_{i=0}^n [\mathcal{B}_{f_i}]$ where $f_i >_{\mathrm{T}} 0$ for each $i \leq n$. Let $\{g_i \mid i \in \omega\}$ be a countable collection of functions all of distinct minimal Turing degree. Let $\mathcal{A} = \bigcup_{i\in\omega} i^\frown\mathcal{B}_{g_i}$ and put $\mathbf{a} = [\mathcal{A}]$. Lemma 3.7 proves that $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$.

Now consider any degree $\prod_{i=0}^n [\mathcal{B}_{f_i}]$, where $f_i >_{\mathrm{T}} 0$ for each $i \leq n$. There is a $j \in \omega$ such that $g_j \not\geq_{\mathrm{T}} f_i$ for each $i \leq n$. Thus for this j, $[\mathcal{B}_{g_j}] \not\geq_{\mathrm{M}} [\mathcal{B}_{f_i}]$ for each $i \leq n$. Therefore $[\mathcal{B}_{g_j}] \not\geq_{\mathrm{M}} \prod_{i=0}^n [\mathcal{B}_{f_i}]$ because $[\mathcal{B}_{g_j}]$ is meet-irreducible. Clearly $[\mathcal{B}_{g_j}] \geq_{\mathrm{M}} \mathbf{a}$, so $\mathbf{a} \not\geq_{\mathrm{M}} \prod_{i=0}^n [\mathcal{B}_{f_i}]$ as well. □

For mass problems $\mathcal{A}_i$, $i \in \omega$, the Medvedev degree $\left[\bigcup_{i\in\omega} i^\frown\mathcal{A}_i\right]$ is not in general the greatest lower bound of the degrees $[\mathcal{A}_i]$, $i \in \omega$. Such greatest lower bounds need not even exist. For example, the degrees $[\mathcal{B}_{g_i}]$, $i \in \omega$ from Theorem 3.8 do not have a greatest lower bound. This follows from results in Dyment [4] which studies when countable collections of degrees have least upper bounds and greatest lower bounds.

If $\mathbf{a}$ is a degree such that $\mathbf{a} \not\geq_{\mathrm{M}} \mathbf{d}$ for all join-irreducible $\mathbf{d} >_{\mathrm{M}} \mathbf{0}'$, then $\mathbf{a} \to \mathbf{d} = \mathbf{d}$ for all join-irreducible $\mathbf{d} >_{\mathrm{M}} \mathbf{0}'$. The degree $\mathbf{a}$ constructed in Theorem 3.8 enjoys a similar property.

Theorem 3.9. *There is a degree $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ such that $\mathbf{a} \to \prod_{i=0}^n \mathbf{d}_i = \prod_{i=0}^n \mathbf{d}_i$ whenever $\mathbf{d}_i >_{\mathrm{M}} \mathbf{0}'$ and is join-irreducible for each $i \leq n$.*

Proof. As in Theorem 3.8, let $\{g_i \mid i \in \omega\}$ be a countable collection of functions all of distinct minimal Turing degree, let $\mathcal{A} = \bigcup_{i\in\omega} i^\frown\mathcal{B}_{g_i}$, and put $\mathbf{a} = [\mathcal{A}]$. Suppose $\mathbf{d}_i >_{\mathrm{M}} \mathbf{0}'$ and is join-irreducible for each $i \leq n$. By Theorem 2.4, for each $i \leq n$ let $\mathcal{I}_i \subseteq \omega^\omega$ be a Turing ideal such that $\mathbf{d}_i = [\omega^\omega - \mathcal{I}_i]$. Thus $\prod_{i=0}^n \mathbf{d}_i = \left[\bigcup_{i=0}^n i^\frown(\omega^\omega - \mathcal{I}_i)\right]$ and

$$\mathbf{a} \to \prod_{i=0}^n \mathbf{d}_i = \left[\left\{ e^\frown g \;\middle|\; \left(\forall f \in \mathcal{A}\right)\left(\Phi_e(f \oplus g) \in \bigcup_{i=0}^n i^\frown(\omega^\omega - \mathcal{I}_i)\right)\right\}\right].$$

We now describe a reduction witnessing $\mathbf{a} \to \prod_{i=0}^n \mathbf{d}_i \geq_{\mathrm{M}} \prod_{i=0}^n \mathbf{d}_i$.

Given $e^\frown g$, for each $i \leq n+1$ search for a string $i^\frown\sigma_i$ such that $\Phi_e((i^\frown\sigma_i) \oplus g)(0)\downarrow$. If there is a $k \leq n$ such that

$$\Phi_e((i^\frown\sigma_i) \oplus g)(0) = \Phi_e((j^\frown\sigma_j) \oplus g)(0) = k$$

for two distinct $i, j \leq n+1$, choose the least such k and output $k^\frown g$. Otherwise output 0.

Suppose we apply this reduction to $e^\frown g \in \mathcal{A} \to \bigcup_{i=0}^n i^\frown(\omega^\omega - \mathcal{I}_i)$. $\Phi_e(f \oplus g)$ must be total for each $f \in \mathcal{A}$, and for each $i \in \omega$ there is an $f \in \mathcal{A}$ with $f(0) = i$. Thus for each $i \leq n+1$ the search finds a string $i^\frown\sigma_i$ such that $\Phi_e((i^\frown\sigma_i) \oplus g)(0)\downarrow$. Moreover, each $i^\frown\sigma_i$ can be extended to a function in $\mathcal{A}$, so $\Phi_e((i^\frown\sigma_i) \oplus g)(0) \leq n$ for each $i \leq n+1$. Therefore there is a least $k \leq n$ for which there are distinct $i, j \leq n+1$ with $\Phi_e((i^\frown\sigma_i) \oplus g)(0) = \Phi_e((j^\frown\sigma_j) \oplus g)(0) = k$. The reduction outputs $k^\frown g$, so we must show that $k^\frown g \in \bigcup_{i=0}^n i^\frown(\omega^\omega - \mathcal{I}_i)$ which means we must show that $g \notin \mathcal{I}_k$. Suppose for a contradiction that $g \in \mathcal{I}_k$. The functions g_i and g_j have distinct minimal degree, so either $g \not\leq_{\mathrm{T}} g_i$ or $g \not\leq_{\mathrm{T}} g_j$ ($g >_{\mathrm{T}} 0$ because $\mathbf{a} \not\geq_{\mathrm{M}} \prod_{i=0}^n \mathbf{d}_i$ by Theorem 3.8). For the sake of argument, suppose $g \not\leq_{\mathrm{T}} g_i$. Then $\sigma_i{}^\frown g \not\leq_{\mathrm{T}} g_i$ as well, so $\sigma_i{}^\frown g \in \mathcal{B}_{g_i}$ and $i^\frown\sigma_i{}^\frown g \in \mathcal{A}$. However, $\Phi_e((i^\frown\sigma_i{}^\frown g) \oplus g) \in k^\frown(\omega^\omega - \mathcal{I}_k)$ by the choice of $i^\frown\sigma_i$. This cannot be because $(i^\frown\sigma_i{}^\frown g) \oplus g \in \mathcal{I}_k$, thus anything it computes is also in $\mathcal{I}_k$. □

By Corollary 4.6 below, the degree $\mathbf{a} = \left[\bigcup_{i\in\omega} i^\frown \mathcal{B}_{g_i}\right]$ used to witness Theorem 3.8 and Theorem 3.9 satisfies $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}$ and so does any degree that bounds it. There are, however, degrees $>_{\mathrm{M}} \mathbf{0}'$ that do not bound any degree of the form $\left[\bigcup_{i\in\omega} i^\frown \mathcal{D}_i\right]$ where $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$ and is join-irreducible for each $i \in \omega$.

Theorem 3.10. *There is a degree* $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ *such that* $\mathbf{a} \not\geq_{\mathrm{M}} \left[\bigcup_{i\in\omega} i^\frown \mathcal{D}_i\right]$ *whenever* $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$ *and is join-irreducible for each* $i \in \omega$.

Proof. Let $\mathcal{D}_i$ be such that $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$ and is join-irreducible for each $i \in \omega$. By Corollary 3.4, for every $i \in \omega$ there is an $f_i >_{\mathrm{T}} 0$ such that $\mathcal{D}_i \geq_{\mathrm{M}} \mathcal{B}_{f_i}$. The mass problem $\mathcal{B}_{f_i}$ is Turing upward-closed for each $i \in \omega$, so $\mathcal{D}_i \subseteq \mathcal{B}_{f_i}$ for each $i \in \omega$. Thus $\bigcup_{i\in\omega} i^\frown \mathcal{D}_i \subseteq \bigcup_{i\in\omega} i^\frown \mathcal{B}_{f_i}$. Hence it suffices to find a degree $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$ that does not bound any degree of the form $\left[\bigcup_{i\in\omega} i^\frown \mathcal{B}_{f_i}\right]$, where $f_i >_{\mathrm{T}} 0$ for each $i \in \omega$.

We use the same construction used in [16] to prove Theorem 3.3. Build mass problems $\mathcal{A}_s \subseteq \{g \mid g >_{\mathrm{T}} 0\}$ such that $\{g \mid g >_{\mathrm{T}} 0\} - \mathcal{A}_s$ is finite for each $s \in \omega$. Set $\mathcal{A}_0 = \{g \mid g >_{\mathrm{T}} 0\}$. At stage $s+1$, choose $h_s >_{\mathrm{T}} 0$ such that h_s does not compute any of the (finitely many) functions in $\{g \mid g >_{\mathrm{T}} 0\} - \mathcal{A}_s$. If $\Phi_s(h_s)$ is total and $>_{\mathrm{T}} 0$, let $g_s = \Phi_s(h_s)$ and set $\mathcal{A}_{s+1} = \mathcal{A}_s - \{g_s\}$. Otherwise set $\mathcal{A}_{s+1} = \mathcal{A}_s$. Put $\mathcal{A} = \bigcap_{s\in\omega} \mathcal{A}_s$ and put $\mathbf{a} = [\mathcal{A}]$.

To see $\mathbf{a} >_{\mathrm{M}} \mathbf{0}'$, observe that by construction $\Phi_s(h_s) \notin \mathcal{A}$ for each $s \in \omega$. Now let $f_i >_{\mathrm{T}} 0$ for each $i \in \omega$. We need to show that $\Phi_e(\mathcal{A}) \not\subseteq \bigcup_{i\in\omega} i^\frown \mathcal{B}_{f_i}$ for every index e. To do this, we first show that the functions in $\{g \mid g >_{\mathrm{T}} 0\} - \mathcal{A}$ have distinct Turing degree. Suppose that g_i leaves $\mathcal{A}$ at stage $i+1$ and g_j leaves $\mathcal{A}$ at stage $j+1$ for $i+1 < j+1$ (i.e., at stage $i+1$ we had $\Phi_i(h_i) = g_i >_{\mathrm{T}} 0$, and at stage $j+1$ we had $\Phi_j(h_j) = g_j >_{\mathrm{T}} 0$). Then $g_i \not\leq_{\mathrm{T}} g_j$ because otherwise $g_i \leq_{\mathrm{T}} g_j \leq_{\mathrm{T}} h_j$, contradicting that h_j was chosen $\not\geq_{\mathrm{T}} g_i$ at stage $j+1$. Now suppose $\Phi_e(g)$ is total for all $g \in \mathcal{A}$. Fix any $\sigma \in \omega^{<\omega}$ such that $\Phi_e(\sigma)(0)\downarrow$, and let n be such that $\Phi_e(\sigma)(0) = n$. $\mathcal{A}$ is missing at most one function $\equiv_{\mathrm{T}} f_n$, so let $g \in \mathcal{A}$ be such that $\sigma \subset g$ and $g \equiv_{\mathrm{T}} f_n$. Then $\Phi_e(g)(0) = n$, but $\Phi_e(g) \notin n^\frown \mathcal{B}_{f_n}$. Hence $\Phi_e(\mathcal{A}) \not\subseteq \bigcup_{i\in\omega} i^\frown \mathcal{B}_{f_i}$. □

Question 3.11. Let $\mathbf{a}$ be the degree constructed in Theorem 3.10. Does $\mathbf{a} \to \left[\bigcup_{i\in\omega} i^\frown \mathcal{D}_i\right] = \left[\bigcup_{i\in\omega} i^\frown \mathcal{D}_i\right]$ whenever $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$ and is join-irreducible for each $i \in \omega$? Is $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}$?

Finally, we note that the answer to Question 1.3 is "no" for $\mathfrak{M}_{\mathrm{w}}$ in place of $\mathfrak{M}$. Let $f >_{\mathrm{T}} 0$ have minimal Turing degree, and let $\mathbf{a} = [\mathcal{B}_f]_{\mathrm{w}}$. Then, in $\mathfrak{M}_{\mathrm{w}}$, $[\mathbf{0}, \mathbf{a}] = \{\mathbf{0}, \mathbf{0}', \mathbf{a}\}$ and $\mathrm{JAN} \subsetneq \mathrm{Th}(\mathfrak{M}_{\mathrm{w}}/\mathbf{a}) \subsetneq \mathrm{CPC}$.

4. New degrees whose corresponding logic is contained in JAN

We extend Theorem 3.1 by proving $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}$ for degrees $\mathbf{a}$ such that $\mathbf{a} \geq_{\mathrm{M}} \left[\bigcup_{i\in\omega} i^\frown \mathcal{D}_i\right]$ for some collection of join-irreducible degrees $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$, $i \in \omega$.

A propositional formula is called *positive* if the connective '$\neg$' does not appear in it. For a logic L let L^+ denote the positive formulas in L, and for a Brouwer algebra $\mathfrak{B}$ let $\mathrm{Th}^+(\mathfrak{B})$ denote the set of positive formulas valid in $\mathfrak{B}$. JAN is the maximum intermediate logic L for which $L^+ = \mathrm{IPC}^+$ [5]. This means that $L^+ = \mathrm{IPC}^+$ implies $L \subseteq \mathrm{JAN}$ for any intermediate logic L. Thus $\mathrm{Th}^+(\mathfrak{B}) = \mathrm{IPC}^+$ implies $\mathrm{Th}(\mathfrak{B}) \subseteq \mathrm{JAN}$ for any Brouwer algebra $\mathfrak{B}$.

Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be Brouwer algebras. An injection $f\colon \mathfrak{B}_1 \to \mathfrak{B}_2$ is called a *B-embedding* if it preserves $\mathbf{0}$, $\mathbf{1}$, $+$, $\times$, and $\to$ (and therefore also $\neg$). An injection $f\colon \mathfrak{B}_1 \to \mathfrak{B}_2$ is called a *B^+-embedding* if it preserves $\mathbf{0}$, $+$, $\times$, and $\to$ (but not necessarily $\mathbf{1}$ or $\neg$). If $\mathfrak{B}_1$ B-embeds into $\mathfrak{B}_2$, then $\mathrm{Th}(\mathfrak{B}_2) \subseteq \mathrm{Th}(\mathfrak{B}_1)$, and if $\mathfrak{B}_1$ B^+-embeds into $\mathfrak{B}_2$, then $\mathrm{Th}^+(\mathfrak{B}_2) \subseteq \mathrm{Th}^+(\mathfrak{B}_1)$. Both of these facts are easily checked in light of [9] Theorem VI.2.4. If $\mathbf{a} \leq \mathbf{b}$ are in a Brouwer algebra $\mathfrak{B}$, then $\mathfrak{B}/\mathbf{a}$ B^+-embeds into $\mathfrak{B}/\mathbf{b}$ by the identity. This implies that $\mathrm{Th}^+(\mathfrak{B}/\mathbf{b}) \subseteq \mathrm{Th}^+(\mathfrak{B}/\mathbf{a})$, and it follows that the $\mathbf{a}$ for which $\mathrm{Th}(\mathfrak{B}/\mathbf{a}) \subseteq \mathrm{JAN}$ is upward-closed in any Brouwer algebra $\mathfrak{B}$.

Our goal is to B^+-embed a certain class of Brouwer algebras into $\mathfrak{M}/\mathbf{a}$. For any set X, let $\mathrm{Fr}(X)$ denote the free distributive lattice generated by X and let $\mathbf{0} \oplus \mathrm{Fr}(X)$ denote $\mathrm{Fr}(X)$ with a new bottom element $\mathbf{0}$. The elements of $\mathrm{Fr}(X)$ are all of the form $\sum_{v\in V}\prod_{u\in U_v}\mathbf{x}_u^v$ where V and the U_v are finite sets of indices and the $\mathbf{x}_u^v$ are all in X (see for example Balbes and Dwinger [1] Section V.3). For such representations, $\sum_{v\in V}\prod_{u\in U_v}\mathbf{x}_u^v \leq \sum_{j\in J}\prod_{i\in I_j}\mathbf{y}_i^j$ if and only if

$$\big(\forall v\in V\big)\big(\exists j\in J\big)\big(\forall i\in I_j\big)\big(\exists u\in U_v\big)\big(\mathbf{x}_u^v\leq \mathbf{y}_i^j\big).$$

If $\mathbf{a},\mathbf{b}\in \mathrm{Fr}(X)$ are such that $\mathbf{a}\not\geq \mathbf{b}$, then $\mathbf{a}\rightarrow\mathbf{b}$ exists. To see this, let $\mathbf{a}=\sum_{v\in V}\prod_{u\in U_v}\mathbf{x}_u^v$ and $\mathbf{b}=\sum_{j\in J}\prod_{i\in I_j}\mathbf{y}_i^j$ be representations for $\mathbf{a}$ and $\mathbf{b}$. Then check

$$\mathbf{a}\rightarrow\mathbf{b}=\sum\Big\{\prod_{i\in I_j}\mathbf{y}_i^j \;\Big|\; \Big(\forall v\in V\Big)\Big(\prod_{i\in I_j}\mathbf{y}_i^j\not\leq\prod_{u\in U_v}\mathbf{x}_u^v\Big)\Big\}.$$

If $\mathbf{a}\geq\mathbf{b}$ are in $\mathrm{Fr}(X)$ for an infinite X, then $\mathbf{a}\rightarrow\mathbf{b}$ fails to exist because in this case $\mathrm{Fr}(X)$ has no least element. We see then that $\mathbf{a}\rightarrow\mathbf{b}$ exists for every $\mathbf{a},\mathbf{b}\in\mathbf{0}\oplus\mathrm{Fr}(X)$. If X is finite, then so are $\mathrm{Fr}(X)$ and $\mathbf{0}\oplus\mathrm{Fr}(X)$. Hence both are Brouwer algebras. Let $\mathrm{Fr}(n)$ denote the free distributive lattice with n generators. The logic $\mathrm{LM}=\bigcap_{n\in\omega}\mathrm{Th}(\mathbf{0}\oplus\mathrm{Fr}(n))$ is called the *Medvedev logic of finite problems*. (LM is usually defined in terms of Brouwer algebras isomorphic to the $\mathbf{0}\oplus\mathrm{Fr}(n)$. See [16] for details.) We take advantage of the fact that $\mathrm{LM}^+=\mathrm{IPC}^+$ [8].

If X is infinite, then $\mathbf{0}\oplus\mathrm{Fr}(X)$ fails to be a Brouwer algebra only because it lacks a top element. Therefore the notion of a B^+-embedding makes sense when we allow $\mathfrak{B}_1$ or $\mathfrak{B}_2$ to be $\mathbf{0}\oplus\mathrm{Fr}(X)$. If we let $\mathbf{0}\oplus\mathrm{Fr}(X)\oplus\mathbf{1}$ denote $\mathrm{Fr}(X)$ with a new bottom element $\mathbf{0}$ and a new top element $\mathbf{1}$, then $\mathbf{0}\oplus\mathrm{Fr}(X)\oplus\mathbf{1}$ is always a Brouwer algebra.

For any partial order $(P,\leq_P)$, let $\mathrm{Fr}(P,\leq_P)$ denote the free distributive lattice generated by $(P,\leq_P)$. $\mathrm{Fr}(P,\leq_P)$ is the quotient $\mathrm{Fr}(P)/\equiv$ where, for $\mathbf{a}=\sum_{v\in V}\prod_{u\in U_v}\mathbf{x}_u^v$ and $\mathbf{b}=\sum_{j\in J}\prod_{i\in I_j}\mathbf{y}_i^j$ in $\mathrm{Fr}(P)$, $\mathbf{a}\equiv\mathbf{b}$ if and only if $(\mathbf{a}\preceq\mathbf{b})\wedge(\mathbf{b}\preceq\mathbf{a})$ and $\mathbf{a}\preceq\mathbf{b}$ if and only if

$$\big(\forall v\in V\big)\big(\exists j\in J\big)\big(\forall i\in I_j\big)\big(\exists u\in U_v\big)\big(\mathbf{x}_u^v\leq_P \mathbf{y}_i^j\big).$$

$\mathrm{Fr}(P,\leq_P)$ is always a distributive lattice, and $\mathbf{0}\oplus\mathrm{Fr}(P,\leq_P)\oplus\mathbf{1}$ is always a Brouwer algebra; also see [13].

The following lemmas facilitate our embeddings. Lemma 4.3 is a slight generalization of the claim in the proof of [13] Lemma 2.3 and of [10] Lemma 6. The embedding is done in Theorem 4.4 which is nearly identical to [14] Theorem 2.11. Part of the reason for reproducing the proof here is to (hopefully) correct the notational inconsistencies in the proof in [14]. We restate [14] Theorem 2.11 for reference.

Theorem 4.1 ([14] Theorem 2.11)**.** *Let $\mathbf{d}=\prod_{i=0}^n\mathbf{d}_i$ where $\mathbf{d}_i>_{\mathrm{M}}\mathbf{0}'$ and $\mathbf{d}_i$ is join-irreducible for each $i\leq n$. Then $\mathbf{0}\oplus\mathrm{Fr}(P,\leq_P)\oplus\mathbf{1}$ B-embeds into $\mathfrak{M}/\mathbf{d}$ for every countable partial order $(P,\leq_P)$.*

(The above theorem is stated more generally in [14]. Each degree $\mathbf{d}_i$ for $i\leq n$ is allowed to be either join-irreducible or $\mathfrak{De}$-irreducible. A degree $\mathbf{a}$ is *dense* if it is of the form $[\mathcal{A}]$ where $\mathcal{A}$ is dense in ω^ω, and a degree $\mathbf{d}$ is *$\mathfrak{De}$-irreducible* if $\mathbf{a}\rightarrow\mathbf{d}=\mathbf{d}$ for all dense degrees $\mathbf{a}$. We do not consider $\mathfrak{De}$-irreducible degrees in our version of [14] Theorem 2.11, which is Theorem 4.4 below, because in Theorem 4.4 we require that the mass problems $\mathcal{D}_i$ (which play the role of the degrees $\mathbf{d}_i$ in [14] Theorem 2.11) are Turing upward-closed. Mass problems that are Turing upward-closed are dense and hence their degrees are not $\mathfrak{De}$-irreducible.)

Lemma 4.2 ([3])**.** *If $\mathcal{X}\not\geq_{\mathrm{M}}\mathcal{Y}$ are mass problems, then there is a $\mathcal{W}\subseteq\mathcal{X}$ with $|\mathcal{W}|\leq\omega$ such that $\mathcal{W}\not\geq_{\mathrm{M}}\mathcal{Y}$.*

Proof. $\mathcal{X}\not\geq_{\mathrm{M}}\mathcal{Y}$ means that there is no Turing functional Φ such that $\Phi(\mathcal{X})\subseteq\mathcal{Y}$. Thus for each functional Φ_e there must be some function $f_e\in\mathcal{X}$ such that $\Phi_e(f_e)\notin\mathcal{Y}$. Let $\mathcal{W}$ consist of a choice of one such $f_e\in\mathcal{X}$ for each functional Φ_e. □

Lemma 4.3. *Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{F}_i$ for $i \in \omega$ be mass problems such that $\bigcup_{i\in\omega} i^\frown\mathcal{F}_i \leq_{\mathrm{M}} \mathcal{U}+\mathcal{V}$ and $\sigma^\frown\mathcal{U} \subseteq \mathcal{U}$ for all $\sigma \in \omega^{<\omega}$. Then there are mass problems $\mathcal{V}_i$ for $i \in \omega$ such that $\bigcup_{i\in\omega} i^\frown\mathcal{V}_i \equiv_{\mathrm{M}} \mathcal{V}$ and $\mathcal{F}_i \leq_{\mathrm{M}} \mathcal{U}+\mathcal{V}_i$ for each $i \in \omega$.*

Proof. Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{F}_i$ for $i \in \omega$ be as in the statement of the lemma. Let Φ be such that $\Phi(\mathcal{U}+\mathcal{V}) \subseteq \bigcup_{i\in\omega} i^\frown\mathcal{F}_i$. For each $i \in \omega$, define $\mathcal{V}_i = \{g \in \mathcal{V} \mid (\exists\sigma \in \omega^{<\omega})(\Phi(\sigma \oplus g)(0) = i)\}$. $\mathcal{V} \leq_{\mathrm{M}} \bigcup_{i\in\omega} i^\frown\mathcal{V}_i$ is clear. $\bigcup_{i\in\omega} i^\frown\mathcal{V}_i \leq_{\mathrm{M}} \mathcal{V}$ by the reduction which, given g, searches for a $\sigma \in \omega^{<\omega}$ such that $\Phi(\sigma \oplus g)(0)\downarrow$ and outputs $\Phi(\sigma \oplus g)(0)^\frown g$. To see $i^\frown\mathcal{F}_i \leq_{\mathrm{M}} \mathcal{U}+\mathcal{V}_i$, consider the reduction which, given $f \oplus g$, searches for a $\sigma \in \omega^{<\omega}$ such that $\Phi(\sigma \oplus g)(0) = i$ and outputs $\Phi((\sigma^\frown f) \oplus g)$. If $f \oplus g \in \mathcal{U}+\mathcal{V}_i$, then such a σ is found, $\sigma^\frown f$ is in $\mathcal{U}$, and $\Phi((\sigma^\frown f) \oplus g)$ is in $i^\frown\mathcal{F}_i$. □

Theorem 4.4. *Let $\mathbf{d} = \left[\bigcup_{i\in\omega} i^\frown\mathcal{D}_i\right]$ where $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$, $[\mathcal{D}_i]$ is join-irreducible, and $\mathcal{D}_i$ is Turing upward-closed for each $i \in \omega$. Then $\mathbf{0} \oplus \mathrm{Fr}(2^\omega)$ B^+-embeds into $\mathfrak{M}/\mathbf{d}$.*

Proof. Let $\mathcal{D}_i$ for $i \in \omega$ be as in the statement of the theorem, let $\mathcal{D} = \bigcup_{i\in\omega} i^\frown\mathcal{D}_i$, and let $\mathbf{d} = [\mathcal{D}]$. Lemma 3.7 proves that $\mathbf{d} >_{\mathrm{M}} \mathbf{0}'$. By Lemma 4.2, let $\mathcal{A} \subseteq \{f \mid f >_{\mathrm{T}} 0\}$ be a countable mass problem such that $\mathcal{A} \not\geq_{\mathrm{M}} \mathcal{D}$. Let $\{f_x \mid x \in 2^\omega\}$ be a collection of functions such that $f_x \mid_{\mathrm{T}} f_y$ for all $x, y \in 2^\omega$ with $x \neq y$ and that $f \not\leq_{\mathrm{T}} f_x$ for all $f \in \mathcal{A}$ and $x \in 2^\omega$. Such a sequence can be constructed via standard recursion-theoretic techniques: build a perfect tree whose paths are Turing incomparable and do not compute any functions in $\mathcal{A}$. See for example [6] Section II.4. Notice that $\mathcal{B}_{f_x} \leq_{\mathrm{M}} \mathcal{A}$ (because $\mathcal{A} \subseteq \mathcal{B}_{f_x}$) for each $x \in 2^\omega$.

Define $G\colon \mathbf{0} \oplus \mathrm{Fr}(2^\omega) \to \mathfrak{M}$ as follows. Let $G(\mathbf{0}) = \mathbf{0}$ and let $G(x) = [\mathcal{B}_{f_x}]$ on the generators $x \in 2^\omega$ of $\mathrm{Fr}(2^\omega)$. Then extend G to all of $\mathbf{0} \oplus \mathrm{Fr}(2^\omega)$ so that $G\big(\sum_{v\in V}\prod_{u\in U_v} x_u^v\big) = \sum_{v\in V}\prod_{u\in U_v} G\big(x_u^v\big)$. G preserves $\mathbf{0}$, $+$, and $\times$ by definition, and G is injective and preserves $\to$ by Lemma 3.2 items (iii) and (iv). Hence G is a B^+-embedding (this is essentially [13] Corollary 2.5). Now define $H\colon \mathbf{0} \oplus \mathrm{Fr}(2^\omega) \to \mathfrak{M}/\mathbf{d}$ by $H(\mathbf{a}) = G(\mathbf{a}) \times \mathbf{d}$ for all $\mathbf{a} \in \mathbf{0} \oplus \mathrm{Fr}(2^\omega)$. This H is the desired B^+-embedding. By definition, H preserves $\mathbf{0}$, $+$, and $\times$. We must show that H is injective and that H preserves $\to$.

Clearly $H(\mathbf{a}) = \mathbf{0}$ if and only if $\mathbf{a} = \mathbf{0}$, so to show that H is injective we let $\mathbf{a}, \mathbf{b} \in \mathrm{Fr}(2^\omega)$ be such that $H(\mathbf{a}) \leq_{\mathrm{M}} H(\mathbf{b})$ and show that $\mathbf{a} \leq \mathbf{b}$. Let $\mathbf{a} = \sum_{v\in V}\prod_{u\in U_v} x_u^v$ be a representation for $\mathbf{a}$ and let $\mathbf{b} = \sum_{j\in J}\prod_{i\in I_j} y_i^j$ be a representation for $\mathbf{b}$. $H(\mathbf{a}) \leq_{\mathrm{M}} H(\mathbf{b})$ means that

$$\sum_{v\in V}\prod_{u\in U_v} G\big(x_u^v\big) \times \mathbf{d} \leq_{\mathrm{M}} \sum_{j\in J}\prod_{i\in I_j} G\big(y_i^j\big) \times \mathbf{d}.$$

Therefore

$$\sum_{v\in V}\prod_{u\in U_v} G\big(x_u^v\big) \times \mathbf{d} \leq_{\mathrm{M}} \sum_{j\in J}\prod_{i\in I_j} G\big(y_i^j\big) = \prod\Big\{\sum_{j\in J} G\big(y_{\alpha(j)}^j\big) \;\Big|\; \alpha \in \prod_{j\in J} I_j\Big\}$$

where the equality is by distributivity ($\prod_{j\in J} I_j$ denotes the Cartesian product of the I_j's). Hence

$$\sum_{v\in V}\prod_{u\in U_v} G\big(x_u^v\big) \times \mathbf{d} \leq_{\mathrm{M}} \sum_{j\in J} G\big(y_{\alpha(j)}^j\big) \text{ for each } \alpha \in \prod_{j\in J} I_j.$$

Each $\sum_{j\in J} G\big(y_{\alpha(j)}^j\big)$ is meet-irreducible by Lemma 3.2 item (ii). Also, $\mathbf{d} \not\leq_{\mathrm{M}} \sum_{j\in J} G\big(y_{\alpha(j)}^j\big)$ for each $\alpha \in \prod_{j\in J} I_j$ because $\sum_{j\in J} G\big(y_{\alpha(j)}^j\big) \leq_{\mathrm{M}} [\mathcal{A}]$ but $\mathbf{d} \not\leq_{\mathrm{M}} [\mathcal{A}]$. Thus

$$\sum_{v\in V}\prod_{u\in U_v} G\big(x_u^v\big) \leq_{\mathrm{M}} \sum_{j\in J} G\big(y_{\alpha(j)}^j\big) \text{ for each } \alpha \in \prod_{j\in J} I_j,$$

and this implies that

$$\sum_{v\in V}\prod_{u\in U_v} G\big(x^v_u\big) \leq_{\mathrm{M}} \prod\Big\{\sum_{j\in J} G\big(y^j_{\alpha(j)}\big)\ \Big|\ \alpha \in \prod_{j\in J} I_j\Big\}.$$

The left-hand side of the above inequality is $G(\mathbf{a})$ and the right-hand side is $G(\mathbf{b})$. G is a B^+-embedding, so we conclude $\mathbf{a} \leq \mathbf{b}$.

If either of $\mathbf{a},\mathbf{b} \in \mathbf{0} \oplus \mathrm{Fr}(2^\omega)$ is $\mathbf{0}$, then clearly $H(\mathbf{a}\rightarrow\mathbf{b}) = H(\mathbf{a})\rightarrow H(\mathbf{b})$. So as before, let $\mathbf{a} = \sum_{v\in V}\prod_{u\in U_v} x^v_u$ and let $\mathbf{b} = \sum_{j\in J}\prod_{i\in I_j} y^j_i$ be in $\mathrm{Fr}(2^\omega)$. We see $H(\mathbf{a}\rightarrow\mathbf{b}) \geq_{\mathrm{M}} H(\mathbf{a})\rightarrow H(\mathbf{b})$ because

$$H(\mathbf{a}\rightarrow\mathbf{b}) + H(\mathbf{a}) = H((\mathbf{a}\rightarrow\mathbf{b}) + \mathbf{a}) \geq_{\mathrm{M}} H(\mathbf{b}).$$

To show that $H(\mathbf{a}\rightarrow\mathbf{b}) \leq_{\mathrm{M}} H(\mathbf{a})\rightarrow H(\mathbf{b})$, we show that if $\mathbf{z} \in \mathfrak{M}$ is such that $H(\mathbf{b}) \leq_{\mathrm{M}} H(\mathbf{a}) + \mathbf{z}$, then $H(\mathbf{a}\rightarrow\mathbf{b}) \leq_{\mathrm{M}} \mathbf{z}$. Suppose $H(\mathbf{b}) \leq_{\mathrm{M}} H(\mathbf{a}) + \mathbf{z}$. That is,

$$\sum_{j\in J}\prod_{i\in I_j} G\big(y^j_i\big) \times \mathbf{d} \leq_{\mathrm{M}} \Big(\sum_{v\in V}\prod_{u\in U_v} G\big(x^v_u\big) \times \mathbf{d}\Big) + \mathbf{z}. \tag{1}$$

Since $\mathbf{a}\rightarrow\mathbf{b} = \sum\big\{\prod_{i\in I_j} y^j_i \mid (\forall v\in V)\big(\prod_{i\in I_j} y^j_i \nleq \prod_{u\in U_v} x^v_u\big)\big\}$, we have

$$\begin{aligned}H(\mathbf{a}\rightarrow\mathbf{b}) &= G(\mathbf{a}\rightarrow\mathbf{b}) \times \mathbf{d}\\ &= \sum\Big\{\prod_{i\in I_j} G\big(y^j_i\big)\ \Big|\ \Big(\forall v\in V\Big)\Big(\prod_{i\in I_j} G\big(y^j_i\big) \nleq_{\mathrm{M}} \prod_{u\in U_v} G\big(x^v_u\big)\Big)\Big\} \times \mathbf{d}.\end{aligned}$$

It suffices to show that, given $j \in J$, if $\prod_{i\in I_j} G\big(y^j_i\big)$ satisfies

$$\Big(\forall v\in V\Big)\Big(\prod_{i\in I_j} G\big(y^j_i\big) \nleq_{\mathrm{M}} \prod_{u\in U_v} G\big(x^v_u\big)\Big),$$

then $\prod_{i\in I_j} G\big(y^j_i\big) \times \mathbf{d} \leq_{\mathrm{M}} \mathbf{z}$. Suppose $\prod_{i\in I_j} G\big(y^j_i\big)$ is such a meet. Then we know

$$\Big(\forall v\in V\Big)\Big(\exists u\in U_v\Big)\Big(\prod_{i\in I_j} G\big(y^j_i\big) \nleq_{\mathrm{M}} G\big(x^v_u\big)\Big).$$

By choosing such a $u \in U_v$ for every $v \in V$ and by appealing to Lemma 3.2 items (i) and (ii), we see that there is an $\alpha \in \prod_{v\in V} U_v$ such that

$$\prod_{i\in I_j} G\big(y^j_i\big) \nleq_{\mathrm{M}} \sum_{v\in V} G\big(x^v_{\alpha(v)}\big). \tag{2}$$

Distributing $\sum_{v\in V}\prod_{u\in U_v} G\big(x^v_u\big)$ in the right-hand side of (1) yields

$$\prod_{i\in I_j} G\big(y^j_i\big) \times \mathbf{d} \leq_{\mathrm{M}} \sum_{v\in V} G\big(x^v_{\alpha(v)}\big) + \mathbf{z}.$$

The degree $\sum_{v\in V} G\big(x^v_{\alpha(v)}\big)$ is a finite join of degrees of the form $[\mathcal{B}_f]$ and thus has a representative $\mathcal{U}$ such that $\sigma^\frown\mathcal{U} \subseteq \mathcal{U}$ for all $\sigma \in \omega^{<\omega}$. So by Lemma 4.3 there are mass problems $\mathcal{Z}_i$ for $i \in I_j$ and

$\widehat{\mathcal{Z}}_i$ for $i \in \omega$ such that

$$\mathbf{z} = \Big(\prod_{i \in I_j} [\mathcal{Z}_i]\Big) \times \Big[\bigcup_{i \in \omega} i^\frown \widehat{\mathcal{Z}}_i\Big],$$

$$G\big(y_i^j\big) \leq_{\mathrm{M}} \sum_{v \in V} G\big(x^v_{\alpha(v)}\big) + [\mathcal{Z}_i] \text{ for each } i \in I_j, \text{ and}$$

$$[\mathcal{D}_i] \leq_{\mathrm{M}} \sum_{v \in V} G\big(x^v_{\alpha(v)}\big) + [\widehat{\mathcal{Z}}_i] \text{ for each } i \in \omega.$$

Each $G\big(y_i^j\big)$ is join-irreducible, and $G\big(y_i^j\big) \nleq_{\mathrm{M}} \sum_{v \in V} G\big(x^v_{\alpha(v)}\big)$ by (2). Thus $G\big(y_i^j\big) \leq_{\mathrm{M}} [\mathcal{Z}_i]$ for each $i \in \omega$, so $\prod_{i \in I_j} G\big(y_i^j\big) \leq_{\mathrm{M}} \prod_{i \in I_j} [\mathcal{Z}_i]$. Each $[\mathcal{D}_i]$ is join-irreducible by assumption, and also $[\mathcal{D}_i] \nleq_{\mathrm{M}} \sum_{v \in V} G\big(x^v_{\alpha(v)}\big)$ because the right-hand side is $\leq_{\mathrm{M}} [\mathcal{A}]$ but the left-hand side is not. It follows that $[\mathcal{D}_i] \leq_{\mathrm{M}} [\widehat{\mathcal{Z}}_i]$ for each $i \in \omega$, and so $\widehat{\mathcal{Z}}_i \subseteq \mathcal{D}_i$ for each $i \in \omega$ because each $\mathcal{D}_i$ is Turing upward-closed. Thus $\bigcup_{i \in \omega} i^\frown \widehat{\mathcal{Z}}_i \subseteq \mathcal{D}$, so $\mathbf{d} \leq_{\mathrm{M}} \big[\bigcup_{i \in \omega} i^\frown \widehat{\mathcal{Z}}_i\big]$. Therefore

$$\prod_{i \in I_j} G\big(y_i^j\big) \times \mathbf{d} \leq_{\mathrm{M}} \big(\prod_{i \in I_j} [\mathcal{Z}_i]\big) \times \big[\bigcup_{i \in \omega} i^\frown \widehat{\mathcal{Z}}_i\big] = \mathbf{z}$$

as desired. □

Corollary 4.5. *If $\mathbf{a} \geq_{\mathrm{M}} \mathbf{d}$ are degrees such that $\mathbf{d} = \big[\bigcup_{i \in \omega} i^\frown \mathcal{D}_i\big]$ where $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$ and is join-irreducible for each $i \in \omega$, then $\mathbf{0} \oplus \mathrm{Fr}(2^\omega)$ B^+-embeds into $\mathfrak{M}/\mathbf{a}$.*

Proof. Let $\mathbf{a}$, $\mathbf{d}$, and $\mathcal{D}_i$ for $i \in \omega$ be as in the statement of the corollary. Let $\mathbf{d}_0 = \big[\bigcup_{i \in \omega} i^\frown C(\mathcal{D}_i)\big]$ and notice that $\mathbf{d} \geq_{\mathrm{M}} \mathbf{d}_0$. $\mathcal{D}_i \equiv_{\mathrm{M}} C(\mathcal{D}_i)$ for each $i \in \omega$ by Lemma 2.3, so $\mathbf{d}_0$ satisfies the hypotheses of Theorem 4.4. Thus $\mathbf{0} \oplus \mathrm{Fr}(2^\omega)$ B^+-embeds into $\mathfrak{M}/\mathbf{d}_0$ which B^+-embeds into $\mathfrak{M}/\mathbf{a}$. □

Corollary 4.6. *If $\mathbf{a} \geq_{\mathrm{M}} \mathbf{d}$ are degrees such that $\mathbf{d} = \big[\bigcup_{i \in \omega} i^\frown \mathcal{D}_i\big]$ where $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$ and is join-irreducible for each $i \in \omega$, then* $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}$.

Proof. The Brouwer algebra $\mathbf{0} \oplus \mathrm{Fr}(n)$ B^+-embeds into $\mathbf{0} \oplus \mathrm{Fr}(2^\omega)$ for each n, and $\mathbf{0} \oplus \mathrm{Fr}(2^\omega)$ B^+-embeds into $\mathfrak{M}/\mathbf{a}$ by Corollary 4.5. Thus $\mathrm{Th}^+(\mathfrak{M}/\mathbf{a}) \subseteq \bigcap_{n \in \omega} \mathrm{Th}^+(\mathbf{0} \oplus \mathrm{Fr}(n)) = \mathrm{LM}^+ = \mathrm{IPC}^+$. So $\mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}$. □

Theorem 4.4 can be modified to B-embed $\mathbf{0} \oplus \mathrm{Fr}(2^\omega) \oplus \mathbf{1}$ into $\mathfrak{M}/\mathbf{d}$ for degrees $\mathbf{d}$ as in the statement of Theorem 4.4. However, if $\mathbf{a} \leq \mathbf{b}$ in a Brouwer algebra $\mathfrak{B}$, it is not in general the case that $\mathfrak{B}/\mathbf{a}$ B-embeds into $\mathfrak{B}/\mathbf{b}$. So the proof of Corollary 4.5 fails for B-embedding $\mathbf{0} \oplus \mathrm{Fr}(2^\omega) \oplus \mathbf{1}$. Theorem 4.4 can also be modified to prove a more precise analogue of [14] Theorem 2.11 (restated as Theorem 4.1 above). Let $\mathbf{d} = \big[\bigcup_{i \in \omega} i^\frown \mathcal{D}_i\big]$ where $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$, $[\mathcal{D}_i]$ is join-irreducible, and $\mathcal{D}_i$ is Turing upward-closed for each $i \in \omega$. Then $\mathbf{0} \oplus \mathrm{Fr}(P, \leq_P) \oplus \mathbf{1}$ B-embeds into $\mathfrak{M}/\mathbf{d}$ for every countable partial order $(P, \leq_P)$.

5. $\mathfrak{F}_{\mathrm{cl}}$ IS NOT PRIME

Recall that a filter $\mathfrak{F}$ in a lattice is called *prime* if $\mathbf{a} + \mathbf{b} \in \mathfrak{F} \rightarrow \mathbf{a} \in \mathfrak{F} \vee \mathbf{b} \in \mathfrak{F}$ for all $\mathbf{a}$ and $\mathbf{b}$ in the lattice. Theorem 2.4 can be phrased as a characterization of the prime principal filters in $\mathfrak{M}$: a degree $\mathbf{a}$ generates a prime filter if and only if $\mathbf{a} = [\omega^\omega - \mathcal{I}]$ for some Turing ideal $\mathcal{I}$. There is little general theory of the non-principal filters in $\mathfrak{M}$, but several specific cases have been studied in Dyment [3], Sorbi [11], Bianchini and Sorbi [2], and Lewis, Shore, and Sorbi [7]. See also [15] for a summary of many of the results appearing in these works. We consider the filters $\mathfrak{F}$ and $\mathfrak{F}_{\mathrm{cl}}$:

Definition 5.1.

- A degree $\mathbf{a}$ is called *dense* (*closed*) if $\mathbf{a} = [\mathcal{A}]$ for an $\mathcal{A}$ that is dense (closed) in ω^ω.
- $\mathfrak{I}$ denotes the ideal generated by $\{\mathbf{a} \mid \mathbf{a} \text{ is dense}\}$.

- $\mathfrak{F}$ denotes $\mathfrak{M} - \mathfrak{I}$.
- $\mathfrak{F}_{\rm cl}$ denotes the filter generated by $\{\mathbf{a} \mid \mathbf{a} >_{\rm M} \mathbf{0}$ and is closed$\}$.

The join and meet of two dense degrees is dense [3], and the join and meet of two closed degrees is closed [2]. Thus $\mathfrak{I} = \{\mathbf{b} \mid (\exists \mathbf{a} \geq_{\rm M} \mathbf{b})(\mathbf{a}$ is dense$)\}$ and $\mathfrak{F}_{\rm cl} = \{\mathbf{b} \mid (\exists \mathbf{a} \leq_{\rm M} \mathbf{b})(\mathbf{a} >_{\rm M} \mathbf{0}$ and is closed$)\}$. The basic properties of $\mathfrak{I}$, $\mathfrak{F}$, and $\mathfrak{F}_{\rm cl}$ are as follows: $\mathfrak{I}$ is a prime ideal [11], $\mathfrak{F}$ is a prime filter [2], $\mathfrak{I}$ is not principal [3], $\mathfrak{F}$ and $\mathfrak{F}_{\rm cl}$ are not principal [2], and $\mathfrak{F}_{\rm cl} \subsetneq \mathfrak{F}$ [2]. Both [2] and [15] ask for a proof that $\mathfrak{F}_{\rm cl}$ is not prime. We provide a proof of this fact now.

Lemma 5.2. *For any $f \in \omega^\omega$ there are $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$ such that $\mathcal{A} + \mathcal{B} \geq_{\rm M} \{f\}$ and, for any closed $\mathcal{C} \subseteq \omega^\omega$, if $\mathcal{A} \geq_{\rm M} \mathcal{C}$ or $\mathcal{B} \geq_{\rm M} \mathcal{C}$, then $\mathcal{C}$ contains a recursive function.*

Proof. Fix a recursive bijection $\omega \leftrightarrow \omega^{<\omega}$. For $e, n \in \omega$, if

$$\forall m \forall \sigma (\exists \tau \supseteq \sigma)(\Phi_e(n^\frown\tau)(m)\downarrow),$$

then define $\eta(e,n,i) \in \omega^{<\omega}$ by induction on $i \in \omega$ as follows. Let $\eta(e,n,0) = n^\frown\sigma$, where σ is the least string such that $\Phi_e(n^\frown\sigma)(0)\downarrow$. Given $\eta(e,n,i)$, let $\eta(e,n,i+1) = \eta(e,n,i)^\frown 0^\frown\sigma$, where σ is the least string such that $\Phi_e(\eta(e,n,i)^\frown 0^\frown\sigma)(i+1)\downarrow$.

Let $f \in \omega^\omega$. We construct $\mathcal{A}$ and $\mathcal{B}$ such that:

- If $g \in \mathcal{A}$, then $g(0)$ has the form
$$g(0) = \langle\, \ell, \langle\, n_0, x_0, y_0 \,\rangle, \ldots, \langle\, n_{\ell-1}, x_{\ell-1}, y_{\ell-1} \,\rangle \,\rangle,$$
where $\ell \in \omega$ and $n_i \in \omega$, $x_i \in \{0,1\}$, and $y_i \in \omega$ for each $i < \ell$.
- If $g \in \mathcal{A}$ and $\langle\, n_e, 0, y_e \,\rangle$ is in the $e^{\rm th}$ position of $g(0)$, then
  - $\exists m \exists \sigma (\forall \tau \supseteq \sigma)(\Phi_e(n_e{}^\frown\tau)(m)\uparrow)$
  - Any $h \in \mathcal{B}$ with $h(0) = n_e$ is of the form $h = n_e{}^\frown\sigma^\frown f$, where $|\sigma| = y_e$.
- If $g \in \mathcal{A}$ and $\langle\, n_e, 1, y_e \,\rangle$ is the $e^{\rm th}$ position of $g(0)$, then
  - $\forall m \forall \sigma (\exists \tau \supseteq \sigma)(\Phi_e(n_e{}^\frown\tau)(m)\downarrow)$
  - Any $h \in \mathcal{B}$ with $h(0) = n_e$ is of the form $h = \eta(e, n_e, i)^\frown 1^\frown f$ for some $i \in \omega$.
- The above properties hold with the roles of $\mathcal{A}$ and $\mathcal{B}$ reversed.

We construct $\mathcal{A}$ and $\mathcal{B}$ in stages. The construction is similar to the construction in Lemma 2.3 in that if g goes into $\mathcal{A}$ before h goes into $\mathcal{B}$, then $h(0)$ codes how to recover f from g, and similarly with the roles of $\mathcal{A}$ and $\mathcal{B}$ reversed. Start at stage 0 with $\mathcal{A} = \emptyset$, $\mathcal{B} = \emptyset$, $s = \langle\,\rangle$, and $t = \langle\,\rangle$.

Stage $e+1$: Set $n_e = e^\frown t$.

Case 1: $\exists m \exists \sigma (\forall \tau \supseteq \sigma)(\Phi_e(n_e{}^\frown\tau)(m)\uparrow)$. Choose such a σ and put $n_e{}^\frown\sigma^\frown f$ in $\mathcal{A}$. Update $s = s^\frown \langle\, n_e, 0, |\sigma| \,\rangle$.

Case 2: $\forall m \forall \sigma (\exists \tau \supseteq \sigma)(\Phi_e(n_e{}^\frown\tau)(m)\downarrow)$. Put the functions $\eta(e, n_e, i)^\frown 1^\frown f$ in $\mathcal{A}$ for each $i \in \omega$. Update $s = s^\frown \langle\, n_e, 1, 0 \,\rangle$.

Repeat the above procedure with the roles of $\mathcal{A}$ and $\mathcal{B}$ reversed and the roles of s and t reversed. This completes stage $e+1$. Then go on to stage $e+2$. This completes the construction.

Suppose $\mathcal{A} \geq_{\rm M} \mathcal{C}$ where $\mathcal{C}$ is closed. We show that $\mathcal{C}$ contains a recursive function. The proof with $\mathcal{B}$ in place of $\mathcal{A}$ is the same. Let $\Phi_e(\mathcal{A}) \subseteq C$. Consider stage $e+1$ of the above construction. Case 1 must not have occurred because otherwise $\mathcal{A}$ would contain a function $n_e{}^\frown\sigma^\frown f$ such that $\Phi_e(n_e{}^\frown\sigma^\frown f)$ is not total. Thus case 2 occurred, and so $\mathcal{A}$ contains the function $\eta(e, n_e, i)^\frown 1^\frown f$ for each $i \in \omega$. Let k be the recursive function $k = n_e{}^\frown\sigma_0{}^\frown 0^\frown\sigma_1{}^\frown 0^\frown\sigma_2{}^\frown 0^\frown \cdots$, where $\eta(e, n_e, i) = n_e{}^\frown\sigma_0{}^\frown 0^\frown \cdots {}^\frown 0^\frown\sigma_i$ for each $i \in \omega$ (think of k as the "limit" of the strings $\eta(e, n_e, i)$ as $i \to \infty$). Then $\Phi_e(\eta(e, n_e, i)^\frown 1^\frown f) \in \mathcal{C}$ and $\Phi_e(\eta(e, n_e, i)^\frown 1^\frown f) \restriction i = \Phi_e(k) \restriction i$ for each $i \in \omega$. Thus $\mathcal{C}$ contains the recursive function $\Phi_e(k)$ because $\mathcal{C}$ is closed.

We now describe a uniform procedure for producing f from $g \oplus h \in \mathcal{A} + \mathcal{B}$. First decode $h(0)$ as $h(0) = \langle\, \ell, \langle\, n_0, x_0, y_0 \,\rangle, \ldots, \langle\, n_{\ell-1}, x_{\ell-1}, y_{\ell-1} \,\rangle \,\rangle$ and look for $g(0)$ among the n_e. If $\langle\, g(0), 0, y_e \,\rangle$ appears in $h(0)$ at position e, then output g from position $y_e + 1$ onward as in this case $g = \sigma^\frown f$ for some string σ of length y_e+1. If $\langle\, g(0), 1, 0\,\rangle$ appears in $h(0)$ at position e, then $g = \eta(e, g(0), i)^\frown 1^\frown f$ for some $i \in \omega$. Compute which i by successively computing the $\eta(e, g(0), j)$, matching them against g, and checking if the next bit of g is 0 (in which case compute $\eta(e, g(0), j+1)$) or 1 (in which case $j = i$). Output f once i is found.

The number $g(0)$ appears among the n_e coded into $h(0)$ if g went into $\mathcal{A}$ before h went into $\mathcal{B}$. Otherwise h went into $\mathcal{B}$ before g went into $\mathcal{A}$, so $h(0)$ appears among the n_e coded in $g(0)$. In this case, switch the roles of g and h and apply the above procedure to compute f. □

Theorem 5.3. *$\mathfrak{F}_{\mathrm{cl}}$ is not prime. In fact, if $\mathfrak{G} \subseteq \mathfrak{F}_{\mathrm{cl}}$, $\mathfrak{G} \neq \{\mathbf{1}\}$ is a filter, then $\mathfrak{G}$ is not prime.*

Proof. Suppose $\mathfrak{G} \subseteq \mathfrak{F}_{\mathrm{cl}}$ is a filter such that $\mathfrak{G} \neq \{\mathbf{1}\}$. Let $f >_{\mathrm{T}} 0$ be such that $[\{f\}] \in \mathfrak{G}$. Let $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$ be as in Lemma 5.2 for this f. Let $\mathbf{a} = [\mathcal{A}]$ and $\mathbf{b} = [\mathcal{B}]$. Then $\mathbf{a}, \mathbf{b} \notin \mathfrak{G}$ because $\mathbf{a}, \mathbf{b} \notin \mathfrak{F}_{\mathrm{cl}}$, but $\mathbf{a}+\mathbf{b} \in \mathfrak{G}$ because $\mathbf{a}+\mathbf{b} \geq_{\mathrm{M}} [\{f\}]$. □

If $\mathbf{x}$ and $\mathbf{y}$ are degrees such that $\mathbf{y}$ is closed and $\mathbf{y} \not\leq_{\mathrm{M}} \mathbf{x}$, then there is no dense degree $\mathbf{z}$ such that $\mathbf{y} \leq_{\mathrm{M}} \mathbf{x}+\mathbf{z}$ [7]. Therefore, if $\mathfrak{G} \subseteq \mathfrak{F}_{\mathrm{cl}}$, $\mathfrak{G} \neq \{\mathbf{1}\}$ is a filter, then any degrees $\mathbf{a}$ and $\mathbf{b}$ witnessing that $\mathfrak{G}$ is not prime must both be in $\mathfrak{F} - \mathfrak{G}$.

The results of Section 3 suggest two new filters to study:

Definition 5.4.

- $\mathfrak{G}$ denotes the filter generated by
$$\{\mathbf{d} \mid \mathbf{d} >_{\mathrm{M}} \mathbf{0}' \text{ and is join-irreducible}\}.$$
- $\mathfrak{H}$ denotes the filter generated by
$$\left\{\left[\bigcup_{i\in\omega} i^\frown \mathcal{D}_i\right] \;\middle|\; (\forall i \in \omega)([\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}' \text{ and is join-irreducible})\right\}.$$

$\mathfrak{G}$ is exactly the set of all degrees $\mathbf{b}$ for which $\mathbf{b} \geq_{\mathrm{M}} \prod_{i=0}^{n} \mathbf{d}_i$ for some join-irreducible degrees $\mathbf{d}_i >_{\mathrm{M}} \mathbf{0}'$, $i \leq n$, and $\mathfrak{H}$ is exactly the set of all degrees $\mathbf{b}$ for which $\mathbf{b} \geq_{\mathrm{M}} \left[\bigcup_{i\in\omega} i^\frown \mathcal{D}_i\right]$ for some join-irreducible degrees $[\mathcal{D}_i] >_{\mathrm{M}} \mathbf{0}'$, $i \in \omega$.

Theorem 5.5. *$\mathfrak{F}_{\mathrm{cl}} \subsetneq \mathfrak{G} \subsetneq \mathfrak{H} \subsetneq \{\mathbf{a} \mid \mathbf{a} >_{\mathrm{M}} \mathbf{0}'\}$. $\mathfrak{G} \not\subseteq \mathfrak{F}$ (hence also $\mathfrak{H} \not\subseteq \mathfrak{F}$). Neither $\mathfrak{G}$ nor $\mathfrak{H}$ is principal.*

Proof. Every closed degree $>_{\mathrm{M}} \mathbf{0}$ bounds a join-irreducible degree $>_{\mathrm{M}} \mathbf{0}'$ [16]. Hence $\mathfrak{F}_{\mathrm{cl}} \subseteq \mathfrak{G}$. $\mathfrak{G} \subseteq \mathfrak{H}$ is clear. To see $\mathfrak{G} \not\subseteq \mathfrak{F}$, observe that every $\mathcal{B}_f$ is dense, so if $f >_{\mathrm{T}} 0$, then $[\mathcal{B}_f] \in \mathfrak{G} - \mathfrak{F}$. This also shows $\mathfrak{G} \not\subseteq \mathfrak{F}_{\mathrm{cl}}$. The degree constructed in Theorem 3.8 witnesses $\mathfrak{H} \not\subseteq \mathfrak{G}$. The degree constructed in Theorem 3.10 witnesses $\{\mathbf{a} \mid \mathbf{a} >_{\mathrm{M}} \mathbf{0}'\} \not\subseteq \mathfrak{H}$. We show that $\mathfrak{G}$ is not principal. The proof for $\mathfrak{H}$ is the same. First, if $\mathcal{A}$ is countable and contains no recursive functions, then there is a function $f >_{\mathrm{T}} 0$ such that $g \not\leq_{\mathrm{T}} f$ for all $g \in \mathcal{A}$. Thus $\mathcal{B}_f \leq_{\mathrm{M}} \mathcal{A}$ (as $\mathcal{A} \subseteq \mathcal{B}_f$) for this f. Every $[\mathcal{B}_f]$ for $f >_{\mathrm{T}} 0$ is in $\mathfrak{G}$, so every $[\mathcal{A}]$ where $\mathcal{A}$ is countable and contains no recursive function is in $\mathfrak{G}$. If $\mathfrak{G}$ were principal, it would be generated by a degree $\mathbf{x}$ such that $\mathbf{x} \leq_{\mathrm{M}} [\mathcal{A}]$ for all countable $\mathcal{A}$ not containing a recursive function. By Lemma 4.2, the only such $\mathbf{x}$ are $\mathbf{0}$ and $\mathbf{0}'$. We know $\mathbf{0}$ and $\mathbf{0}'$ are not in $\mathfrak{G}$, so $\mathfrak{G}$ cannot be principal. □

We end with a question.

Question 5.6.

- Is $\mathfrak{F} \subseteq \mathfrak{G}$? Is $\mathfrak{F} \subseteq \mathfrak{H}$?
- Is $\mathfrak{G}$ prime? Is $\mathfrak{H}$ prime?
- Is $\{\mathbf{a} \mid \mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}\}$ a filter?

To prove that $\{\mathbf{a} \mid \mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}\}$ is a filter, it suffices to prove that $\mathrm{Th}(\mathfrak{M}/(\mathbf{a}\times\mathbf{b})) \subseteq \mathrm{JAN}$ whenever both $\mathrm{Th}(\mathfrak{M}/\mathbf{a})$ and $\mathrm{Th}(\mathfrak{M}/\mathbf{b})$ are $\subseteq$ JAN because $\{\mathbf{a} \mid \mathrm{Th}(\mathfrak{M}/\mathbf{a}) \subseteq \mathrm{JAN}\}$ is upward-closed in $\mathfrak{M}$.

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