MENGER'S THEOREM IN Π^1_1-CA_0

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ABSTRACT. We prove Menger's theorem for countable graphs in Π^1_1-CA_0. Our proof in fact proves a stronger statement, which we call extended Menger's theorem, that is equivalent to Π^1_1-CA_0 over RCA_0.

1. Introduction

König's duality theorem for finite bipartite graphs is a classic theorem in graph theory and one of the pillars of matching theory. It expresses a duality between matchings and covers in bipartite graphs. Let (X, Y, E) be a bipartite graph. A *matching* is a set of edges $M \subseteq E$ such that no two edges in M share a vertex. A *cover* is a set of vertices $C \subseteq X \cup Y$ such that every edge in E has a vertex in C. Finite König's duality theorem says that the cardinalities of matchings and the cardinalities of covers meet in the middle.

Finite König's Duality Theorem. *In every finite bipartite graph, the maximum cardinality of a matching equals the minimum cardinality of a cover.*

Finite Menger's theorem generalizes finite König's duality theorem from bipartite graphs to arbitrary graphs. Let G be a graph with vertices $V(G)$ and edges $E(G)$. A *web* is a triple (G, A, B) where G is a graph and A and B are distinguished sets of vertices $A, B \subseteq V(G)$. The notion of a matching in a bipartite graph is generalized by the notion of a *set of disjoint A-B paths*[1] in a web. An A-B path in a web (G, A, B) is a path that starts in A and ends in B. Two paths are disjoint if they have no vertices in common. The notion of a cover in a bipartite graph is generalized by the notion of an *A-B separator* in a web. An A-B separator in a web (G, A, B) is a set of vertices $C \subseteq V(G)$ such that every A-B path in G contains a vertex of C (so that removing C from the graph separates A from B).

Finite Menger's Theorem. *In every finite web (G, A, B), the maximum cardinality of a set of disjoint A-B paths equals the minimum cardinality of an A-B separator.*

Finite Menger's theorem is itself a special case of the famous max-flow min-cut theorem for network flows. See [5] Section 2.1 for a full treatment of finite König's duality theorem, [5] Section 3.3 for finite Menger's theorem, and [5] Section 6.2 for the max-flow min-cut theorem.

The conclusions of finite König's duality theorem and finite Menger's theorem remain true for infinite bipartite graphs and infinite webs, but they are more an exercise in cardinal arithmetic than they are in combinatorics. To deepen the combinatorial content of these theorems, Erdős conjectured that there always exist a matching and a cover that simultaneously witness each other's optimality. His reformulations are what we now call König's duality theorem and Menger's theorem.

König's Duality Theorem. *In every bipartite graph (X, Y, E), there is a matching M and a cover C such that C consists of exactly one vertex from each edge in M.*

Menger's Theorem. *In every web (G, A, B), there is a set of disjoint A-B paths M and an A-B separator C such that C consists of exactly one vertex from each path in M.*

This research was partially supported by NSF grants DMS-0554855 and DMS-0852811.

[1]For us, "path" always means "simple path," that is, no repeated vertices.

The most general case, Menger's theorem for webs of arbitrary cardinality, is now known to be true. The proof took more than forty years to discover. The first progress was by Podewski and Steffens, who proved König's duality theorem for countable bipartite graphs [7]. Aharoni next proved König's duality theorem for arbitrary bipartite graphs [1]. He then proved Menger's theorem for countable webs [2]. Finally, Aharoni and Berger proved Menger's theorem for arbitrary webs [3].

The question motivating our work is the following.

Question 1.1. What is the axiomatic strength of Menger's theorem for countable webs in the context of second-order arithmetic?

Aharoni, Magidor, and Shore [4] and Simpson [8] answered Question 1.1 for König's duality theorem for countable bipartite graphs. Aharoni, Magidor, and Shore noticed that Aharoni's proof of König's duality theorem in [1] actually proves a stronger statement, which they call *extended König's duality theorem*. They proved that extended König's duality theorem is equivalent to $\Pi^1_1\text{-}\mathtt{CA}_0$ over $\mathtt{RCA}_0$, and they proved that König's duality theorem implies $\mathtt{ATR}_0$ over $\mathtt{RCA}_0$ [4]. Simpson produced a new proof of König's duality theorem in $\mathtt{ATR}_0$ by exploiting the fact that $\mathtt{ATR}_0$ proves the existence of models of $\Sigma^1_1\text{-}\mathtt{AC}_0$ [8]. Therefore König's duality theorem for countable bipartite graphs is equivalent to $\mathtt{ATR}_0$ over $\mathtt{RCA}_0$.

A priori, Menger's theorem for countable webs implies $\mathtt{ATR}_0$ over $\mathtt{RCA}_0$ because it implies König's duality theorem for countable bipartite graphs over $\mathtt{RCA}_0$. Here we provide a proof Menger's theorem for countable webs in $\Pi^1_1\text{-}\mathtt{CA}_0$. The general plan for our proof is inspired by Aharoni's proof in [2] and Diestel's presentation of it in [5] Section 8.4. As with König's duality theorem, we notice that this proof in fact proves a stronger statement, which we call *extended Menger's theorem*, that is equivalent to $\Pi^1_1\text{-}\mathtt{CA}_0$ over $\mathtt{RCA}_0$. By general considerations, Menger's theorem cannot imply $\Pi^1_1\text{-}\mathtt{CA}_0$ over $\mathtt{RCA}_0$. Menger's theorem can be written as a Π^1_2 sentence in the language of second-order arithmetic, and no true Π^1_2 sentence implies $\Pi^1_1\text{-}\mathtt{CA}_0$, even over $\mathtt{ATR}_0$ (see [4] Proposition 4.17). Question 1.1 now becomes more specific.

Question 1.2. Is Menger's theorem for countable webs provable in $\mathtt{ATR}_0$?

This paper is organized as follows. Section 2 explains the background graph-theoretic primitives and subsystems of second-order arithmetic needed for this work. Section 3 develops in $\mathtt{ACA}_0$ the tools needed to prove Menger's theorem in $\Pi^1_1\text{-}\mathtt{CA}_0$. Section 4 gives a proof of Menger's theorem for countable webs in $\Pi^1_1\text{-}\mathtt{CA}_0$. Section 5 introduces extended Menger's theorem and proves that it is equivalent to $\Pi^1_1\text{-}\mathtt{CA}_0$ over $\mathtt{RCA}_0$.

2. Background

2.1. **Graph theory basics and conventions.** All the graphs that we consider are countable because we are working in second-order arithmetic. All the graphs that we consider are directed. Menger's theorem for undirected graphs follows from Menger's theorem for directed graphs by the usual trick of replacing an undirected edge by two directed edges. Henceforth a "graph" is a countable directed graph.

As defined in the introduction, a web is a triple (G, A, B) where G is a graph and A and B are distinguished sets of vertices $A, B \subseteq V(G)$. We often abuse this notation by writing G for (G, A, B). For convenience, we always assume that there are no edges directed into A, that there are no edges directed out of B, and that $A \cap B = \emptyset$.

If H and H' are subgraphs of a graph G, then $H \cup H'$ is the subgraph of G induced by $V(H) \cup V(H')$, and $G - H$ is the subgraph induced by $V(G) - V(H)$.

Let G be a graph. If P is a path in G, we write $\mathrm{in}(P)$ for the first vertex of P (if it exists) and $\mathrm{ter}(P)$ for the last vertex of P (if it exists). If P is a path with $\mathrm{in}(P) \in A$ and $\mathrm{ter}(P) \in B$ for some $A, B \subseteq V(G)$, then we call P an A-B path. If P is a path and $x \in V(P)$, then Px denotes the subpath of P consisting of all the vertices up to and including x, and $\overline{Px}$ denotes the subpath of P consisting of all the vertices up to and not including x. Similarly, xP denotes the subpath of P consisting of all the vertices following x and including x, and $\underline{xP}$ denotes the subpath of P consisting of all the all the vertices following x and not including x. If P and Q are paths with $V(P) \cap V(Q) = \{x\}$, then PxQ is the path obtained by concatenating the paths Px and xQ. If $V(P) \cap V(Q) = \{\text{ter}(P)\} = \{\text{in}(Q)\}$, then PQ denotes $P\,\text{ter}(P)Q$, the concatenation of the paths P and Q.

For our purposes, a *tree* is a directed acyclic graph T that has a distinguished root $r \in V(T)$ such that for any $x \in V(T)$ there is a unique path in T from r to x. The path in a tree T from its root to an $x \in V(T)$ is denoted Tx. If P is a finite path, a *tree with trunk* P is a tree T of the form $P \cup T'$ where T' is a tree rooted at $\text{ter}(P)$. A tree with trunk P has root $\text{in}(P)$. If $G = (G, A, B)$ is a web, an A-B *tree in* G is a subgraph of G that is a tree with root in A and exactly one vertex in B.

2.2. **Reverse mathematics.** Reverse mathematics, introduced by Friedman [6], is an analysis of the logical strength of the theorems of ordinary mathematics in the context of second-order arithmetic. A result in reverse mathematics typically has the form "T is equivalent to `strong system` over `weak system`," where `strong system` and `weak system` are subsystems of second-order arithmetic and T is some theorem from ordinary mathematics. This means that T is provable in `strong system` and that all the axioms of `strong system` are provable in `weak system`$\cup\{T\}$. The proof of `strong system` from `weak system`$\cup\{T\}$ is called a *reversal*.

We now describe the axiomatic systems that we will use to analyze Menger's theorem. We follow [9], the standard reference for reverse mathematics. Also see [4] Section 2 for a thorough introduction to most of the systems we consider and for computability-theoretic interpretations of these systems.

Before we describe the systems, we need to know that the *basic axioms* are the sentences

$$\begin{aligned}
&\forall m(m+1 \neq 0)\\
&\forall m \forall n(m+1 = n+1 \rightarrow m = n)\\
&\forall m(m+0 = m)\\
&\forall m \forall n(m+(n+1) = (m+n)+1)\\
&\forall m(m \times 0 = 0)\\
&\forall m \forall n(m \times (n+1) = (m \times n)+m)\\
&\forall m \neg(m < 0)\\
&\forall m \forall n(m < n+1 \leftrightarrow (m < n \vee m = n)),
\end{aligned}$$

that the *induction axiom* is the sentence

$$\forall X((0 \in X \wedge \forall n(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n(n \in X)),$$

and that the *comprehension scheme* consists of all universal closures of formulas of the form

$$\exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where φ can be any formula in the language of second-order arithmetic in which X does not occur freely. Full second-order arithmetic consists of the basic axioms, the induction axiom, and the comprehension scheme.

RCA_0 (for *recursive comprehension axiom*) consists of the basic axioms, the Σ^0_1 *induction scheme*, and the Δ^0_1 *comprehension scheme*. The Σ^0_1 induction scheme consists of all universal closures of formulas of the form

$$(\varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n \varphi(n)$$

where φ is Σ^0_1. The Δ^0_1 comprehension scheme consists of all universal closures of formulas of the form

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \varphi(n))$$

where φ is Σ^0_1, ψ is Π^0_1, and X does not occur freely in φ. $\mathtt{RCA}_0$ is the standard `weak system` for the purpose of reversals. $\mathtt{RCA}_0$ proves that the function $\langle i, j \rangle \mapsto (i+j)^2 + i$ is injective (see [9] Section II.2). For $X \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, we define

$$(X)_n = \{i \mid \langle i, n \rangle \in X\} \text{ and}$$
$$(X)^n = \{\langle i, m \rangle \mid \langle i, m \rangle \in X \wedge m < n\}.$$

$\mathtt{RCA}_0$ proves that if X exists, then so do $(X)_n$ and $(X)^n$. We interpret $(X)_n$ as the $(n+1)^{\text{th}}$ column of X and $(X)^n$ as set of the first n columns of X.

$\mathtt{ACA}_0$ (for *arithmetical comprehension axiom*) consists of the basic axioms, the induction axiom, and the *arithmetical comprehension scheme*. The arithmetical comprehension scheme is the restriction of the comprehension scheme to formulas φ that are arithmetical.

$\mathtt{ATR}_0$ (for *arithmetical transfinite recursion*) consists of $\mathtt{ACA}_0$ plus an axiom scheme that says if a set can be constructed by iterating arithmetical comprehension along an existing well-order, then that set exists. Let $\mathrm{LO}(X, <_X)$ be a formula that says "$<_X$ is a linear order on the set X," and let $\mathrm{WO}(X, <_X)$ be a formula that says "$<_X$ is a well-order on the set X." Given a formula $\theta(n, Y)$, let $\mathrm{H}_\theta(X, <_X, Y)$ be a formula that says "$\mathrm{LO}(X, <_X)$ and $Y = \{\langle n, j \rangle \mid j \in X \wedge \theta(n, \{\langle m, i \rangle \in Y \mid i <_X j\})\}$." The axioms of $\mathtt{ATR}_0$ consist of those of $\mathtt{ACA}_0$ plus all universal closures of formulas of the form

$$\forall X \forall <_X (\mathrm{WO}(X, <_X) \rightarrow \exists Y\, \mathrm{H}_\theta(X, <_X, Y))$$

where θ is arithmetical. An easier-to-understand equivalent of $\mathtt{ATR}_0$ is the system Σ^1_1 *separation*, which consists of the axioms of $\mathtt{RCA}_0$ plus the all universal closures of formulas of the form

$$\neg \exists n(\varphi_0(n) \wedge \varphi_1(n)) \rightarrow \exists Z \forall n((\varphi_0(n) \rightarrow n \in Z) \wedge (\varphi_1(n) \rightarrow n \notin Z)),$$

where φ_0 and φ_1 are Σ^1_1 and Z does not occur freely in either φ_0 or φ_1 (see [9] Theorem V.5.1).

Σ^1_1-$\mathtt{DC}_0$ (for Σ^1_1 *dependent choice*) consists of $\mathtt{ACA}_0$ and the scheme of Σ^1_1 *dependent choice*. The scheme of Σ^1_1 dependent choice consists of all universal closures of formulas of the form

$$\forall n \forall X \exists Y \eta(n, X, Y) \rightarrow \exists Z \forall n \eta(n, (Z)^n, (Z)_n)$$

where η is Σ^1_1 and Z does not occur freely in η.

Π^1_1-$\mathtt{CA}_0$ (for Π^1_1 *comprehension axiom*) consists of the basic axioms, the induction axiom, and the Π^1_1 *comprehension scheme*. The Π^1_1 comprehension scheme is the restriction of the comprehension scheme to formulas φ that are Π^1_1.

$\mathtt{RCA}_0$ is strictly weaker than $\mathtt{ACA}_0$, which is strictly weaker than both $\mathtt{ATR}_0$ and Σ^1_1-$\mathtt{DC}_0$. $\mathtt{ATR}_0$ and Σ^1_1-$\mathtt{DC}_0$ are independent over $\mathtt{RCA}_0$. However, $\mathtt{ATR}_0$ proves the consistency of Σ^1_1-$\mathtt{DC}_0$. Both $\mathtt{ATR}_0$ and Σ^1_1-$\mathtt{DC}_0$ are strictly weaker than Π^1_1-$\mathtt{CA}_0$.

Our proof of Menger's theorem in Π^1_1-$\mathtt{CA}_0$ relies on two key meta-mathematical facts. The first key fact concerns the existence of β-models. The second key fact concerns the existence of models of Σ^1_1-$\mathtt{DC}_0$.

Definition 2.1. A *countable coded ω-model* is a set $X \subseteq \mathbb{N}$ viewed as coding the structure $\mathcal{M} = (\mathbb{N}, \{(X)_n \mid n \in \mathbb{N}\}, +, \times, 0, 1, <)$.

We usually identify a countable coded ω-model X with the structure $\mathcal{M}$ that it codes.

Definition 2.2. A *countable coded β-model* is a countable coded ω-model $\mathcal{M}$ that is absolute for Σ^1_1 formulas with parameters from $\mathcal{M}$. That is, if φ is a Σ^1_1 formula with parameters from $\mathcal{M}$, then $\mathcal{M} \models \varphi$ if and only if φ is true.

Theorem 2.3 (see [9] Theorem VII.2.10)**.** *The statement "for every X there is a countable coded β-model $\mathcal{M}$ with $X \in \mathcal{M}$" is equivalent to Π^1_1-CA_0 over ACA_0.*

It is helpful to keep in mind that ACA_0 proves that every countable coded β-model is a model of ATR_0 (see [9] Theorem VII.2.7).

Theorem 2.4 (see [9] Theorem VIII.4.20)**.** ATR_0 *proves that for every X there is a countable coded ω-model $\mathcal{M}$ of Σ^1_1-DC_0 with $X \in \mathcal{M}$.*

The statement "for every X there is a countable coded ω-model $\mathcal{M}$ of Σ^1_1-DC_0 with $X \in \mathcal{M}$" is in fact equivalent to ATR_0 over RCA_0. See [9] Lemma VIII.4.15 for the reversal.

3. Warps, waves, and alternating walks

In this section we use ACA_0 to develop the basic tools we need to prove Menger's theorem in Π^1_1-CA_0. Our notation and terminology mostly follows [2] with some ideas borrowed from [5] Section 8.4.

Definition 3.1. A *warp* in a web $G = (G, A, B)$ is a subgraph W of G such that

- $A \subseteq V(W)$,
- every $x \in V(W)$ has $\text{in-deg}_W(x) \leq 1$ and $\text{out-deg}_W(x) \leq 1$, and
- every $x \in V(W)$ is reachable from some $a \in A$ by a path in W.

A warp is thus a collection of disjoint paths in G with each path starting at a distinct vertex in A and such that for every $a \in A$ there is a path in the warp starting at a. Such paths may be one-way infinite. It is often convenient to think of a warp W as the collection of its component paths $\{P_a \mid a \in A \wedge \text{in}(P_a) = a\}$ with the understanding that this collection is coded by the set $\{\langle a, \langle n, x \rangle \rangle \mid x \text{ is the } n^{\text{th}} \text{ vertex of } P_a\}$. "$P$ is a path in W" always means that P is one of these component paths.

If W is a warp, then let $\text{ter}(W) = \{x \in V(W) \mid \text{out-deg}_W(x) = 0\}$. That is, $\text{ter}(W)$ is the set of terminal vertices of the paths in W. The statement "if W is a warp then $\text{ter}(W)$ exists" is equivalent to ACA_0 over RCA_0, hence our assumption of ACA_0 throughout this section.

Definition 3.2. A *wave* in a web $G = (G, A, B)$ is a warp W such that $\text{ter}(W)$ is an A-B separator.

It is important to note that "X is an A-B separator in (G, A, B)" is an arithmetical property. A-B paths are finite, and quantification over them can be coded by quantification over $\mathbb{N}$. Thus "W is a wave in G" is also an arithmetical property.

The warp $\{P_a \mid a \in A\}$ in which each path P_a is the trivial path (a) is always a wave, and we call it the *trivial wave*.

Definition 3.3. For warps W and Y in a web $G = (G, A, B)$, Y is an *extension* of W (written $W \leq Y$) if and only if W is a subgraph of Y.

Definition 3.4. If $(W_i \mid i \in \mathbb{N})$ is a sequence of warps such that $W_i \leq W_{i+1}$ for each $i \in \mathbb{N}$, then $\bigcup_{i\in\mathbb{N}} W_i$ denotes the *limit warp* defined by $V\big(\bigcup_{i\in\mathbb{N}} W_i\big) = \bigcup_{i\in\mathbb{N}} V(W_i)$ and $E\big(\bigcup_{i\in\mathbb{N}} W_i\big) = \bigcup_{i\in\mathbb{N}} E(W_i)$.

It is easy to check in RCA_0 that a limit warp, if it exists, is indeed a warp. However, the statement "if $(W_i \mid i \in \mathbb{N})$ is a sequence of warps such that $W_i \leq W_{i+1}$ for each $i \in \mathbb{N}$, then $\bigcup_{i\in\mathbb{N}} W_i$ exists" is equivalent to ACA_0 over RCA_0.

Definition 3.5. Let $W = \{P_a \mid a \in A\}$ be a wave in a web $G = (G, A, B)$. Then

- P_a is *W-essential in G* if and only if P_a is finite and there is a $\text{ter}(P_a)$-B path in G disjoint from $V(W) - \{\text{ter}(P_a)\}$,
- $a \in A$ is *W-essential in G* if and only if P_a is W-essential in G, and

- $\mathrm{ess}_G(W) = \{a \in A \mid a \text{ is } W\text{-essential in } G\}$.

The motivation behind the definition of W-essential in G is that if P is a path in a wave W that is W-essential in G, then $\mathrm{ter}(W)$ needs $\mathrm{ter}(P)$ to separate A from B. If Q is a $\mathrm{ter}(P)$-B path disjoint from $V(W) - \{\mathrm{ter}(P)\}$, then PQ is an A-B path disjoint from $\mathrm{ter}(W) - \{\mathrm{ter}(P)\}$. One readily checks that $\{\mathrm{ter}(P) \mid P \text{ is a } W\text{-essential path in } G\}$ is an A-B separator.

Definition 3.6. If W and Y are waves in a web $G = (G, A, B)$ with $W \leq Y$, then Y is a *good* extension of W if and only if $\mathrm{ess}_G(W) = \mathrm{ess}_G(Y)$ and Y is a *bad* extension of W if and only if $\mathrm{ess}_G(W) \neq \mathrm{ess}_G(Y)$.

If W and Y are waves in a web $G = (G, A, B)$ with $W \leq Y$, then it is always the case that $\mathrm{ess}_G(Y) \subseteq \mathrm{ess}_G(W)$. Thus Y is a good extension of W if and only if $\mathrm{ess}_G(W) \subseteq \mathrm{ess}_G(Y)$.

Lemma 3.7 (in ACA_0; see [2] Lemma 2.5)**.** *If $(W_i \mid i \in \mathbb{N})$ is a sequence of waves in a web $G = (G, A, B)$ such that $W_i \leq W_{i+1}$ for each $i \in \mathbb{N}$, then $\bigcup_{i\in\mathbb{N}} W_i$ is a wave in G.*

Proof. Let $W = \bigcup_{i\in\mathbb{N}} W_i$. As mentioned above, it is easy to check that W is a warp. We need to show that $\mathrm{ter}(W)$ is an A-B separator. Let P be an A-B path, and let $X = \{\langle x, i\rangle \mid x \in V(P) \cap \mathrm{ter}(W_i)\}$ which exists by arithmetical comprehension. Each W_i is a wave, hence X is infinite. As $V(P)$ is finite, there must be an $x \in V(P)$ such that $\{i \mid x \in \mathrm{ter}(W_i)\}$ is infinite. Then $x = \mathrm{ter}(Q)$ for the path Q in W containing x. If not, then there is a vertex following x on Q, the corresponding edge must appear in W_n for some n, and so $x \notin \mathrm{ter}(W_i)$ for all $i \geq n$. ☐

Definition 3.8. Let W and Y be warps in a web $G = (G, A, B)$ with $W \leq Y$. Let Q be a finite path in both Y and W (i.e., the path Q is in W and is not properly extended in Y). A $(Y - W)$-*alternating walk from* $\mathrm{ter}(Q)$ is a walk $R = x_0e_0x_1e_1 \cdots e_{n-1}x_n$ such that

(i) $x_0 = \mathrm{ter}(Q)$,
(ii) for all $i \leq n$, $x_i \in (V(G) - V(W)) \cup \mathrm{ter}(W)$,
(iii) for all $i < n$, if $e_i \notin E(Y)$, then $e_i = (x_i, x_{i+1})$,
(iv) for all $i < n$, if $e_i \in E(Y)$, then $e_i = (x_{i+1}, x_i)$ (i.e., R traverses e_i backwards),
(v) for all $i, j \leq n$ with $i \neq j$, if $x_i = x_j$, then $x_i \in V(Y)$, and
(vi) for all $i, j \leq n$ with $i \neq j$, $e_i \neq e_j$.
(vii) for all $0 < i \leq n$, if $x_i \in V(Y)$, then either e_{i-1} or e_i is in $E(Y)$.

Note that if x_n is the last vertex on a $(Y - W)$-alternating walk from $\mathrm{ter}(Q)$ and $x_n \in V(Y)$, then item (vii) implies that $e_{n-1} \in E(Y)$, and by item (iv) it must also be that $e_{n-1} = (x_n, x_{n-1})$. A $(Y - W)$-alternating walk from $\mathrm{ter}(Q)$ is similar to a Y-walk as defined in [2] and to a walk which alternates with respect to Y as defined in [5] Section 3.3. The difference is that a $(Y - W)$-alternating walk from $\mathrm{ter}(Q)$ is not allowed to use the vertices in $V(W) - \mathrm{ter}(W)$, hence the notation "$Y - W$."

Definition 3.9. Let $W = \{P_a \mid a \in A\}$ and $Y = \{Q_a \mid a \in A\}$ be warps in a web $G = (G, A, B)$ with $W \leq Y$. Let Q_{a_0} be a finite path in both Y and W. Then $\mathrm{alt}_G(Y - W, \mathrm{ter}(Q_{a_0}))$ denotes the warp $\{Q'_a \mid a \in A\}$ where $Q'_a = Q_a x$ if x is the last vertex on Q_a which lies on a $(Y - W)$-alternating walk from $\mathrm{ter}(Q_{a_0})$ and $Q'_a = P_a$ if no such x exists.

Our definition of $\mathrm{alt}_G(Y - W, \mathrm{ter}(Q_{a_0}))$ is analogous to the definition of $M(a_0, W)$ in [2]. Also, note that $W \leq \mathrm{alt}_G(Y - W, \mathrm{ter}(Q_{a_0})) \leq Y$. The first inequality is by Definition 3.8 item (ii) and the second inequality is clear.

The crucial lemma from this section is Lemma 3.12 below. Lemma 3.10 and Lemma 3.11 are used to prove Lemma 3.12.

Lemma 3.10 (in ACA_0)**.** *Let W and Y be warps in a web $G = (G, A, B)$ with $W \leq Y$. Let Q be a finite path in both Y and W. Let R be a $(Y - W)$-alternating walk from* $\mathrm{ter}(Q)$. *Then there is a warp $Z \geq W$ in G with* $\mathrm{ter}(Z) = (\mathrm{ter}(Y) - \{\mathrm{ter}(Q)\}) \cup \{\mathrm{ter}(R)\}$.

Proof. Let $R = x_0e_0x_1e_1\cdots e_{n-1}x_n$ where $x_0 = \text{ter}(Q)$ and $x_n = \text{ter}(R)$. Assume $n > 0$, for otherwise we may take $Z = Y$. Let Z' be the subgraph of G with $E(Z') = E(Y) \triangle E(R)$ and $V(Z') = A \cup \{x \mid (\exists e \in E(Z'))(x \text{ is a vertex of } e)\}$. One readily checks the following equalities:

- if $x \in V(Y) - V(R)$, then $\text{in-deg}_{Z'}(x) = \text{in-deg}_Y(x)$ and $\text{out-deg}_{Z'}(x) = \text{out-deg}_Y(x)$,
- for $0 < i < n$, if $x_i \in V(R) - V(Y)$, then $\text{in-deg}_{Z'}(x_i) = 1$ and $\text{out-deg}_{Z'}(x_i) = 1$,
- for $0 < i < n$, if $x_i \in V(R) \cap V(Y)$ is in $V(Z')$, then $\text{in-deg}_{Z'}(x_i) = \text{in-deg}_Y(x_i)$ and $\text{out-deg}_{Z'}(x_i) = \text{out-deg}_Y(x_i)$,
- $\text{in-deg}_{Z'}(x_0) = \text{in-deg}_Y(x_0)$ and $\text{out-deg}_{Z'}(x_0) = 1$, and
- $\text{in-deg}_{Z'}(x_n) = 1$ and $\text{out-deg}(x_n) = 0$.

It follows that $\text{in-deg}_{Z'}(x) \leq 1$ and $\text{out-deg}_{Z'}(x) \leq 1$ for all $x \in V(Z')$, which means that every component of Z' is either a path or a cycle. Let Z be the subgraph of Z' consisting of the component paths of Z' (i.e., Z is the subgraph of Z' induced by $\{x \in V(Z') \mid x \text{ is not on a cycle in } Z'\}$). Z contains every vertex $x \in V(Z')$ with $\text{in-deg}_{Z'}(x) = 0$ or $\text{out-deg}_{Z'}(x) = 0$. In particular, $A \subseteq V(Z)$ and $\text{ter}(Z) = (\text{ter}(Y) - \{\text{ter}(Q)\}) \cup \{\text{ter}(R)\}$. To show that Z is a warp, we need only show that $\text{in}(P)$ exists and is in A for every path P in Z. The above equations imply that if $x \in V(P) - A$, then $\text{in-deg}_Z(x) \neq 0$ and hence that x has an immediate predecessor on P. This fact together with the fact that R is finite implies that there is an $x \in V(P)$ such that $(V(Px) \cap V(R)) - A = \emptyset$. Thus the edges of Px must all be edges of Y, which means that Px must be an initial segment of some path in Y. Hence $\text{in}(P)$ exists and is in A. Finally, $Z \geq W$ by Definition 3.8 item (ii). ☐

Lemma 3.11 (in ACA_0; see [2] Lemma 2.7)**.** *Let W and Y be waves in a web $G = (G, A, B)$ with $W \leq Y$. Let Q be a finite path in both Y and W. Then $\text{alt}_G(Y - W, \text{ter}(Q))$ is a wave.*

Proof. Let $U = \text{alt}_G(Y - W, \text{ter}(Q))$. Suppose for a contradiction that P is an A-B path disjoint from $\text{ter}(U)$. Let w be the last vertex on P that is in $V(W)$, and let S be the path in W containing w. It must be that $w = \text{ter}(S)$, for otherwise SwP is an A-B path disjoint from $\text{ter}(W)$, contradicting that W is a wave. The path SwP is, however, an A-B path disjoint from $\text{ter}(U)$. Y is a wave, so wP intersects $\text{ter}(Y)$, which must happen at a vertex in $V(Y) - V(U)$. Let y be the first vertex on wP in $V(Y) - V(U)$. Let z be the last vertex on wPy in $V(U)$, which exists because $w \in V(U)$.

Claim. *There is a $(Y - W)$-alternating walk from $\text{ter}(Q)$ ending at z.*

Proof of claim. Let Q' be the path in Y containing z. We show that there is a $(Y - W)$-alternating walk R from $\text{ter}(Q)$ that meets Q' at a vertex r which is past z on Q'. If r is the first such vertex on R, then $RrQ'z$ (following the edges of Q' backwards) is the desired walk. If $z = w$, then Q' extends S, so if there is no such walk R then by Definition 3.9 S is a path in U which contradicts that P is disjoint from $\text{ter}(U)$. On the other hand, if $z \neq w$, then $z \notin V(W)$ by choice of w. As $z \in V(U) - V(W)$ and $z \notin \text{ter}(U)$, again by Definition 3.9 it must be the case that some $(Y - W)$-alternating walk R from $\text{ter}(Q)$ meets Q' at a vertex past z. ☐

Now let R be the walk provided by the claim, let r be the last vertex of zPy on R, and let y' be the vertex immediately preceding y on the path in Y containing y. Then $RrPy(y', y)y'$ is a $(Y - W)$-alternating walk from $\text{ter}(Q)$ on which y lies which contradicts $y \notin V(U)$. ☐

Lemma 3.12 (in ACA_0; see [2] Lemma 2.8)**.** *Let W be a wave in a web $G = (G, A, B)$ that has no bad extensions in G. Let $x \in V(G) - V(W)$ be such that there is a wave $Y \geq W$ in $G - \{x\} = (G - \{x\}, A - \{x\}, B - \{x\})$ with $\text{ess}_{G-\{x\}}(Y) \subsetneq \text{ess}_G(W)$. Then there is a wave $Z \geq W$ in G with $x \in \text{ter}(Z)$.*

Proof. Let $G = (G, A, B)$, $W = \{P_a \mid a \in A\}$, x, and $Y = \{Q_a \mid a \in A\}$ be as in the statement of the lemma. Let $a_0 \in \text{ess}_G(W) - \text{ess}_{G-\{x\}}(Y)$. If we replace Q_{a_0} with P_{a_0} in Y, then we retain that this path is not Y-essential in $G - \{x\}$. Thus we may assume $Q_{a_0} = P_{a_0}$. In particular, Q_{a_0} is a finite path that is not Y-essential in $G - \{x\}$.

Claim. *In G, there is an alternating $(Y-W)$-walk from* $\mathrm{ter}(Q_{a_0})$ *ending at x.*

Proof of claim. If there is a $\mathrm{ter}(Q_{a_0})$-x path disjoint from $V(Y)-\{\mathrm{ter}(Q_{a_0})\}$, then this path is the desired walk. So suppose instead there is no such path. Let $U = \mathrm{alt}_{G-\{x\}}(Y-W, \mathrm{ter}(Q_{a_0}))$. U is a wave in $G-\{x\}$ by Lemma 3.11. Furthermore, $a_0 \notin \mathrm{ess}_{G-\{x\}}(Y)$ implies that $a_0 \notin \mathrm{ess}_{G-\{x\}}(U)$ because if P is a $\mathrm{ter}(Q_{a_0})$-B path in $G-\{x\}$, then the first vertex on P in $V(Y)-\{\mathrm{ter}(Q_{a_0})\}$ is also in $V(U)$. We prove that U is not a wave in G. To do this, it suffices to show that every $\mathrm{ter}(Q_{a_0})$-B path in G intersects $V(U)-\{\mathrm{ter}(Q_{a_0})\}$. Therefore if U were a wave in G, it would be a bad extension of W in G because a_0 would be in $\mathrm{ess}_G(W)-\mathrm{ess}_G(U)$. This is a contradiction. Consider a $\mathrm{ter}(Q_{a_0})$-B path P. If $x \notin V(P)$, then P is a path in $G-\{x\}$ and hence P intersects $V(U)-\{\mathrm{ter}(Q_{a_0})\}$ because $a_0 \notin \mathrm{ess}_{G-\{x\}}(U)$. If $x \in V(P)$, then by assumption Px intersects $V(Y)-\{\mathrm{ter}(Q_{a_0})\}$. Again, the first vertex on P in $V(Y)-\{\mathrm{ter}(Q_{a_0})\}$ is also in $V(U)$.

We now know that U is a wave in $G-x$ but not in G. Thus there is an A-B path S in G avoiding $\mathrm{ter}(U)$, and x must lie on S. Let z be the last vertex of Sx that is in $V(U)$. It must be that $z \in ((V(U)-V(W)) \cup \mathrm{ter}(W)) - \mathrm{ter}(U)$. Hence there must be an alternating $(Y-W)$-walk R from $\mathrm{ter}(Q_{a_0})$ to z. Let y be the last vertex of zSx which lies on R. Then $RySx$ is the desired alternating $(Y-W)$-walk from $\mathrm{ter}(Q_{a_0})$ to x. ☐

By the claim, let R be an alternating $(Y-W)$-walk from $\mathrm{ter}(Q_{a_0})$ ending at x. Apply Lemma 3.10 to get a warp $Z \geq W$ in G with $\mathrm{ter}(Z) = (\mathrm{ter}(Y)-\{\mathrm{ter}(Q_{a_0})\}) \cup \{x\}$. Z is a wave because $\mathrm{ter}(Y)-\{\mathrm{ter}(Q_{a_0})\}$ is an $(A-\{x\})$-$(B-\{x\})$ separator in $G-\{x\}$, thus $(\mathrm{ter}(Y)-\{\mathrm{ter}(Q_{a_0})\})\cup\{x\}$ is an A-B separator in G. ☐

4. Menger's theorem in Π^1_1-CA_0

We plan to prove Menger's theorem as follows. Given a web $G = (G, A, B)$, start with W a $\leq$-maximal wave in G. Let C be the terminal vertices of the paths in W that are W-essential. Then extend these W-essential paths to be the collection of disjoint A-B paths M. Lemma 4.1 below provides the $\leq$-maximal wave W, and Lemma 4.2 below is the tool we use to extend the W-essential paths to a collection of disjoint A-B paths. The proof of Lemma 4.1 is the only argument in which we employ the full strength of Π^1_1-CA_0.

Lemma 4.1 (in Π^1_1-CA_0; see [2] Corollary 2.5a)**.** *In every web there is a $\leq$-maximal wave.*

Proof. Let $G = (G, A, B)$ be a web, let $(g_n \mid n \in \mathbb{N})$ be an enumeration of $V(G)$, and by Theorem 2.3 let $\mathcal{M}$ be a countable coded β-model with $(G, A, B) \in \mathcal{M}$. Using ACA_0 outside $\mathcal{M}$, we construct a sequence of integers $(i_n \mid n \in \mathbb{N})$ such that $(\mathcal{M})_{i_n}$ is a wave for each $n \in \mathbb{N}$ and $(\mathcal{M})_{i_n} \leq (\mathcal{M})_{i_{n+1}}$ for each $n \in \mathbb{N}$. Let i_0 be an index such that $(\mathcal{M})_{i_0}$ is the trivial wave $\{(a) \mid a \in A\}$. Suppose we have $i_0, \ldots, i_n$. If there is an $i \in \mathbb{N}$ such that $(\mathcal{M})_i$ is a wave with $(\mathcal{M})_i \geq (\mathcal{M})_{i_n}$ and $g_n \in V((\mathcal{M})_i)$, then let i_{n+1} be such an i. Otherwise let $i_{n+1} = i_n$. With the desired sequence $(i_n \mid n \in \mathbb{N})$ in hand, let W be the limit $W = \bigcup_{n \in \mathbb{N}} (\mathcal{M})_{i_n}$, which is a wave by Lemma 3.7. This W is $\leq$-maximal in (G, A, B). If not, there is a wave $Y \geq W$ with some $g_n \in V(Y)-V(W)$. As $(\mathcal{M})_{i_n} \leq W \leq Y$, at stage $n+1$ in the construction the Σ^1_1 formula $(\exists Y)(Y \text{ is a wave} \wedge Y \geq (\mathcal{M})_{i_n} \wedge g_n \in V(Y))$ is true and hence is true in $\mathcal{M}$ because $\mathcal{M}$ is a β-model. Therefore we chose i_{n+1} so that $g_n \in V((\mathcal{M})_{i_{n+1}})$, contradicting $g_n \notin V(W)$. ☐

In [2] and [5], Lemma 4.1 is obtained by a simple application of Zorn's lemma. Our proof above is the most effective proof possible, in the sense that Lemma 4.1 is equivalent to Π^1_1-CA_0 over RCA_0 (see Corollary 5.3 below).

The following Lemma 4.2 is the key tool used to complete the proof of Menger's theorem. We first give a proof of Lemma 4.2 in the style of ordinary mathematics for the sake of clarity. We then explain how to formalize Lemma 4.2 in a way that will allow us to complete the proof of Menger's theorem in Π^1_1-CA_0.

Lemma 4.2 (see [2] Theorem 3.4). *Let W be a wave in $G = (G, A, B)$ that has no bad extensions in G, let $a \in \mathrm{ess}_G(W)$, and let P_a be the component path of W starting at a. Then there is a finite a-B tree T with trunk P_a such that $V(T) \cap (V(W) - V(P_a)) = \emptyset$, and there is a wave Y in $G - T$ such that $Y \geq W - P_a$, $\mathrm{ess}_{G-T}(Y) = \mathrm{ess}_G(W) - \{a\}$, and Y has no bad extensions in $G - T$.*

Proof. We assume that the conclusion of the lemma is false and construct a bad extension of W in G, which is a contradiction.

Let $(Q_n \mid n \in \mathbb{N})$ list all the $\mathrm{ter}(P_a)$-B paths with each path occurring on the list infinitely often. We construct sequences $(T_n \mid n \in \mathbb{N})$ and $(Y_n \mid n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$

(i) T_n is a finite tree in G with trunk P_a,
(ii) Y_n is a wave in $G - T_n$,
(iii) $T_{n-1} \subseteq T_n$ (if $n > 0$),
(iv) $Y_{n-1} \leq Y_n$ (if $n > 0$),
(v) $\mathrm{ess}_{G-T_n}(Y_n) = \mathrm{ess}_G(W) - \{a\}$, and
(vi) Y_n has no bad extensions in $G - T_n$.

Start with $T_0 = P_a$ and $Y_0 = W - P_a$. Items (i), (ii), and (v) are easily checked for $n = 0$. Furthermore, if Y were a bad extension of Y_0 in $G - T_0$, then $Y \cup \{P_a\}$ would be a bad extension of W in G. Hence we have item (vi) for $n = 0$ as well.

Suppose we have constructed T_n and Y_n. If $V(Q_n) \cap V(Y_n) \neq \emptyset$ or $T = T_n \cup Q_n$ is not a tree with trunk P_a, set $T_{n+1} = T_n$ and $Y_{n+1} = Y_n$. Otherwise $V(Q_n) \cap V(Y_n) = \emptyset$ and $T = T_n \cup Q_n$ is a tree with trunk P_a. The situation now is that

- T is a finite a-B tree with trunk P_a such that $V(T) \cap (V(W) - V(P_a)) = \emptyset$ and
- Y_n is a wave in $G - T$ such that $Y_n \geq W - P_a$.

We are assuming that the lemma is false, so either $\mathrm{ess}_{G-T}(Y_n) \subsetneq \mathrm{ess}_G(W) - \{a\}$ or Y_n has a bad extension in $G - T$. Both cases imply the existence of a wave $Y \geq Y_n$ in $G - T$ with $\mathrm{ess}_{G-T}(Y) \subsetneq \mathrm{ess}_G(W) - \{a\} = \mathrm{ess}_{G-T_n}(Y_n)$ (where the equality is by item (v)). Let x be the first vertex on Q_n such that there exists a wave $Y \geq Y_n$ in $G - (T_n \cup Q_n x)$ with $\mathrm{ess}_{G-(T_n \cup Q_n x)}(Y) \subsetneq \mathrm{ess}_{G-T_n}(Y_n)$ (note $x \neq \mathrm{ter}(P_a)$ by item (vi)). Let $T_{n+1} = T_n \cup \overline{Q_n x}$. By the choice of x, Y_n has no bad extensions in $G - T_{n+1}$, but there is an extension $Y \geq Y_n$ in $(G - T_{n+1}) - x$ with $\mathrm{ess}_{(G-T_{n+1})-x}(Y) \subsetneq \mathrm{ess}_{G-T_{n+1}}(Y_n)$. Thus by Lemma 3.12 there is a wave $Y_{n+1} \geq Y_n$ in $G - T_{n+1}$ with $x \in \mathrm{ter}(Y_{n+1})$. With this T_{n+1} and Y_{n+1}, items (i)-(iv) are clear for $n + 1$. Item (v) is by the choice of x, which implies that $\mathrm{ess}_{G-T_{n+1}}(Y_{n+1}) = \mathrm{ess}_{G-T_n}(Y_n) = \mathrm{ess}_G(W) - \{a\}$. Item (vi) is again by the choice of x because a bad extension $Y \geq Y_{n+1}$ in $G - T_{n+1}$ would be a $Y \geq Y_n$ in $G - T_{n+1}$ with $\mathrm{ess}_{G-T_{n+1}}(Y) \subsetneq \mathrm{ess}_{G-T_n}(Y_n)$.

Let $T = \bigcup_{n \in \mathbb{N}} T_n$, and let $Y = \bigcup_{n \in \mathbb{N}} Y_n$. By construction, $V(Y_n) \cap V(T_m) = \emptyset$ for all $n, m \in \mathbb{N}$, which means that each Y_n is a wave in $G - T$. Therefore Y is a wave in $G - T$ by Lemma 3.7. It remains to show that $\mathrm{ter}(Y)$ is an A-B separator in G. Our desired contradiction follows because then $Y \cup \{P_a\}$ would be a bad extension of W in G because $a \in \mathrm{ess}_G(W) - \mathrm{ess}_G(Y \cup \{P_a\})$.

Let P be an A-B path in G. To show that P intersects $\mathrm{ter}(Y)$, we show that there is a final segment S of P that lies in $G - T$ and intersects $V(Y)$. This suffices to finish the proof because if x is the last vertex of S in $V(Y)$ and Q is the component path of Y containing x, then QxS is an A-B path in $G - T$ which means that xS (and hence P) must intersect $\mathrm{ter}(Y)$. Thus let x be the last vertex of P on T (if there is no such x, then P is a path in $G - T$ and thus intersects $\mathrm{ter}(Y)$), let n be such that $x \in T_n$, and let $m > n$ be such that $Q_m = \mathrm{ter}(P_a)T_n xP$. Consider stage $m + 1$ of the construction. If $V(Q_m) \cap V(Y_m) \neq \emptyset$, then it must be that $\underline{xP}$ intersects Y_m and hence intersects Y as desired. Otherwise $V(Q_m) \cap V(Y_m) = \emptyset$ and $T_m \cup Q_m$ is a tree with trunk P_a. Thus we choose Y_{m+1} to contain a vertex of $\underline{xP}$, so Y intersects $\underline{xP}$ as desired. □

Lemma 4.3 (in ACA_0). *If $\mathcal{M}$ is a countable coded ω-model of Σ^1_1-DC_0, then Lemma 4.2 holds in $\mathcal{M}$.*

Proof. Consider the formula $\varphi(G, Y, P, x)$ which says there exists a number z such that

(i) z codes a finite subset of $V(P)$,

(ii)

$$\forall y(\forall Y'(Y' \text{ is a wave} \geq Y \text{ in } G - \overline{Py} \rightarrow \mathrm{ess}_{G-\overline{Py}}(Y') = \mathrm{ess}_G(Y)) \rightarrow y \in z),$$

(iii)

$$\begin{aligned} &\forall s(s \text{ codes a finite set} \\ &\quad \wedge \forall y(\exists Y'(Y' \text{ is a wave} \geq Y \text{ in } G - \overline{Py} \wedge \mathrm{ess}_{G-\overline{Py}}(Y') \subsetneq \mathrm{ess}_G(Y)) \rightarrow y \in s) \\ &\rightarrow V(P) - z \subseteq s), \text{ and} \end{aligned}$$

(iv) x is the first vertex on P not in z.

The reason for the somewhat convoluted definition of φ is that prenexing this φ yields a Σ^1_1 formula.

Claim. *In $\mathcal{M}$, suppose that $G = (G, A, B)$ is a web, Y is a wave in G, and P is a finite path in G disjoint from $V(Y)$ such that Y has no bad extensions in G but there exists a wave $Y' \geq Y$ in $G - P$ with $\mathrm{ess}_{G-P}(Y') \subsetneq \mathrm{ess}_G(Y)$. Then $\mathcal{M} \models \varphi(G, Y, P, x)$ if and only if, in $\mathcal{M}$, x is the first vertex on P such that there exists a wave $Y' \geq Y$ in $G - Px$ with $\mathrm{ess}_{G-Px}(Y') \subsetneq \mathrm{ess}_G(Y)$.*

Proof of claim. For the forward direction, using $\mathtt{ACA}_0$ outside $\mathcal{M}$, let

$$Z = \{y \in V(P) \mid \mathrm{ess}_{G-\overline{Py}}(Y') = \mathrm{ess}_G(Y) \text{ for all waves } Y' \geq Y \text{ in } G - \overline{Py} \text{ that are in } \mathcal{M}\}.$$

Let z be a number coding Z and let s be a number coding $V(P) - Z$. In $\mathcal{M}$, z and s code the same sets that they do outside of $\mathcal{M}$, and $\mathcal{M}$ interprets that

$$z \text{ codes } \{y \in V(P) \mid \mathrm{ess}_{G-\overline{Py}}(Y') = \mathrm{ess}_G(Y) \text{ for all waves } Y' \geq Y \text{ in } G - \overline{Py}\} \text{ and}$$
$$s \text{ codes } \{y \in V(P) \mid \mathrm{ess}_{G-\overline{Py}}(Y') \subsetneq \mathrm{ess}_G(Y) \text{ for some wave } Y' \geq Y \text{ in } G - \overline{Py}\}.$$

Hence in $\mathcal{M}$, this z is the only z which satisfies items (i)-(iii). Thus if $\varphi(G, Y, P, x)$ holds in $\mathcal{M}$, x must be the first vertex on P not in z for this z. Thus x must be the first vertex on P such that, in $\mathcal{M}$, there exists a wave $Y' \geq Y$ in $G - Px$ with $\mathrm{ess}_{G-Px}(Y') \subsetneq \mathrm{ess}_G(Y)$.

For the converse, by using $\mathtt{ACA}_0$ outside of $\mathcal{M}$, let x be the first vertex on P such that, in $\mathcal{M}$, there exists a wave $Y' \geq Y$ in $G - Px$ with $\mathrm{ess}_{G-Px}(Y') \subsetneq \mathrm{ess}_G(Y)$. Let z be a number coding $V(\overline{Px})$. In $\mathcal{M}$, z also codes $V(\overline{Px})$, and this z witnesses $\mathcal{M} \models \varphi(G, Y, P, x)$. □

Suppose for a contradiction that Lemma 4.2 is false in $\mathcal{M}$ and, in $\mathcal{M}$, let W, $G = (G, A, B)$, and P_a be a counterexample to Lemma 4.2. We use Σ^1_1-$\mathtt{DC}_0$ in $\mathcal{M}$ to run the construction from Lemma 4.2. This produces in $\mathcal{M}$ a bad extension of W in G, which is a contradiction.

We apply Σ^1_1-$\mathtt{DC}_0$ to the formula $\eta(n, X, Y)$ below. Our η has fixed parameters G, W, P_a, and $(Q_n \mid n \in \mathbb{N})$ (a list of all $\mathrm{ter}(P_a)$-B paths with each occurring infinitely often). We think of a set $Y \subseteq \mathbb{N}$ as coding a pair $Y = (tY, wY)$ where wY is a wave in $G - tY$. Formally, $tY = (Y)_0$ and $wY = (Y)_1$. Our formula $\eta(n, X, Y)$ says that if $t(X)_{n-1}$ is the tree and $w(X)_{n-1}$ is the wave constructed at stage $n-1$ in Lemma 4.2, then tY is the tree and wY is the wave constructed at stage n in Lemma 4.2. Formally, $\eta(n, X, Y)$ says:

- If $n = 0$, then $tY = P_a$ and $wY = W - P_a$.
- If $n > 0$, $t(X)_{n-1}$ is a finite tree with trunk P_a such that $V(t(X)_{n-1}) \cap (V(W) - V(P_a)) = \emptyset$, $w(X)_{n-1} \geq W - P_a$ is a wave in $G - t(X)_{n-1}$, $\mathrm{ess}_{G-t(X)_{n-1}}(w(X)_{n-1}) = \mathrm{ess}_G(W) - \{a\}$, and $w(X)_{n-1}$ has no bad extensions in $G - t(X)_{n-1}$, then
  - if $V(Q_{n-1}) \cap V(w(X)_{n-1}) \neq \emptyset$ or $t(X)_{n-1} \cup Q_{n-1}$ is not a tree, then $tY = t(X)_{n-1}$ and $wY = w(X)_{n-1}$, and
  - if $V(Q_{n-1}) \cap V(w(X)_{n-1}) = \emptyset$ and $t(X)_{n-1} \cup Q_{n-1}$ is a tree, then there is an x such that $tY = t(X)_{n-1} \cup \overline{Q_{n-1}x}$, wY is a wave in $G - tY$, $x \in \mathrm{ter}(wY)$, $wY \geq w(X)_{n-1}$, and $\varphi(G - t(X)_{n-1}, w(X)_{n-1}, Q_{n-1}, x)$.

Prenexing η yields a Σ^1_1 formula. To see this, observe that all the subformulas of η are arithmetical, with the exception of "$w(X)_{n-1}$ has no bad extensions in $G - t(X)_{n-1}$," which is Π^1_1 and appears in the antecedent of η, and φ, which is Σ^1_1 and appears in the consequent of η.

We show that $\mathcal{M} \models \forall n \forall X \exists Y \eta(n, X, Y)$. The interesting case is when $n > 0$ and $X \in \mathcal{M}$ is such that, in $\mathcal{M}$,

- $t(X)_{n-1}$ is a finite tree in G with trunk P_a such that $V(t(X)_{n-1}) \cap (V(W) - V(P_a)) = \emptyset$,
- $w(X)_{n-1} \geq W - P_a$ is a wave in $G - t(X)_{n-1}$,
- $\mathrm{ess}_{G-t(X)_{n-1}}(w(X)_{n-1}) = \mathrm{ess}_G(W) - \{a\}$,
- $w(X)_{n-1}$ has no bad extensions in $G - t(X)_{n-1}$,
- $V(Q_{n-1}) \cap V(w(X)_{n-1}) = \emptyset$, and
- $t(X)_{n-1} \cup Q_{n-1}$ is a tree in G with trunk P_a.

By applying ACA_0 outside $\mathcal{M}$, let x be the first vertex on Q_{n-1} such that, in $\mathcal{M}$, there is a wave $Z \geq w(X)_{n-1}$ in $G - (t(X)_{n-1} \cup Q_{n-1}x)$ with $\mathrm{ess}_{G-(t(X)_{n-1} \cup Q_{n-1}x)}(Z) \subsetneq \mathrm{ess}_{G-t(X)_{n-1}}(w(X)_{n-1})$. Such an x exists by the assumption that G, W, and P_a are a counterexample to Lemma 4.2 in $\mathcal{M}$. By the claim, $\varphi(G - t(X)_{n-1}, w(X)_{n-1}, Q_{n-1}, x)$ holds in $\mathcal{M}$. As $\mathcal{M} \models \mathsf{ACA}_0$, apply Lemma 3.12 inside $\mathcal{M}$ to get a wave $Z \geq w(X)_{n-1}$ in $G - (t(X)_{n-1} \cup \overline{Q_{n-1}x})$ with $x \in \mathrm{ter}(Z)$. Set $tY = t(X)_{n-1} \cup \overline{Q_{n-1}x}$ and $wY = Z$ to get a $Y \in \mathcal{M}$ witnessing $\exists Y \eta(n, X, Y)$.

Now apply Σ^1_1-DC_0 inside $\mathcal{M}$ to conclude that $\mathcal{M} \models \exists Z \forall n \eta(n, (Z)^n, (Z)_n)$, and let $Z \in \mathcal{M}$ be such a Z. By induction, for all $n \in \mathbb{N}$,

- $t(Z)_n$ is a finite tree in G with trunk P_a such that $V(t(Z)_n) \cap (V(W) - V(P_a)) = \emptyset$,
- $w(Z)_n \geq W - P_a$ is a wave in $G - t(Z)_n$,
- $t(Z)_{n-1} \subseteq t(Z)_n$ (if $n > 0$),
- $w(Z)_{n-1} \leq w(Z)_n$ (if $n > 0$),
- $\mathrm{ess}_{G-t(Z)_n}(w(Z)_n) = \mathrm{ess}_G(W) - a_0$,
- $w(Z)_n$ has no bad extensions in $G - t(Z)_n$ that are in $\mathcal{M}$,

and additionally if $V(Q_n) \cap V(w(Z)_n) = \emptyset$ and $t(Z)_n \cup Q_n$ is a finite tree in G with trunk P_a, then $\mathrm{ter}(w(Z)_{n+1})$ contains a vertex of Q_n. Notice that although the statement "$w(Z)_n$ has no bad extensions in $G - t(Z)_n$ that are in $\mathcal{M}$" is Π^1_1 from $\mathcal{M}$'s perspective, it is arithmetical outside of $\mathcal{M}$ because quantifying over the sets inside $\mathcal{M}$ from outside $\mathcal{M}$ amounts to quantifying over (the indices of) the columns of the set coding $\mathcal{M}$. Thus the above-listed properties can be proven (in ACA_0) outside of $\mathcal{M}$ by induction using an appropriate arithmetical formula.

Inside $\mathcal{M}$, let $T = \bigcup_{n\in\mathbb{N}} t(Z)_n$ and $Y = \bigcup_{n\in\mathbb{N}} w(Z)_n$. Just as in the proof of Lemma 4.2, Y is a wave in $G - T$ and $\mathrm{ter}(Y)$ is an A-B separator in G. Thus $Y \cup \{P_a\} \in \mathcal{M}$ is the desired bad extension of W in G, which gives the contradiction. □

Theorem 4.4. *Menger's theorem for countable webs is provable in Π^1_1-CA_0.*

Proof. Let $G = (G, A, B)$ be a countable web. By Lemma 4.1, let $W = \{P_a \mid a \in A\}$ be a $\leq$-maximal wave in G. Let $C = \{\mathrm{ter}(P_a) \mid a \in \mathrm{ess}_G(W)\}$. We extend the paths in $\{P_a \mid a \in \mathrm{ess}_G(W)\}$ to be a collection of disjoint A-B paths M. M and C then witness Menger's theorem for G.

By Theorem 2.4, let $\mathcal{M}$ be a countable coded ω-model of Σ^1_1-DC_0 containing G and W. By Lemma 4.3, Lemma 4.2 holds in $\mathcal{M}$. Also, $\mathcal{M} \models$ "W is a $\leq$-maximal wave", therefore $\mathcal{M} \models$ "W has no bad extensions in G" because W has no proper extensions in G whatsoever. Let $(a_n \mid n \in \mathbb{N})$ enumerate $\mathrm{ess}_G(W)$. Outside $\mathcal{M}$, we construct sequences $(X_n \mid n \in \mathbb{N})$, $(Y_n \mid n \in \mathbb{N})$, and $(Q_n \mid n \in \mathbb{N})$ such that, for all $n \in \mathbb{N}$,

- $X_n \in \mathcal{M}$, $Y_n \in \mathcal{M}$, and $Q_n \in \mathcal{M}$,
- $X_n \subseteq V(G)$ is a finite set, $X_n \cap A = \{a_i \mid i \leq n\}$, and $X_n \subseteq X_{n+1}$,
- Y_n is a wave in $G - X_n$ such that $Y_n \geq W - \bigcup_{i\leq n} P_{a_i}$, Y_n has no bad extensions in $G - X_n$, and $\mathrm{ess}_{G-X_n}(Y_n) = \{a_i \mid i > n\}$, and
- Q_n is an A-B path extending P_{a_n}.

To get started, by Lemma 4.3 let $T \in \mathcal{M}$ be a finite a_0-B tree in G with trunk P_{a_0} such that $V(T) \cap (V(W) - V(P_{a_0})) = \emptyset$, and let $Y_0 \in \mathcal{M}$ be a wave in $G - T$ such that $Y_0 \geq W - P_{a_0}$, $\mathrm{ess}_{G-T}(Y_0) = \{a_i \mid i > 0\}$, and Y_0 has no bad extensions in $G - T$. Let $X_0 = T$, and let Q_0 be the a_0-B path in T. Suppose we have X_n, Y_n and Q_n. Let $P'_{a_{n+1}}$ be the path in Y_n starting at a_{n+1}, and note that $P'_{a_{n+1}}$ extends $P_{a_{n+1}}$. By Lemma 4.3, let $T \in \mathcal{M}$ be a finite a_{n+1}-B tree in $G - X_n$ with trunk $P'_{a_{n+1}}$ such that $V(T) \cap (V(Y_n) - V(P'_{a_{n+1}})) = \emptyset$, and let $Y_{n+1} \in \mathcal{M}$ be a wave in $G - (X_n \cup T)$ such that $Y_{n+1} \geq Y_n - P'_{a_{n+1}}$, $\mathrm{ess}_{G-(X_n \cup T)}(Y_{n+1}) = \{a_i \mid i > n+1\}$, and Y_{n+1} has no bad extensions in $G - (X_n \cup T)$. Let $X_{n+1} = X_n \cup T$ and let Q_{n+1} be the a_{n+1}-B path in T. In the end, the collection $M = \{Q_n \mid n \in \mathbb{N}\}$ consists of disjoint A-B paths in G, and $C = \{\mathrm{ter}(P_a) \mid a \in \mathrm{ess}_G(W)\}$ is an A-B separator containing exactly one vertex from each path in M. □

5. Extended Menger's theorem

Although Menger's theorem for countable webs cannot be equivalent to $\Pi^1_1\text{-}\mathsf{CA}_0$ over RCA_0 as discussed in the introduction, the proof given in Theorem 4.4 is equivalent to $\Pi^1_1\text{-}\mathsf{CA}_0$ in the sense that it proves a stronger statement, called extended Menger's theorem, that is equivalent to $\Pi^1_1\text{-}\mathsf{CA}_0$ over RCA_0. This additional strength comes from our application of Lemma 4.1.

Extended Menger's Theorem. *Let (G, A, B) be a countable web. Then there is a set of disjoint A-B paths M and an A-B separator C such that C consists of exactly one vertex from each path in M. Furthermore, C is the set of terminal vertices of the essential paths in a $\leq$-maximal wave.*

Let G be a graph. For $x \in V(G)$ and $X \subseteq V(G)$, $N(x) = \{y \in V(G) \mid (x, y) \in E(G)\}$ denotes the set of *neighbors* of x and $D(X) = \{y \in V(G) \mid N(y) \subseteq X\}$ denotes the *demand* of X. In the proof of König's duality theorem for countable bipartite graphs in [4], the following lemma plays the role that Lemma 4.1 plays in the proof of Menger's theorem for countable webs given in Theorem 4.4.

Lemma 5.1 ([4] Lemma 3.2)**.** *Let (X, Y, E) be a countable bipartite graph. Then there is a $\subseteq$-maximum $Y_0 \subseteq Y$ for which there is a matching of Y_0 into $D(Y_0)$.*

The application of Lemma 5.1 yields a stronger form of König's duality theorem, called extended König's duality theorem.

Extended König's Duality Theorem. *Let (X, Y, E) be a countable bipartite graph. Then there is a matching M and a cover C such that C consists of exactly one vertex from each edge in E. Furthermore, for every $y \in Y$, $y \in C$ if and only if there is a $Y_0 \subseteq Y$ containing y and a matching of Y_0 into $D(Y_0)$.*

Extended König's duality theorem is equivalent to $\Pi^1_1\text{-}\mathsf{CA}_0$ over RCA_0 by [4] Theorem 4.18. In fact, Lemma 5.1 itself is equivalent to $\Pi^1_1\text{-}\mathsf{CA}_0$ over RCA_0 by [4] Corollary 4.20. In contrast, recall from the introduction that König's duality theorem is equivalent to ATR_0 over RCA_0. We show that the existence of a $\leq$-maximal wave, that is, Lemma 4.1, implies Lemma 5.1 over RCA_0. It follows that both Lemma 4.1 and extended Menger's theorem are equivalent to $\Pi^1_1\text{-}\mathsf{CA}_0$ over RCA_0.

Lemma 5.2. *Lemma 4.1 implies Lemma 5.1 over* RCA_0.

Proof. We prove the lemma in two steps. First, we prove that Lemma 4.1 implies ACA_0 over RCA_0. Second, we prove that Lemma 4.1 implies Lemma 5.1 over ACA_0.

First work in RCA_0. We use the fact that ACA_0 is equivalent to the statement "for every injection $f \colon \mathbb{N} \to \mathbb{N}$ there is a $Z \subseteq \mathbb{N}$ such that $\forall n(n \in Z \leftrightarrow \exists m(f(m) = n))$" (see [9] Lemma III.1.3). So let $f \colon \mathbb{N} \to \mathbb{N}$ be an injection. Let (X, Y, E) be the bipartite graph with sides $X = \{x_n \mid n \in \mathbb{N}\}$ and $Y = \{y_n \mid n \in \mathbb{N}\}$ and edges $E = \{(x_m, y_n) \mid f(m) = n\}$. Let G be the web $G = ((X, Y, E), X, Y)$, and by Lemma 4.1 let W be a $\leq$-maximal wave in G. Let $Z = \{n \mid y_n \in V(W)\}$. We show that $\forall n(n \in Z \leftrightarrow \exists m(f(m) = n))$. If $f(m) = n$, then (x_m, y_n) is the only edge incident to either x_n or

y_n because f is an injection. Thus the path in W starting at x_m is either the trivial path (x_m) or the path (x_m, y_n). If the path is (x_m), then the path could be extended to (x_m, y_n), giving a proper extension of the wave W and contradicting maximality. Thus the path is (x_m, y_n), hence $y_n \in V(W)$ and $n \in Z$. Conversely, if $n \in Z$, then $y_n \in V(W)$ so (x_m, y_n) must be an edge for some $m \in \mathbb{N}$. This can only happen if $f(m) = n$.

Now work in $\mathtt{ACA}_0$. Let (X, Y, E) be a countable bipartite graph. By Lemma 4.1, let W be a $\leq$-maximal wave in the web $G = ((X, Y, E), X, Y)$. Let $Y_0 = Y \cap \mathrm{ter}(W)$. We show that Y_0 witnesses Lemma 5.1 for (X, Y, E). Let M be the matching consisting of the paths in W of length 1. If $y \in Y_0$, then by choice of Y_0 and M there is an $x \in X$ such that (x, y) in M. If $x \notin D(Y_0)$, then there is a $y' \in Y - Y_0$ such that $(x, y') \in E$. Clearly $y' \notin \mathrm{ter}(W)$, and $x \notin \mathrm{ter}(W)$ as well because (x, y) is a path in W. Thus (x, y') is an X-Y path in G avoiding $\mathrm{ter}(W)$, contradicting that W is a wave. Therefore M is a matching of Y_0 into $D(Y_0)$.

To see that Y_0 is $\subseteq$-maximum, suppose for a contradiction that there is a $Y' \subseteq Y$ and a matching M' of Y' into $D(Y')$ such that $Y' \nsubseteq Y_0$. Let W' be the subgraph of (X, Y, E) with vertices $V(W) \cup Y'$ and edges $E(W) \cup \{(x, y) \in M' \mid y \notin Y_0\}$. W is a proper subgraph of W', so if we can show that W' is a wave, then we have that $W < W'$, contradicting the maximality of W. Consider an edge $(x, y) \in M'$ with $y \notin Y_0$. It must be that $x \in \mathrm{ter}(W)$ because otherwise (x, y) would be an X-Y path in G avoiding $\mathrm{ter}(W)$. It follows that W' is a warp. To see that $\mathrm{ter}(W')$ is an X-Y separator, consider an edge $(x, y) \in E$. We know $\mathrm{ter}(W)$ is an X-Y separator, so either $x \in \mathrm{ter}(W)$ or $y \in \mathrm{ter}(W)$. If $y \in \mathrm{ter}(W)$ then $y \in \mathrm{ter}(W')$, so assume $x \in \mathrm{ter}(W)$. If $x \notin \mathrm{ter}(W')$, then there must have been an edge $(x, y') \in M'$ for some $y' \in Y' - Y_0$. By assumption, M' is a matching from Y' into $D(Y')$, so $x \in D(Y')$ because M' matches y' and x. Therefore $y \in Y'$ because $x \in D(Y')$ and (x, y) is an edge. Clearly $Y' \subseteq \mathrm{ter}(W')$, so $y \in \mathrm{ter}(W')$ as desired. □

Corollary 5.3. *Lemma 4.1 is equivalent to* Π^1_1-$\mathtt{CA}_0$ *over* $\mathtt{RCA}_0$.

Proof. The given proof of Lemma 4.1 is in Π^1_1-$\mathtt{CA}_0$. By Lemma 5.2, Lemma 4.1 implies Lemma 5.1 over $\mathtt{RCA}_0$. By [4] Corollary 4.20, Lemma 5.1 is equivalent to Π^1_1-$\mathtt{CA}_0$ over $\mathtt{RCA}_0$. □

Corollary 5.4. *Extended Menger's theorem is equivalent to* Π^1_1-$\mathtt{CA}_0$ *over* $\mathtt{RCA}_0$.

Proof. Theorem 4.4 proves extended Menger's theorem in Π^1_1-$\mathtt{CA}_0$. Extended Menger's theorem asserts the existence of a $\leq$-maximal wave, which is equivalent to Π^1_1-$\mathtt{CA}_0$ over $\mathtt{RCA}_0$ by Corollary 5.3. □

Acknowledgements

Thanks to my advisor Richard A. Shore for introducing me to reverse mathematics and for many helpful discussions.

References

1. Ron Aharoni, *König's duality theorem for infinite bipartite graphs*, Journal of the London Mathematical Society **29** (1984), no. 2, 1–12.
2. ———, *Menger's theorem for countable graphs*, Journal of Combinatorial Theory, Series B **43** (1987), no. 3, 303–313.
3. Ron Aharoni and Eli Berger, *Menger's theorem for infinite graphs*, Inventiones Mathematicae **176** (2009), no. 1, 1–62.
4. Ron Aharoni, Menachem Magidor, and Richard A. Shore, *On the strength of König's duality theorem for infinite bipartite graphs*, Journal of Combinatorial Theory, Series B **54** (1992), no. 2, 257–290.
5. Reinhard Diestel, *Graph theory*, Springer, 2006.
6. Harvey Friedman, *Some systems of second order arithmetic and their use*, Proceedings of the International Congress of Mathematicians, Canadian Mathematical Congress, 1975, pp. 235–242.
7. Klaus-Peter Podewski and Karsten Steffens, *Injective choice functions for countable families*, Journal of Combinatorial Theory, Series B **21** (1976), no. 1, 40–46.

8. Stephen G. Simpson, *On the strength of König's duality theorem for countable bipartite graphs*, Journal of Symbolic Logic **59** (1994), no. 1, 113–123.
9. ———, *Subsystems of Second Order Arithmetic*, Cambridge University Press, 2009.

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