Reflection algebras for theories of iterated truth definitions

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- Previous work on stronger theories of truth with *Evgeny Dashkov* (unfinished).
- Influenced by:
  - Feferman and Schütte's analysis of predicativity;
  - Ulf Schmerl's fine structure theorems for iterated reflection principles;
  - Kotlarski's et al. study of inductive satisfaction classes.

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Why truth predicates?

- Truth predicates are tightly related to reflection principles and are convenient in our framework.
- Theories of iterated truth are mutually interpretable with various standard theories of predicative strength (ramified analysis, iterated Π^0_1-comprehension).
- The framework remains first order and many ingredients are preserved from the treatment of PA.

Languages with truth predicates

- $\mathcal{L}$ first order language extending that of PA by (finitely many) predicate letters
- $\mathcal{L}_\alpha := \mathcal{L} \cup \{\mathsf{T}_\beta : \beta < \alpha\}$ new unary predicates
  (An elementary ordering representing ordinals up to α induces a Gödel numbering of $\mathcal{L}_\alpha$.)
- $\mathsf{T}_\alpha(\ulcorner\varphi\urcorner)$ means *"φ is a true $\mathcal{L}_\alpha$-sentence"*.

Uniform Tarski biconditionals

- Axioms UTB_α in $\mathcal{L}_{\alpha+1}$:
  U1 $\forall \vec{x}\, (\varphi(\vec{x}) \leftrightarrow \mathsf{T}_\alpha(\ulcorner \varphi(\underline{\vec{x}}) \urcorner))$, for each $\varphi(\vec{x}) \in \mathcal{L}_\alpha$;
  U2 $\neg \mathsf{T}_\alpha(\underline{n})$, if n is not a G.n. of an $\mathcal{L}_\alpha$-sentence.
- $\mathrm{UTB}_{<\alpha} := \bigcup_{\beta<\alpha} \mathrm{UTB}_\beta$ in $\mathcal{L}_\alpha$

Fact. $\mathrm{UTB}_{<\alpha}$ conservatively extends $\mathrm{UTB}_{<\beta}$ if $\beta < \alpha$.

(A model of $\mathrm{UTB}_{<\beta}$ can be extended to a model of $\mathrm{UTB}_{<\alpha}$.)

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Formula classes

Arithmetical hierarchy:

- $\Delta_0^{\mathcal{L}} = \Pi_0^{\mathcal{L}} = \Sigma_0^{\mathcal{L}}$ closure of atomic $\mathcal{L}$-formulas under $\wedge$, $\neg$ and bounded quantifiers;
- $\Pi_{n+1}^{\mathcal{L}} := \forall \vec{x}\, \varphi$ where $\varphi \in \Sigma_n^{\mathcal{L}}$;
- $\Sigma_{n+1}^{\mathcal{L}} := \exists \vec{x}\, \varphi$ where $\varphi \in \Pi_n^{\mathcal{L}}$.

Formula classes

Hyperarithmetical hierarchy:

- $\Pi_\alpha := \Pi^{\mathcal{L}}_{1+n}$ if $\alpha = n < \omega$;
- $\Pi_\alpha := \Pi^{\mathcal{L}_{\beta+1}}_{n+1}$ if $\alpha = \omega(1+\beta)+n$;
- $\Pi_{<\lambda} := \bigcup_{\alpha<\lambda} \Pi_\alpha$ if $\lambda \in \mathrm{Lim}$.

Rem. Π_α-formulas define $\Pi_1(0^{(\alpha)})$-sets in $\mathbb{N}$.

- $\Pi_{<\omega}$ arithmetical (in $\mathcal{L}$) sets;
- $\Pi_\omega = \Pi_1(0^{(\omega)}) = \Pi_1(\mathrm{T}_0)$.

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Reflection principles

Let S be Gödelian and $S \vdash \mathsf{EA}$.
$\Box_S$ is the provability predicate for S.

- $R_\alpha(S) := \{\forall \vec{x}\,(\Box_S \varphi(\underline{\vec{x}}) \to \varphi(\vec{x})) : \varphi \in \Pi_\alpha\}$;
- $R_{<\lambda}(S) := \{R_\alpha(S) : \alpha < \lambda\}$.

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Π_α-conservativity

We fix an elementary well-ordering $(\Lambda, <)$ and hence:

- language $\mathcal{L}_\Lambda$;
- formula classes Π_α, for all $\alpha < \omega(1 + \Lambda)$;
- basic theory of iterated Tarski biconditionals
  $IB := \mathsf{EA}^+ + \mathsf{UTB}_{<\Lambda}$ where $\mathsf{EA}^+ := I\Delta_0 + \mathsf{Supexp}$.

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Iterating reflection

A.M. Turing (1939), G. Kreisel (1950s), S. Feferman (1962)

Let $(\Omega, \prec)$ be an elementary well-ordering.
Let $R_\alpha^\beta(S)$ denote β times iterated R_α along $(\Omega, \prec)$:

$$R_\alpha^\beta(S) \equiv \bigcup\{R_\alpha(R_\alpha^\gamma(S)) : \gamma \prec \beta\}.$$

Proof-theoretic analysis by iterated reflection

Given U and α, we are interested in finding ordinal notations β such that

$$U \equiv_\alpha R_\alpha^\beta(\mathsf{EA}),$$

especially for $\alpha = 0, 1, \omega$.

These notations characterize

- $\alpha = 0$: Π^0_1-consequences, consistency strength
- $\alpha = 1$: Π^0_2-consequences, provably total computable functions
- $\alpha = \omega$: (pseudo) Π^1_1-consequences, provably well-founded orderings

Conservativity spectra

- *Π_α-ordinal of S*, denoted $ord_\alpha(S)$, is the sup of all $\beta \in \Omega$ such that $S \vdash R_\alpha^\beta(\mathrm{EA})$;
- *Conservativity spectrum of S* is the sequence $(ord_\beta(S))_{\beta<\Omega}$.

Examples of spectra:

$I\Sigma_1 : \ (\omega^\omega, \omega, 1, 0, 0, \dots)$

$\mathrm{PA} : \quad (\varepsilon_0, \varepsilon_0, \varepsilon_0, \dots)$

$\mathrm{PA} + PH : \quad (\varepsilon_0^2, \varepsilon_0 \cdot 2, \varepsilon_0, \varepsilon_0, \dots)$

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Two basic conservation results

Theorem

Let U be $\Pi_{\alpha+1}$-axiomatized extension of IB and $S \vdash U$. Over U, $R_{\alpha+1}(S) \equiv_\alpha R_\alpha^\omega(S)$.

- Essentially known in the context of first-order arithmetic with an almost identical proof using cut-elimination.
- A well-known particular case is the Parsons–Mints–Takeuti theorem on the Π_2^0-conservativity of $I\Sigma_1$ over PRA.

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Theorem

Let $\lambda \in \mathrm{Lim}$. Then, over IB, $R_\lambda(S) \equiv_{<\lambda} R_{<\lambda}(S)$.

We build a local $\Pi_{<\lambda}$-preserving interpretation of $IB + R_\lambda(S)$ into $IB + R_{<\lambda}(S)$.

Cor. $\mathrm{IB} + \mathrm{RFN}_{\Pi_1(\mathrm{T})}(S)$ is conservative over $PA + \mathrm{RFN}(S)$ for arithmetical sentences.

Rem. Both conservation theorems are formalizable in EA^+.

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Kotlarski theorem

Compositional truth axioms CT:

- $\forall\varphi\,(At[\varphi] \to (T[\varphi] \leftrightarrow T_0[\varphi]))$;
- $\forall\varphi,\psi\,(T[\varphi \wedge \psi] \leftrightarrow (T[\varphi] \wedge T[\psi]))$;
- $\forall\varphi\,(T[\neg\varphi] \leftrightarrow \neg T[\varphi])$;
- $\forall\varphi\,(T[\forall x\,\varphi(x)] \leftrightarrow \forall x\; T[\varphi(\underline{x})])$.

Cor. (Kotlarski) $\mathrm{PA} + \mathrm{CT} + I\Delta_0(\mathrm{T})$ is conservative over $\mathrm{PA} + \mathrm{RFN}^{\omega}(\mathrm{PA})$.

Proof.

Kotlarski theory is contained in $\mathrm{EA} + \mathrm{UTB} + R_1(\mathrm{EA} + \mathrm{UTB})$. Then apply Theorems 1 and 2. □

Semilattice of Gödelian theories

Def. $\mathfrak{G}_{\mathsf{IB}}$ is the set of all Gödelian extensions of IB mod $=_{\mathsf{IB}}$.

$S \leq_{\mathsf{IB}} T \iff \mathsf{IB} \vdash \forall x\, (\Box_T(x) \to \Box_S(x))$;

$S =_{\mathsf{IB}} T \iff (S \leq_{\mathsf{IB}} T \text{ and } T \leq_{\mathsf{IB}} S)$.

Then $(\mathfrak{G}_{\mathsf{IB}}, \wedge_{\mathsf{IB}})$ is a lower semilattice with $S \wedge_{\mathsf{IB}} T := S \cup T$ (defined by the disjunction of the numerations of S and T)

- Each R_α acts on $\mathfrak{G}_{\mathsf{IB}}$: $S \longmapsto \mathsf{IB} + R_\alpha(S)$;
- $(\mathfrak{G}_{\mathsf{IB}}, \wedge_{\mathsf{IB}}, (R_\alpha)_{\alpha<\Lambda})$ semilattice with a family of monotone operators.

Fact. Over IB the schemata $R_\alpha(S)$ are finitely axiomatizable.

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Reflection calculus RC_Λ

Language: $A ::= \top \mid p \mid (A \wedge A) \mid \alpha A$ for $\alpha < \Lambda$
Sequents: $A \vdash B$

RC_Λ rules:

1. $A \vdash A$; $A \vdash \top$; if $A \vdash B$ and $B \vdash C$ then $A \vdash C$;
2. $A \wedge B \vdash A, B$; if $A \vdash B$ and $A \vdash C$ then $A \vdash B \wedge C$;
3. if $A \vdash B$ then $\alpha A \vdash \alpha B$; $\alpha\alpha A \vdash \alpha A$;
4. $\alpha A \vdash \beta A$ for $\alpha > \beta$;
5. $\alpha A \wedge \beta B \vdash \alpha(A \wedge \beta B)$ for $\alpha > \beta$.

Ex. $3\top \wedge 23p \vdash 3(\top \wedge 23p) \vdash 323p$.

Soundness of RC_Λ

An arithmetical interpretation is a map from RC_Λ-formulas to $\mathfrak{G}_{\mathsf{IB}}$ satisfying: $\top^* = \top$; $(A \wedge B)^* = A^* \wedge_{\mathsf{IB}} B^*$; $(\alpha A)^* = R_\alpha(A^*)$.

Th. If $A \vdash_{\mathrm{RC}_\Lambda} B$ then $A^* \leq_{\mathsf{IB}} B^*$, for any interpretation $*$.

RC_Λ as an ordinal notation system

Define: $A <_\alpha B$ iff $B \vdash \alpha A$.

- $\mathbb{W}$ is the set of all variable-free RC_Λ formulas.
- $\mathbb{W}_\alpha$ is the restriction of $\mathbb{W}$ to the signature $\{\beta : \alpha \leq \beta < \Lambda\}$.

Facts.

1. Every $A \in \mathbb{W}$ is equivalent to a word (formula without $\wedge$);
2. $(\mathbb{W}_\alpha, <_\alpha)$ is a well-ordering modulo equivalence in RC_Λ;
3. Its order type can be characterized in terms of Veblen φ function.

Ex. The order type of $(\mathbb{W}, <_0)$ in $\mathrm{RC}_{\omega^\alpha}$ is $\varphi_\alpha(0)$.

Veblen functions

- $\varphi_0(\beta) := \omega^{1+\beta}$;
- $\varphi_{\alpha+1}(\beta) :=$ β-th fixed point of φ_α;
- $\varphi_\lambda(\beta) :=$ β-th simultaneous fixed point of $\{\varphi_\alpha : \alpha < \lambda\}$, if $\lambda \in \mathrm{Lim}$.
- $\Gamma_0 :=$ the least ordinal > 0 closed under $\varphi_\alpha(\beta)$.

Fact. The order types of elements of $\mathbb{W}_\alpha \setminus \{\top\}$ within $(\mathbb{W}_0, <_0)$ are enumerated by φ_α.

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Schmerl-type formulas

Recall that $\equiv_\alpha$ denotes conservativity w.r.t. Π_α.
A^*_S denotes the interpretation of A in $\mathfrak{G}_S$.

Theorem

Let S be a $\Pi_{\alpha+1}$-axiomatizable extension of IB. In $\mathfrak{G}_S$, for all $A \in \mathbb{W}_\alpha$,

$$A^*_S \equiv_\alpha R_\alpha^{o_\alpha(A)}(S).$$

Cor. For any ordinal notations $\alpha, \beta, \gamma < \Gamma_0$,
$R^\gamma_{\alpha+\omega^\beta}(S) \equiv_\alpha R_\alpha^{\varphi_\beta(\gamma)}(S)$.

This holds, because $o_\alpha(A) = \varphi_\beta(o_{\alpha+\omega^\beta}(A))$ for $A \in \mathbb{W}_{\alpha+\omega^\beta}$.

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A few examples

1. Peano arithmetic: $\mathrm{PA} \equiv_{\Pi^0_{n+1}} R_n^{\varepsilon_0}(\mathrm{EA}^+)$.
2. $\mathrm{ACA} := \mathrm{PA}$ + arithmetical comprehension + full induction. Well-known: $\mathrm{ACA} \equiv \mathrm{PA}(\mathrm{T}_0) \equiv \mathrm{IB} + R_{<\omega 2}(\mathrm{IB})$.

   $\mathrm{ACA} \equiv_\omega \mathrm{IB} + R_\omega^{\varepsilon_0}(\mathrm{IB}) \equiv_{<\omega} \mathrm{IB} + R_{<\omega}^{\varepsilon_0}(\mathrm{IB})$;

   $\mathrm{ACA} \equiv_n \mathrm{IB} + R_n^{\varepsilon_{\varepsilon_0}}(\mathrm{IB})$ for $n < \omega$.
3. $\mathrm{ACA}^+ := \mathrm{ACA} + \forall X\, \exists Y\; Y = X^{(\omega)}$. Then $\mathrm{ACA}^+ \equiv \mathrm{PA}(\mathrm{T}_0, \mathrm{T}_1, \dots, \mathrm{T}_\omega) \equiv \mathrm{IB} + R_{<\omega^2+\omega}(\mathrm{IB})$. Hence, $\mathrm{ACA}^+ \equiv_\omega \mathrm{IB} + R_\omega^{\varphi_2(\varepsilon_0)}(\mathrm{IB})$.

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Iterated arithmetical comprehension

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