Vapnik-Chervonenkis Dimension and Density on Johnson and Hamming Graphs

https://arxiv.org/abs/2007.16042

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October 8, 2020

Set systems

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Let $\phi(\vec{x}; \vec{y})$ be a formula, we call $\vec{x}$ the object variables and $\vec{y}$ the parameter variables of $\phi(\vec{x}; \vec{y})$. A set system for a formula $\phi(\vec{x}; \vec{y})$ with m object variables and n parameter variables in a model M with universe set M is a set system (M^m, S_ϕ) where:

$$S_\phi = \{\{\vec{a} \in M^m : M \models \phi(\vec{a}; \vec{b})\} : \vec{b} \in M^n\}$$

Example

Let $\phi(x, y)$ be the formula in the language of graphs stating that there is an edge between x and y.
Then the set system for $\phi(x, y)$ on a graph G is the collection:

$$(V(G), (N(v))_{v \in V(G)})$$

Shatter function

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Let $(X, \mathcal{S})$ be a set system. The *shatter function* $\pi_{(X,\mathcal{S})} : \mathbb{N} \to \mathbb{N}$ for $(X, \mathcal{S})$ is:

$$\pi_{(X,\mathcal{S})}(n) := \max\{|\{T \cap A : T \in \mathcal{S}\}| : A \subseteq X \wedge |A| = n\}$$

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The shatter function $\pi_{\mathcal{C}} : \mathbb{N} \to \mathbb{N}$ for a class $\mathcal{C}$ of set systems:

$$\pi_{\mathcal{C}}(n) := \max\{\pi_{(X,\mathcal{S})}(n) : (X, \mathcal{S}) \in \mathcal{C})\}$$

VC-Characteristics

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The *VC-Density* of a (class of) set system is the infimum r (if one exists) such that $\pi(n) \in \mathcal{O}(n^r)$

Lemma (Sauer-Shelah)

If $(X, \mathcal{S})$ *has finite VC-dimension* d *then* $\pi_{\mathcal{S}}(n) \leq \sum_{i=0}^{d} \binom{n}{i}$.

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Corollary

VC-Density $\leq$ VC-Dimension.

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Extremal combinatorics.

Johnson Graphs

Definition

Let n and k be natural numbers $n \geq k$. The Johnson graph $J(n, k)$ is a graph whose vertices correspond to the k-element subsets of a set of size n and two vertices are adjacent if their corresponding sets intersect in all but one element, i.e. their symmetric difference has size 2.

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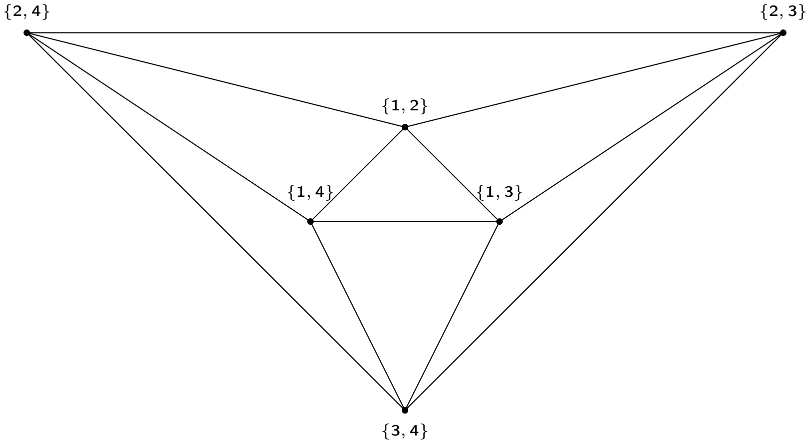
$J(n, 1) = K_n$.

The Johnson graph $J(4,2)$.



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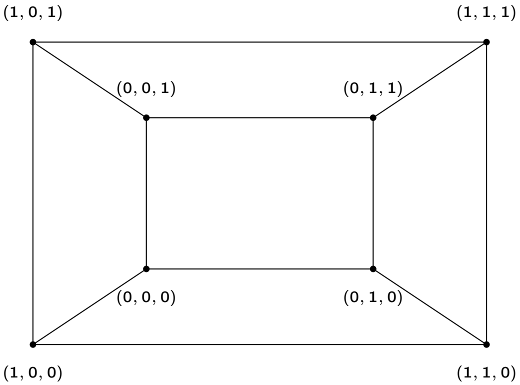
The Hamming graph $H(3, 2)$.



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A result by Adler and Adler stating that nowhere dense classes of graphs have stable (and therefore dependent) theories.
By fixing one parameter Johnson graphs and Hamming graphs give classes with a dependent limit theory.
Yet Johnson and Hamming graphs graphs are clearly somewhere dense.
Furthermore they have unbounded local clique-width.

Main results

Theorem

The edge relation has:

- *VC-dimension 4 on the class of all Johnson graphs.*
- *VC-dimension 3 on the class of all Hamming graphs.*
- *VC-density 2 on the class of all Johnson graphs.*
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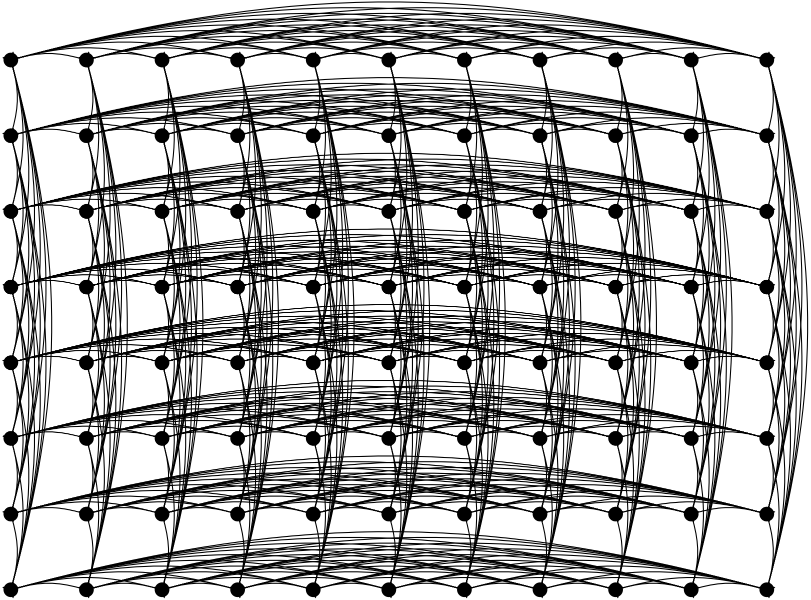
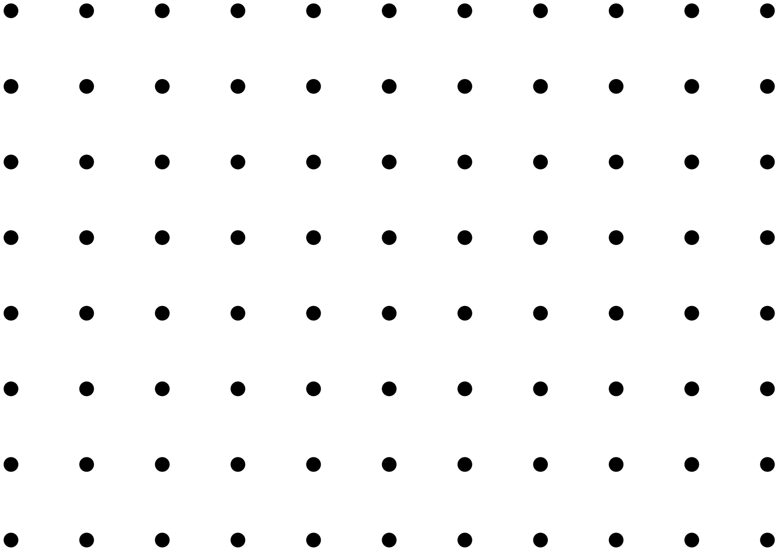
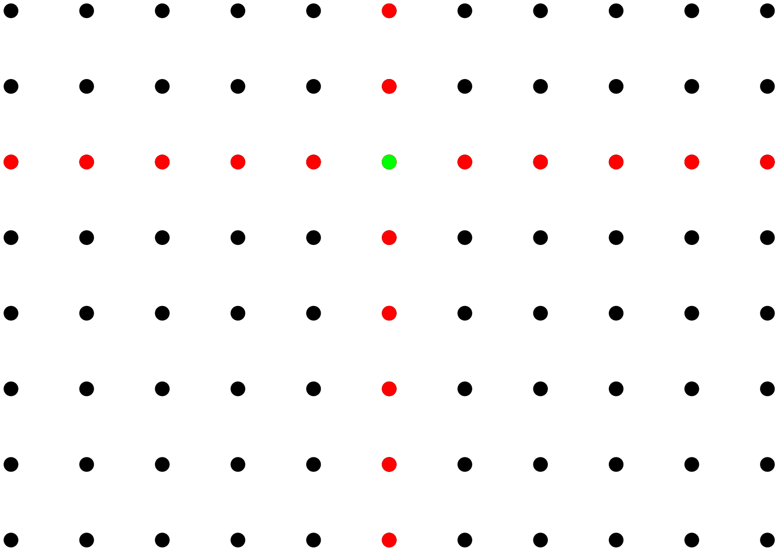
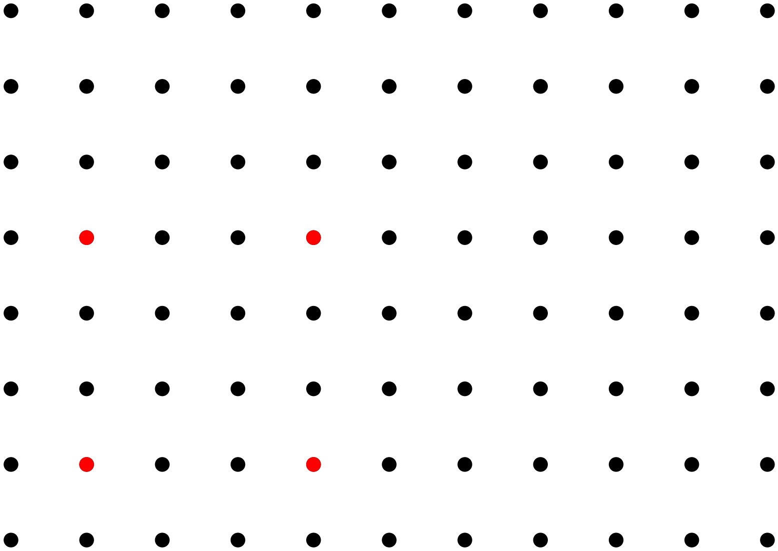
Observation.

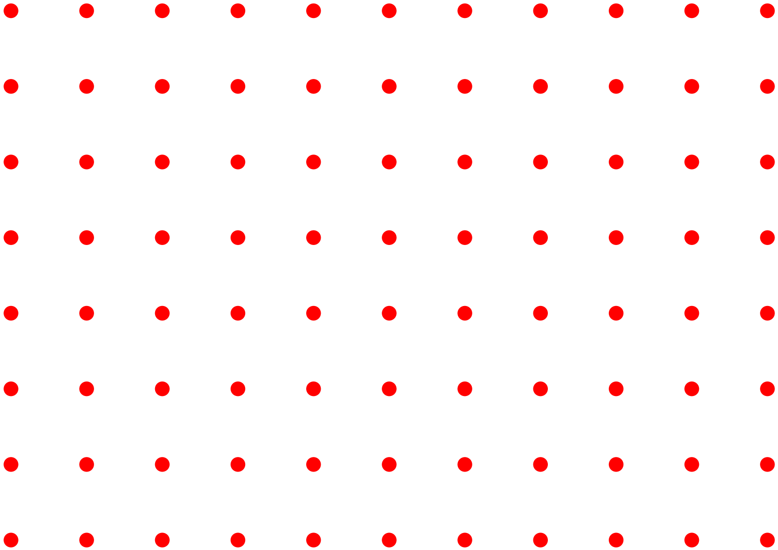
In the Johnson graph $J(m,k)$ we have $N(v) = \{(v \setminus \{a\}) \cup \{x\} | a \in v \wedge x \notin v\}$. This induces the $k \times (m-k)$ rook's graph.











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Let

$$S(A) := \{N(u) \cap A; u \in V(G)\},$$
$$C_1(A) := \{N \in S(A) : N \text{ is a clique}\}.$$
$$C_2(A) := \{N \in S(A) : N \text{ is not a clique}\}.$$
$$\pi(n) = |S(A)| = |C_1(A)| + |C_2(A)|.$$

Counting elements of $C_1(A)$

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For every vertex $u \in A$ we have that A intersects at most $|A|$ rows and at most $|A|$ columns of the rook's graph induced by $N(u)$.
So u can be a member of at most $2|A|$ maximal cliques of $J(m, k)$ that intersect A in more than two vertices.
So the number of maximal cliques of $J(m, k)$ that intersect A in more than two vertices is at most $2|A|^2$

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