On low for speed oracles

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Lowness for speed

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Complexity theory: $\mathbf{P}^A$, $\mathbf{NP}^A$, …

Lowness for speed

Relativization in complexity can shed light on important questions:

Theorem (Baker, Gill, Solovay)

There are oracles A such that $\mathbf{P}^A = \mathbf{NP}^A$, and oracles B such that $\mathbf{P}^B \neq \mathbf{NP}^B$

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Definition (Allender)

X is **low for speed (l.f.s)** if every *decidable* set/language L that can be computed with oracle X in time f can be computed without oracle in time $poly(f)$.

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This basically amounts to collapsing all time classes simultaneously: $\mathbf{P}^A = \mathbf{P}$, $\mathbf{NP}^A = \mathbf{NP}$, $\mathbf{EXPTIME}^A = \mathbf{EXPTIME}$, ...

Does such an *A* exist? Obviously yes: take *A* to be in PTIME-computable! (note: *X* computable but EXPTIME-complete would not work, so lowness for speed is **not** closed under $\equiv_T$).

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Proof is a priority argument. One constructs A to be sparse, so that at stage t there are few candidates for $A \upharpoonright t$, thus for a functional Φ one can try to simulate all possible Φ^A in parallel (+ some very nice twist to handle Friedberg-Muchnik requirements).

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2. What is the situation outside c.e. sets? How big is the set LFS in terms of cardinality/category/measure? (category answered by Bayer and Slaman)
3. Closing under $\equiv_T$: what are the X that are equivalent to some low for speed? (note: every degree contains a non low for speed). Are such X closed downwards? under join?

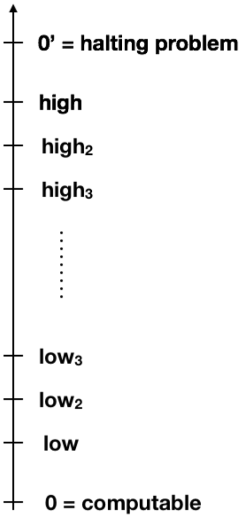
Within c.e. sets

Can we characterize the c.e. sets in LFS? Seems very hard, but one can get partial results.

One way to study LFS inside c.e. sets is with respect to the high/low hierarchy:

- A is *low* if $A' = \mathbf{0}'$; A is *low*$_n$ if $A^{(n)} = \mathbf{0}^{(n)}$.
- A is *high* if $A' = \mathbf{0}''$; A is *high*$_n$ if $A^{(n)} = \mathbf{0}^{(n+1)}$.

Strength


0' = halting problem
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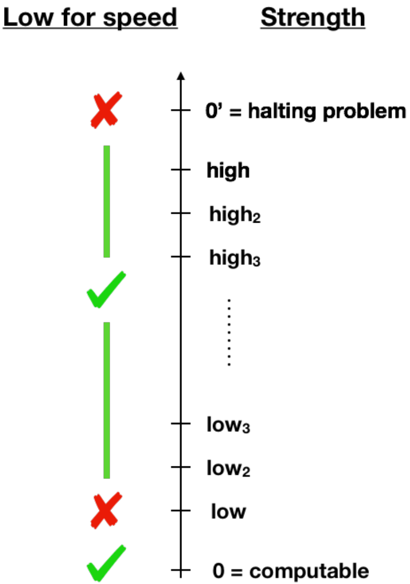
Theorem (BD)
If A is c.e., non-computable but low, it is necessarily **not** l.f.s. (!).

Theorem (BD)
However, there is a c.e. set A which is non-computable, low_2, and l.f.s. .



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Outside the c.e. world

How common are low for speed sets? Is the set LFS uncountable? co-meager? of measure 1?

Quick definitions

A set $\mathcal{U}$ of infinite binary sequence is **open** for the product topology if it can be written as:

$$\mathcal{U} = \bigcup_{\sigma \in W} [\sigma]$$

where *W* is a (countable) set of binary strings and $[\sigma]$ is the set of infinite binary sequences that start with σ.

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We say that $\mathcal{U}$ is **effectively open** if W can be chosen to be computably enumerable (or computable)

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- A sequence X is **weakly 1-generic** if for every dense effectively open set $\mathcal{U}$, we have that X is in $\mathcal{U}$.
- A sequence X is **1-generic** if for every effectively open set $\mathcal{U}$, we have $X \in \mathcal{U}$ or X is in the interior of the complement of $\mathcal{U}$.

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Effective measure theory:

- A sequence X is **Martin-Löf random** if for every sequence of uniformly effectively open sets $(\mathcal{U}_n)$ such that $\mu(\mathcal{U}_n) \leq 2^{-n}$, we have $X \notin \bigcap_n \mathcal{U}_n$
- A sequence X is **Schnorr random** if for every sequence of uniformly effectively open sets $(\mathcal{U}_n)$ such that $\mu(\mathcal{U}_n) = 2^{-n}$, we have $X \notin \bigcap_n \mathcal{U}_n$

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Well...... it's complicated....

Theorem (Bayer-Slaman)
LFS is meager **if and only if** P $\neq$ NP.

So we might not know for a while whether LFS is meager or co-meager.

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However,

Theorem (BD)
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- Every non-computable c.e. set *computes* a l.f.s. set.
- Almost every oracle (in the measure sense) *computes* a l.f.s. set.
- There is a low Δ_2^0 set that is low for speed.

Randomness vs lowness for speed

Like for generics, one could expect a conditional behaviour of randoms w.r.t. lowness for speed, for example a dependance on the answer to $\mathrm{P} = \mathrm{BPP}$. This is not the case:

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Theorem (BD)
If A has Martin-Löf random **degree** (in fact, DNC degree is enough), it is not low for speed.

Proof inspired by Blum's speedup theorem.

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Proof: extend the earlier result to show that a low c.e. *degree* does not contain any l.f.s. set. Take a non-computable c.e. set X which is l.f.s. and apply Sack's splitting theorem to get a low c.e. Y with $0 <_T Y <_T X$.

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How does lowness interact with minimality? We were able to prove

Theorem (BD)

There exists a minimal Turing degree which does not contain any l.f.s. set.

We conjectured that there is also a l.f.s. oracle of minimal Turing degree.......


BREAKING
NEWS

Turing degrees and LFS

A very recent result:

Theorem (Harrison-Trainor, Downey)
There exists a l.f.s. oracle of minimal Turing degree.

Thank you!