Quantified Reflection Calculus with one modality

Ana de Almeida Borges Joost J. Joosten

Universitat de Barcelona

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In this talk we...

- Discuss known shortcomings of quantified provability logic
- Introduce QRC_1 as a candidate solution
- Explore some famous proofs
- State obtained results about QRC_1
- Sketch a couple of new proofs

Provability Logics

- Interpret $\Box$ as "is provable"
- Interpret $\Diamond$ as "is consistent"

Examples:

- GL is $K4 + \Box(\Box\varphi \to \varphi) \to \Box\varphi$ (Löb's axiom)
- GLP is a polymodal version of GL, with $[0], [1], \ldots$ as modalities
  - Decidability is PSPACE-complete
- RC is the strictly positive fragment of GLP, with statements of the form $\varphi \vdash \psi$, where φ, ψ are in the language built from $\top$, p, $\wedge$, $\langle 0 \rangle, \langle 1 \rangle, \ldots$
  - E.g. $\langle 1 \rangle p \vdash \langle 0 \rangle p$
  - Decidability is in PTIME

Arithmetical realizations

It is possible to express Gödel's provability predicate in PA:

$$\mathrm{Prov}_{\mathrm{PA}}(\varphi) := \exists p\, \mathrm{Proof}_{\mathrm{PA}}(p, \varphi)$$

Let $\mathcal{L}_\Box$ be the language of GL.
An arithmetical realization is any function $(\cdot)^\star$ taking:

$$\begin{aligned} \text{formulas in } \mathcal{L}_\Box &\to \text{sentences in } \mathcal{L}_{\mathrm{PA}} \\ \text{propositional variables} &\to \text{arithmetical sentences} \\ \text{boolean connectives} &\to \text{boolean connectives} \\ \Box &\to \mathrm{Prov}_{\mathrm{PA}} \end{aligned}$$

Solovay's Theorem

Theorem (Solovay, 1976)

Let $\varphi \in \mathcal{L}_\Box$. *Then:*

$$\mathsf{GL} \vdash \varphi$$
$$\Updownarrow$$
$$\mathsf{PA} \vdash (\varphi)^\star \textit{ for any arithmetical realization } (\cdot)^\star$$

This can be written as:

$$\mathsf{GL} = \{\varphi \in \mathcal{L}_\Box \mid \text{for any } (\cdot)^\star, \text{ we have } \mathsf{PA} \vdash (\varphi)^\star\}$$

Solovay for quantified modal logic?

Let $\mathcal{L}_{\Box,\forall}$ be the language of relational quantified modal logic:

$$\top\text{, relation symbols, boolean connectives, } \forall x\text{, and } \Box$$

Define arithmetical realizations $(\cdot)^\bullet$ for $\mathcal{L}_{\Box,\forall}$:

$$\begin{aligned}
\text{formulas in } \mathcal{L}_{\Box,\forall} &\to \text{formulas in } \mathcal{L}_{\mathsf{PA}} \\
n\text{-ary relation symbols} &\to \text{arithmetical formulas with } n \text{ free variables} \\
\text{boolean connectives} &\to \text{boolean connectives} \\
\forall x &\to \forall x \\
\Box &\to \mathsf{Prov}_{\mathsf{PA}}
\end{aligned}$$

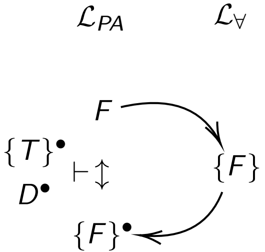
Theorem (Vardanyan, 1986)

$$\{\textit{closed } \varphi \in \mathcal{L}_{\Box,\forall} \mid \textit{for any } (\cdot)^\bullet, \ \textit{we have } \mathsf{PA} \vdash (\varphi)^\bullet\}$$

is Π^0_2*-complete. Thus it is not recursively axiomatizable.*

Artemov's Lemma

- Let $F \in \mathcal{L}_{\mathrm{PA}}$ be a formula
- Replace arithmetical symbols $0, +1, +, \times, =$ with predicates Z, S, A, M, E, obtaining $\{F\} \in \mathcal{L}_\forall$
- Go back to $\mathcal{L}_{\mathrm{PA}}$ with a realization $(\cdot)^\bullet$



When are F and $F^\bullet$ equivalent?

- Under $\{T\}^\bullet$ to get arithmetical axioms...
- ... and under $D^\bullet$ to get recursive $A^\bullet$ and $M^\bullet$
- By Tennenbaum's Theorem the model induced by $(\cdot)^\bullet$ is standard

$$
\begin{aligned}
D := \Diamond\top \wedge {} & \\
& \forall x\,(Z(x) \to \Box Z(x)) \wedge \forall x\,(\neg Z(x) \to \Box\neg Z(x)) \wedge \\
& \cdots S \cdots A \cdots M \cdots E
\end{aligned}
$$

Artemov's Theorem

Theorem (Artemov, 1985)

$$\mathcal{A} := \{\textit{closed } \varphi \in \mathcal{L}_{\Box,\forall} \mid \textit{for any } (\cdot)^\bullet,\ \textit{we have } \mathbb{N} \vDash (\varphi)^\bullet\}$$

is not arithmetical.

- By Tarski's Undefinability Theorem the class of true arithmetical sentences $\mathcal{V}$ is not arithmetical
- We provide a bijection ! between $\mathcal{V}$ and $\mathcal{A}$
- For $F \in \mathcal{L}_{\mathsf{PA}}$, let $F! := \{T\} \wedge D \to \{F\}$
- We see that F is true iff $F!$ is always true
- ($\Rightarrow$) If F is true, pick any $(\cdot)^\bullet$ and see that $\{T\}^\bullet \wedge D^\bullet \to \{F\}^\bullet$ is true
- ($\Leftarrow$) If $F!$ is always true, pick $(\cdot)^\bullet$ as the "normal" interpretation and see that $\{F\}^\bullet$ - and hence F - are true

Escape to Vardanyan's Theorem?

Restrict $\mathcal{L}_{\Box,\forall}$ to the strictly positive fragment $\mathcal{L}_{\Diamond,\forall}$:

$$\text{Terms} ::= \text{Variables} \mid \text{Constants}$$

$$\mathcal{L}_{\Diamond,\forall} ::= \top \mid \text{relation symbols applied to Terms} \mid \varphi \wedge \varphi \mid \forall x\, \varphi \mid \Diamond\varphi$$

The arithmetical realizations $(\cdot)^*$ for $\mathcal{L}_{\Diamond,\forall}$ send:

$$\text{formulas in } \mathcal{L}_{\Diamond,\forall} \to \text{axiomatizations of theories in } \mathcal{L}_{\mathsf{PA}}$$

Define a calculus QRC$_1$ with statements $\varphi \vdash \psi$ where:

$$\varphi, \psi \in \mathcal{L}_{\Diamond,\forall}$$

Prove arithmetical soundness and completeness for QRC$_1$:

$$\mathsf{QRC}_1 \stackrel{?}{=} \{\varphi \vdash \psi \mid \text{for any } (\cdot)^*, \text{ we have } \mathsf{PA} \vdash (\varphi \vdash \psi)^*\}$$

QRC_1: Axioms and rules

$$\varphi \vdash \top \qquad \varphi \wedge \psi \vdash \varphi \qquad \Diamond\Diamond\varphi \vdash \Diamond\varphi \qquad \frac{\varphi \vdash \psi}{\Diamond\varphi \vdash \Diamond\psi}$$

$$\varphi \vdash \varphi \qquad \varphi \wedge \psi \vdash \psi$$

$$\frac{\varphi \vdash \psi \quad \psi \vdash \chi}{\varphi \vdash \chi} \qquad \frac{\varphi \vdash \psi \quad \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi} \qquad \frac{\varphi \vdash \psi}{\varphi \vdash \forall x\, \psi} \qquad \frac{\varphi[x\leftarrow t] \vdash \psi}{\forall x\, \varphi \vdash \psi}$$

$x \notin \mathrm{fv}\, \varphi$ $\qquad$ t free for x in φ

$$\frac{\varphi \vdash \psi}{\varphi[x\leftarrow t] \vdash \psi[x\leftarrow t]}$$

t free for x in φ and ψ

$$\frac{\varphi[x\leftarrow c] \vdash \psi[x\leftarrow c]}{\varphi \vdash \psi}$$

c not in φ nor ψ

Arithmetical semantics

The arithmetical realizations $(\cdot)^*$ for $\mathcal{L}_{\Diamond,\forall}$:

$$
\begin{aligned}
\text{formulas in } \mathcal{L}_{\Diamond,\forall} &\to \text{axiomatizations of theories in } \mathcal{L}_{\mathsf{PA}} \\
\text{constants } c_i &\to \text{variables } y_i \\
\text{variables } x_i &\to \text{variables } z_i \\
(\top)^* &:= \tau_{\mathsf{I\Sigma_1}}(u) \\
(S(c,x))^* &:= \sigma(y,z,u) \vee \tau_{\mathsf{I\Sigma_1}}(u) \\
(\psi(c,x) \wedge \delta(c,x))^* &:= (\psi(c,x))^* \vee (\delta(c,x))^* \\
(\Diamond\psi(c,x))^* &:= \tau_{\mathsf{I\Sigma_1}}(u) \vee (u = \mathrm{Con}_{(\psi(c,x))^*}(\top)) \\
(\forall x\, \psi(c,x))^* &:= \exists z\, (\psi(c,x))^* \\
(\varphi(c,x) \vdash \psi(c,x))^* &:= \forall \theta, y, z\, (\Box_{\psi^*(y,z)}\theta \to \Box_{\varphi^*(y,z)}\theta)
\end{aligned}
$$

Arithmetical soundness

Theorem (Arithmetical soundness)

$$\mathrm{QRC}_1 \subseteq \{\varphi \vdash \psi \mid \textit{for any } (\cdot)^*, \textit{ we have } \mathrm{I}\Sigma_1 \vdash \forall\, \theta, y, z\, (\Box_{\psi^*(y,z)}\theta \to \Box_{\varphi^*(y,z)}\theta)\}$$

By induction on the QRC_1-proof. Here is the case of $\Diamond\Diamond\varphi \vdash \Diamond\varphi$:

- Pick any $(\cdot)^*$, reason in $\mathrm{I}\Sigma_1$, and let θ, y, z be arbitrary
- Assume $\Box_{(\Diamond\varphi)^*}\theta$
- Then $\Box_\tau(\mathrm{Con}_{\varphi^*}(\top) \to \theta)$
- By provable Σ_1-completeness, $\Box_\tau(\mathrm{Con}_\tau(\mathrm{Con}_{\varphi^*}(\top)) \to \mathrm{Con}_{\varphi^*}(\top))$
- Then $\Box_\tau(\mathrm{Con}_\tau(\mathrm{Con}_{\varphi^*}(\top)) \to \theta)$
- We conclude $\Box_{(\Diamond\Diamond\varphi)^*}\theta$

Arithmetical completeness

Conjecture (Arithmetical completeness)

$$\mathsf{QRC}_1 \supseteq \{\varphi \vdash \psi \mid \text{for any } (\cdot)^*, \text{ we have } \mathsf{I\Sigma_1} \vdash (\varphi \vdash \psi)^*\}$$

Solovay's completeness proof

Theorem (Solovay, 1976)

$$\text{GL} \supseteq \{\varphi \in \mathcal{L}_\Box \mid \textit{for any } (\cdot)^\star, \ \textit{we have } \text{PA} \vdash (\varphi)^\star\}$$

- Assume $\text{GL} \nvdash \varphi$
- Take a (finite, transitive, conversely well-founded, rooted) Kripke model $\mathcal{M}$ not satisfying φ at world 1 (the root)
- Embed $\mathcal{M}$ (with an extra world 0 pointing to the root) into the language of arithmetic, obtaining a formula λ_i representing each world i
- Define $S^\star$ as the disjunction of the λ_i such that $i \Vdash S$
- Prove a Truth Lemma stating that (for $i > 0$ and χ a subformula of φ) if $i \Vdash \chi$ then $\text{PA} \vdash \lambda_i \to \chi^\star$ and if $i \nVdash \chi$ then $\text{PA} \vdash \lambda_i \to \neg\chi^\star$

Solovay's completeness proof (cont'ed)

Theorem (Solovay, 1976)

$$\mathsf{GL} \supseteq \{\varphi \in \mathcal{L}_\Box \mid \textit{for any } (\cdot)^\star,\ \textit{we have } \mathsf{PA} \vdash (\varphi)^\star\}$$

- ...
- Prove a Truth Lemma stating that (for $i > 0$ and χ a subformula of φ) if $i \Vdash \chi$ then $\mathsf{PA} \vdash \lambda_i \to \chi^\star$ and if $i \nVdash \chi$ then $\mathsf{PA} \vdash \lambda_i \to \neg\chi^\star$
- Then $\mathsf{PA} \vdash \lambda_1 \to \neg\varphi^\star$
- Prove $\mathbb{N} \vDash \lambda_0$
- Prove $\mathsf{PA} \vdash \lambda_0 \to \Diamond\lambda_1$.
- Then $\mathsf{PA} \vdash \lambda_0 \to \Diamond(\neg\varphi^\star)$
- Then $\mathbb{N} \vDash \neg\Box\varphi^\star$
- Then $\mathsf{PA} \nvdash \varphi^\star$

How to adapt Solovay's proof to QRC_1?

- Kripke completeness for QRC_1
- Counter models should be finite, transitive, irreflexive and rooted
- Find an appropriate embedding of such models in arithmetic, preserving the nice properties of the λ_i
- We think the relational properties of the models can be encoded with the same λ_i, while independently encoding information about the domains some other way

Relational models

Kripke models where:

- each world w is a first-order model with a finite domain
- each constant symbol c and relational symbol S has a denotation at each world
- there is a transitive relation R between worlds
- the domains are inclusive: if wRv, then $\text{domain}(w) \subseteq \text{domain}(v)$
- the constants have concordant interpretations: if wRv, then $\text{denotation}_v(c) = \text{denotation}_w(c)$
- we use w-assignments $g : \text{Variables} \to \text{domain}(w)$ to interpret variables
- we abuse notation and define $g(c) := \text{denotation}_w(c)$ for all w-assignments g and constants c

Satisfaction

Let g be a w-assignment.

$$\mathcal{M}, w \Vdash^g S(t,u) \iff \langle g(t), g(u) \rangle \in \text{denotation}_w(S)$$

$$\mathcal{M}, w \Vdash^g \Diamond\varphi \iff$$
$$\text{there is a world } v \text{ such that } wRv \text{ and } \mathcal{M}, v \Vdash^g \varphi$$

$$\mathcal{M}, w \Vdash^g \forall x\, \varphi \iff$$
$$\text{for all } w\text{-assignments } h \sim_x g, \text{ we have } \mathcal{M}, w \Vdash^h \varphi$$

Relational soundness and completeness

Theorem (Relational soundness)

If $\varphi \vdash \psi$, then for any model $\mathcal{M}$, world w, and w-assignment g:

$$\mathcal{M}, w \Vdash^g \varphi \implies \mathcal{M}, w \Vdash^g \psi.$$

Theorem (Relational completeness)

If $\varphi \nvdash \psi$, then there is a finite model $\mathcal{M}$, a world w, and a w-assignment g such that:

$$\mathcal{M}, w \Vdash^g \varphi \quad \text{and} \quad \mathcal{M}, w \nVdash^g \psi.$$

Since QRC_1 has the finite model property, it is decidable.

Proving relational completeness

- Given $\varphi \nvdash \psi$, build a counter-model
- The standard is to use term models: each world is the set of formulas true at that world
- We also want to know which formulas are *not* true at given worlds
- Our worlds are pairs of "positive" (true) and "negative" (false) formulas:

$$w = \langle w^+, w^- \rangle \qquad \text{e.g. } \langle \{\varphi\}, \{\psi\} \rangle$$

- Worlds should be *well-formed* pairs though...

Well-formed pairs

Let Λ be a set of formulas.

- $\Gamma \vdash \delta$ is shorthand for $(\bigwedge_{\gamma \in \Gamma} \gamma) \vdash \delta$
- A pair p is *closed* if every formula in p is closed
- A pair p is *consistent* if for every $\delta \in p^-$ we have $p^+ \nvdash \delta$
- A pair p is Λ-*maximal* if for every $\varphi \in \Lambda$, either $\varphi \in p^+$ or $\varphi \in p^-$
- A pair p is *fully witnessed* if for every formula $\forall x \, \varphi \in p^-$ there is a constant c such that $\varphi[x \leftarrow c] \in p^-$
- A pair p is Λ-*well-formed* if it is closed, Λ-maximal, consistent and fully witnessed

Building a world from an incomplete pair

- Start with the closed consistent pair $p = \langle p^+, p^- \rangle$
- Let C be a finite set of constants containing the constants in p and some new constants
- Let Λ be the closure under (closed) subformulas of p, and such that if $\forall x\, \varphi \in \Lambda$, then for every $c \in C$ we have $\varphi[x{\leftarrow}c] \in \Lambda$
- Goal: end with a Λ-well-formed pair w containing p

Method

- Some formulas in Λ are consequences of p^+, and thus must be added to w^+ to preserve consistency
- We put all the other formulas of Λ in p^-

This Method works!

Lemma

If the number of new constants in C is the maximum $\forall$-depth of formulas in p, the Method produces a Λ-well-formed pair w containing p.

- w is consistent because $\varphi \in w^+$ if and only if $p^+ \vdash \varphi$
- w is fully-witnessed because...

$$\forall x\, \varphi \in w^-$$

$$\Downarrow$$

there is some new $c \in C$ s.t. c doesn't appear in $\forall x\, \varphi$

$$\Downarrow$$

$$p^+ \nvdash \varphi[x\leftarrow c]$$

$$\Downarrow$$

$$\varphi[x\leftarrow c] \in w^-$$

Building a counter-model

- Start with $\varphi \nvdash \psi$ (both closed)
- Build a (well-formed!) world w s.t. $\varphi \in w^+$ and $\psi \in w^-$
- Let domain(w) be the set of constants C from that construction
- Let the denotation of relation symbols at w correspond to their membership in w^+
- If $\Diamond\chi \in w^+$, create a new world v_χ seen from w by completing

$$\langle \{\chi\}, \{\delta, \Diamond\delta \mid \Diamond\delta \in w^-\} \cup \{\Diamond\chi\} \rangle$$

- Define the domain and the denotation at v_χ like with w
- Repeat until all $\Diamond$-formulas are witnessed

Putting it together

Lemma (Truth lemma)

Let $\mathcal{M}$ be the counter-model we just built. Then for any world w, w-assignment g, and formula $\chi^g \in \Lambda$:

$$\mathcal{M}, w \Vdash^g \chi \iff \chi^g \in w^+,$$

where χ^g is χ with every free variable x replaced by $g(x)$.

Theorem (Relational completeness)

If $\varphi \nvdash \psi$, then there is a finite model $\mathcal{M}$, a world w, and a w-assignment g such that:

$$\mathcal{M}, w \Vdash^g \varphi \quad \text{and} \quad \mathcal{M}, w \nVdash^g \psi.$$

In summary

QRC_1:

- quantified, strictly positive provability logic
- sound w.r.t arithmetical semantics
- complete w.r.t arithmetical semantics? (work in progress)
- sound and complete w.r.t. relational semantics
- decidable

Thank you

ana de almeida gabriel @ ub . edu

Further Reading

A.A.B. and J.J. Joosten (2020)
Quantified Reflection Calculus with one modality
Advances in Modal Logic 13

R. Goldblatt (2011)
Quantifiers, propositions and identity: admissible semantics for quantified modal and substructural logics
Cambridge University Press

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Arithmetic complexity of predicate logics of provability and their fragments
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Soviet Mathematics Doklady 33, 569–572 (English)