Some old and new uses of the tree labeling method

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Reverse math, in one slide

Reverse mathematics is a foundational program for calibrating the computable and proof-theoretic content of mathematical principles.

Various subsystems of Z_2 are used as benchmarks against which to test the strength of theorems we are interested in: RCA_0, WKL, ACA_0, ...

RCA_0 consists of the algebraic axioms about the natural numbers, plus Δ^0_1-comprehension and Σ^0_1-induction.

A model of RCA_0 is a pair $(N, \mathcal{S})$, where N is a (possibly nonstandard) first-order structure, and $\mathcal{S} \subseteq \mathcal{P}(N)$ is closed under Δ^0_1-definability.

An ω-model is a model $(N, \mathcal{S})$ with $N = \omega$, which can thus be identified just with $\mathcal{S}$. If $\mathcal{S} \models RCA_0$ then $\mathcal{S}$ is a Turing ideal.

The computability-theoretic perspective

We are interested in statements of the form

$$\forall X\,[\Phi(X) \to \exists Y\,\Psi(X, Y)],$$

where Φ and Ψ are some kind of properties of X and Y.

We think of this as a problem, "given X satisfying Φ, find Y satisfying Ψ".

We call the X such that $\Phi(X)$ holds the instances of the problem,
and the Y such that $\Psi(X, Y)$ holds the solutions to X for this problem.

Typically, we look at problems whose instances and solutions are subsets of $\mathbb{N}$,
and where the properties Φ and Ψ are arithmetical.

Basic question. Given an instance of a problem, how complex are its solutions?

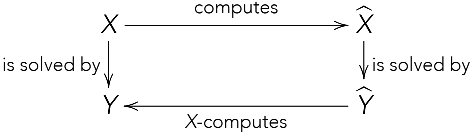
Computable reducibility

Let P and Q be problems.

P is computably reducible to Q, written P $\leq_c$ Q, if

- every instance X of P computes an instance $\widehat{X}$ of Q,
- every Q-solution $\widehat{Y}$ to $\widehat{X}$, together with X, computes a P-solution Y to X.

So the following diagram commutes:



(Dzhafarov '15; Hirschfeldt and Jockusch '16).

Weihrauch reducibility

Let P and Q be problems.

P is Weihrauch reducible to Q, written P $\leq_W$ Q, if

- every instance X of P *uniformly* computes an instance $\widehat{X}$ of Q,
- every Q-solution $\widehat{Y}$ to $\widehat{X}$, together with X, *uniformly* computes a P-solution Y to X.

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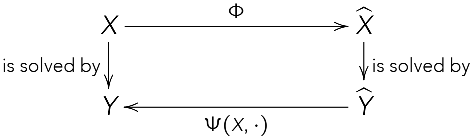

(Weihrauch '92; Brattka; Gherardi and Marcone '08; DDHMS '16).

Strong forms

Let P and Q be problems.

P is strongly computably reducible to Q, written P $\leq_{sc}$ Q, if

- every instance X of P computes an instance $\widehat{X}$ of Q,
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Some examples

Ramsey's theorem. For $n, k \geq 1$, RT^n_k is the following problem:

- instances are all colorings $c : [\omega]^n \to k$;
- solutions are all infinite sets H homogeneous for c (i.e., c constant on $[H]^n$).

Theorem (Dorais, Dzhafarov, Hirst, Mileti, and Shafer).

If $n \geq 1$ and $k > j$, then $\mathsf{RT}^n_k \not\leq_{\mathrm{sW}} \mathsf{RT}^n_j$.

Theorem (Hirschfeld and Jockusch; Brattka and Rakotoniania).

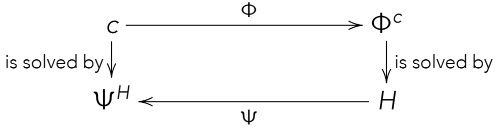
If $n \geq 1$ and $k > j$, then $\mathsf{RT}^n_k \not\leq_{\mathrm{W}} \mathsf{RT}^n_j$.

Theorem (Dzhafarov). If $k > j$, then $\mathsf{RT}^1_k \not\leq_{\mathrm{sc}} \mathsf{RT}^1_j$.

Theorem (Patey). If $n \geq 2$ and $k > j$, then $\mathsf{RT}^n_k \not\leq_{\mathrm{c}} \mathsf{RT}^n_j$.

Example: $\mathsf{RT}^1_3 \not\leq_{\mathrm{sW}} \mathsf{RT}^1_2$

Fix Turing functionals Φ and Ψ.



We must construct:

- a coloring $c : \omega \to 3$,
- an infinite homogeneous set H for $\Phi^c : \omega \to 2$ such that either

  (↑) there are only finitely many x such that $\Psi^H(x) \downarrow = 1$, or

  (↓) there exists $x < y$ such that $\Psi^H(x) \downarrow = \Psi^H(y) \downarrow 1$ and $c(x) \neq c(y)$.

Example: $\mathsf{RT}^1_3 \not\leq_{\mathsf{sW}} \mathsf{RT}^1_2$

Theorem (Seetapun). One of the following is true:

- there is an infinite set I such that no $F \subseteq I$ satisfies $(\exists x)[\Psi^F(x) \downarrow= 1]$,
- there are finite sets $F_0, \ldots, F_n$ such that $\Psi^{F_i}(x) \downarrow= 1$ for some x, and for every every $d : \omega \to 2$ there is an i such that F_i is homogeneous for d.

In the first case, let c be arbitrary. Take any homogeneous set $H \subseteq I$ for Φ^c. (↑)

In the second, find all the F_i, and for each, fix x with $\Psi^H(x) \downarrow= 1$:

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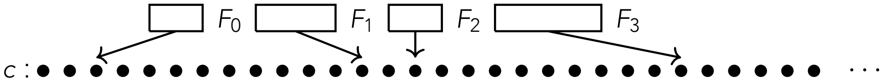

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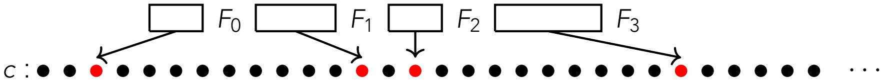

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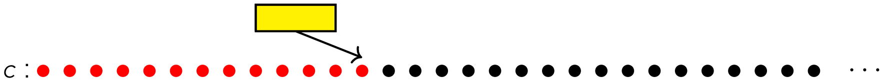

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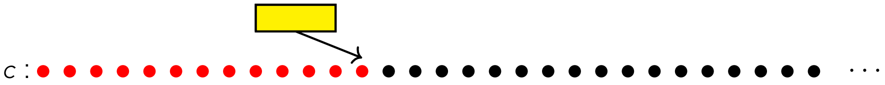

Repeat for each of the other colors allowable for c. Obtain ($\downarrow$).

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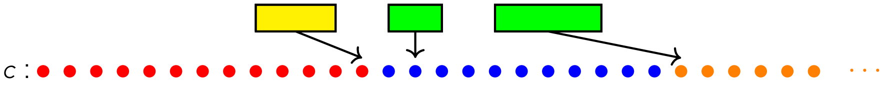


Repeat for each of the other colors allowable for c. Obtain (↓).

A more complicated problem: SRT^2_2

A coloring $c : [\omega]^2 \to 2$ is stable if for every x, $\lim_y c(x, y)$ exists.

- The value of $\lim_y c(x, y)$ is the limit color of x.
- The least n s.t. $(\forall z > n)[c(x, z) = \lim_y c(x, y)]$ is the stabilization point of x.

Stable Ramsey's theorem. SRT^2_2 is the restriction of RT^2_2 to stable colorings.

Combinatorially, solutions to SRT^2_2 have global structure and local structure.

- The global structure ensures all elements have the same limit color.
- The local structure ensures all pairs of elements have the same color.

Typically: apply RT^1_2 to get global structure, then thin to get local structure.

Seetapun's combinatorial trick only works for global structure, not local.

Example: $\mathsf{RT}^1_3 \not\leq_{\mathbf{sc}} \mathsf{SRT}^2_2$

We must construct:

- a coloring $c : \omega \to 3$,
- for each Φ, if $\Phi^c : [\omega]^2 \to 2$ is stable, an infinite homogeneous set H such that for every Ψ, either

  ($\uparrow$) there are only finitely many x such that $\Psi^H(x) \downarrow= 1$, or

  ($\downarrow$) there exists $x < y$ such that $\Psi^H(x) \downarrow= \Psi^H(y) \downarrow 1$ and $c(x) \neq c(y)$.

In the ($\downarrow$) case, we can no longer postpone defining c until we find diagonalization opportunities.

This causes a serious tension: defining c to make some finite set F homogeneous (local structure) may make Ψ^F homogeneous for c.

The tree labeling construction

Let T be the set of all increasing $\alpha \in \omega^{<\omega}$ such that
for all finite $F \subseteq \operatorname{ran}(\alpha \upharpoonright |\alpha| - 1)$, it is not the case that $\Psi^F(x) \downarrow = 1$ for some x.

Case 1. If T is not well-founded, let I be a path through T.
Now commit to building H inside $\operatorname{ran}(I)$, and obtain $(\uparrow)$.

The tree labeling construction

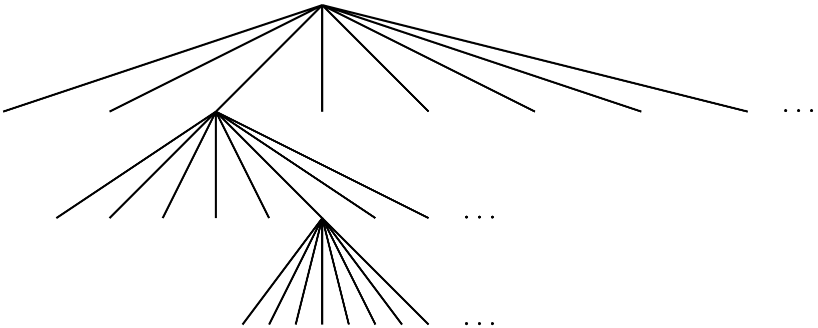
Let T be the set of all increasing $\alpha \in \omega^{<\omega}$ such that
for all finite $F \subseteq \mathrm{ran}(\alpha \restriction |\alpha| - 1)$, it is not the case that $\Psi^F(x) \downarrow = 1$ for some x.

Case 2. Suppose T is well-founded.

We label each $\alpha \in T$, either by some $x \in \omega$ or by the symbol ∞.

- If α is a leaf, its label is the least x such that $\Psi^F(x) = 1$ for some $F \subseteq \mathrm{ran}(\alpha)$.
- If α is not a leaf and infinitely many αi have the same label $x \in \omega$, label α by the least such x.
- If α is not a leaf, and no $x \in \omega$ appears as the label of infinitelay many αi, label α by ∞.

The tree labeling construction



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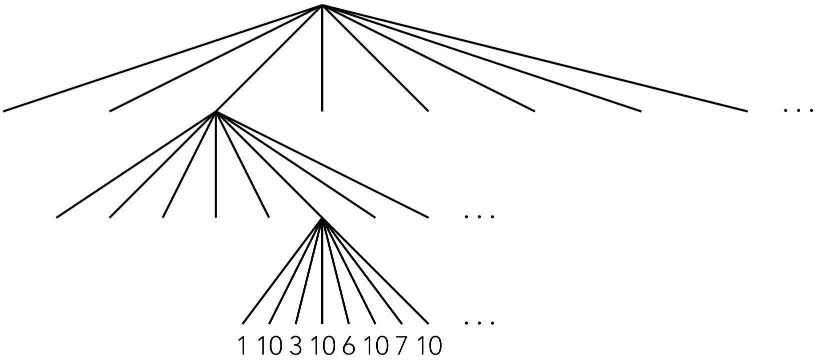

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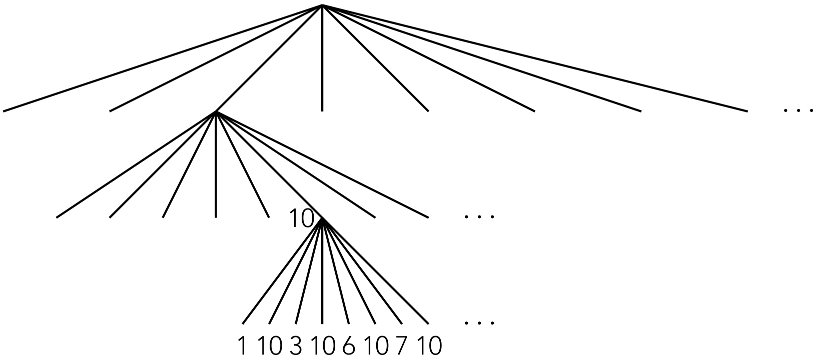

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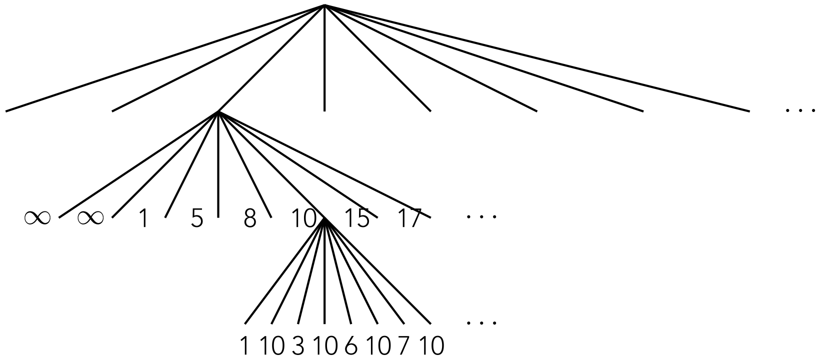

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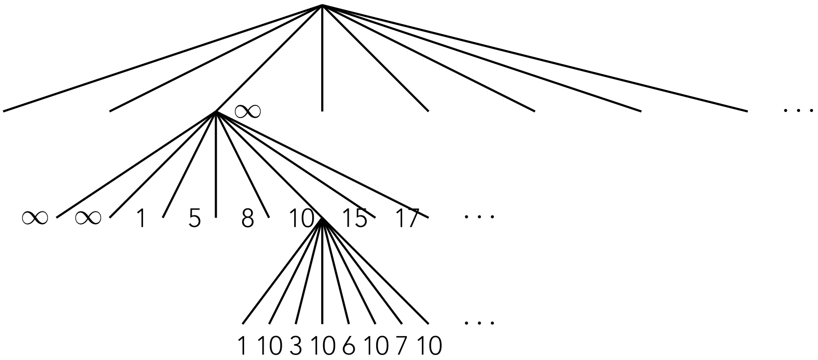

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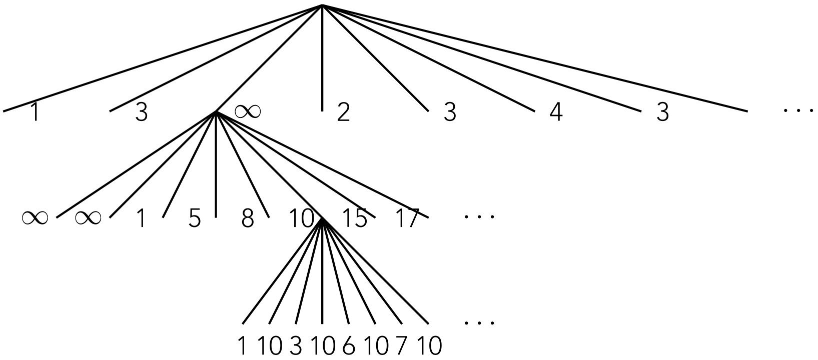

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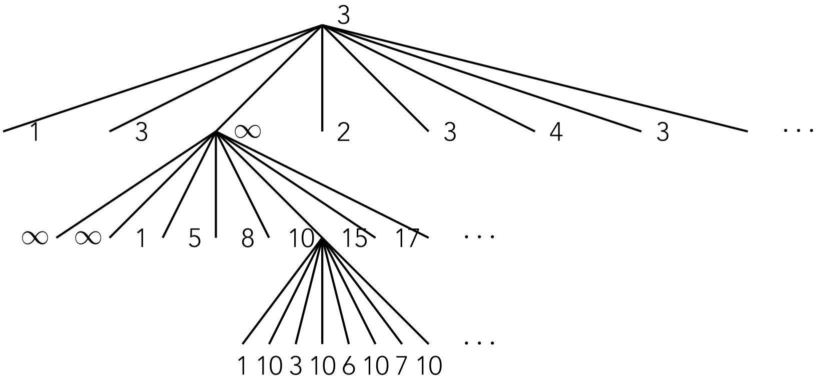

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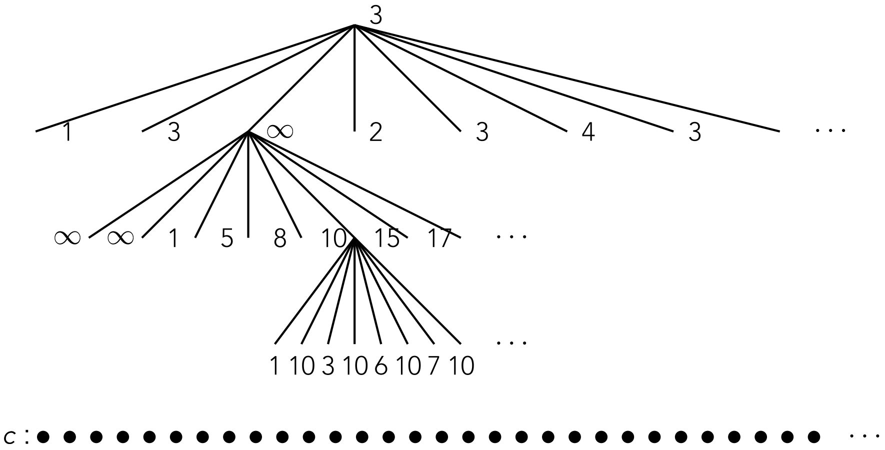

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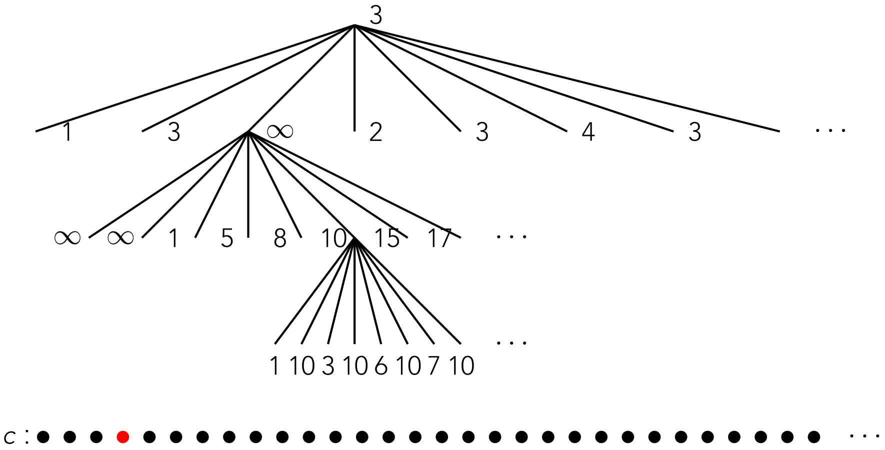

Make progress towards (↓), as before.

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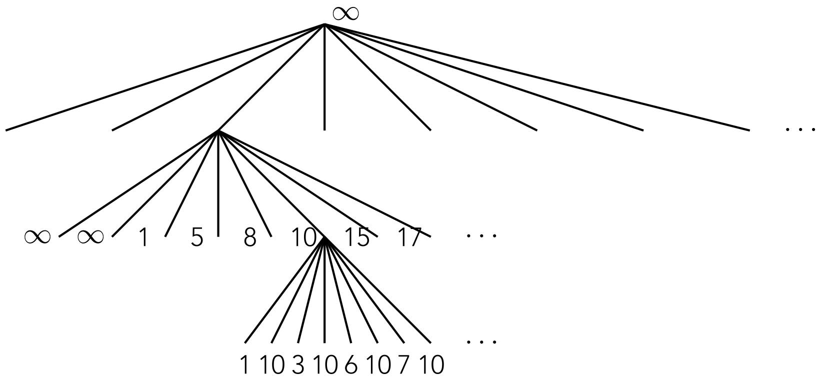

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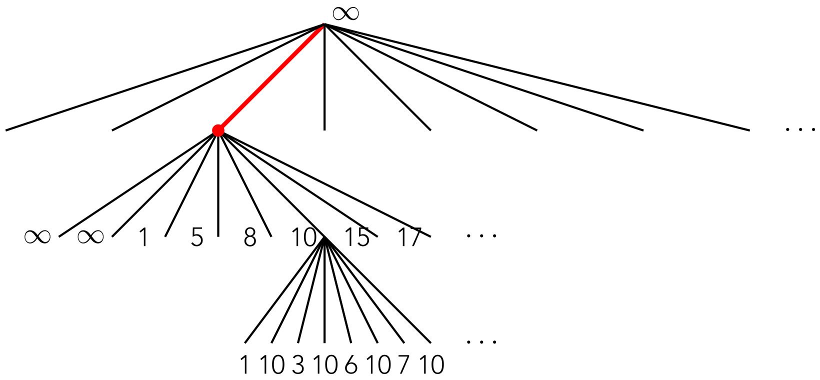

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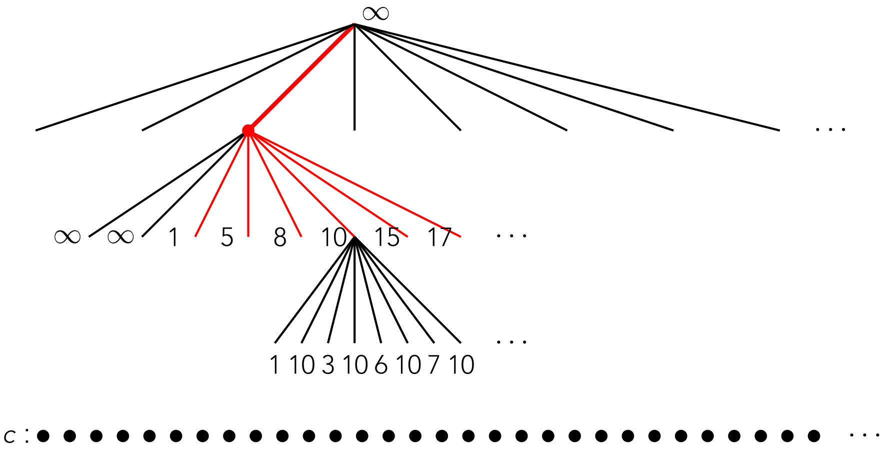

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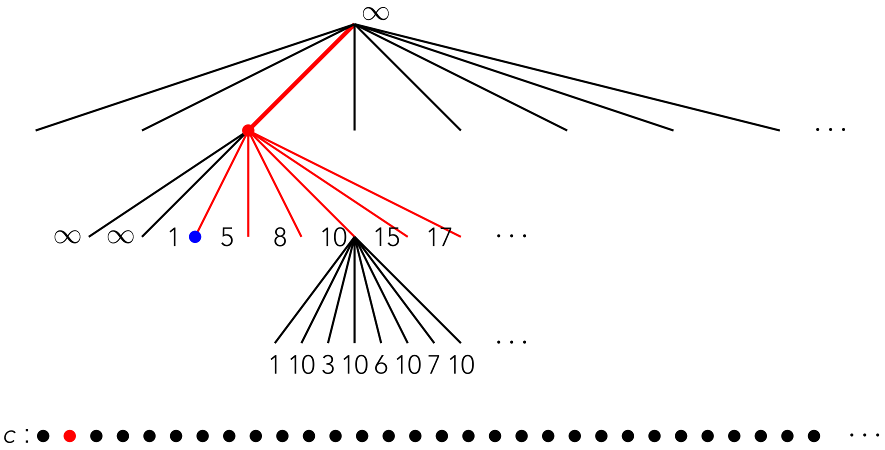

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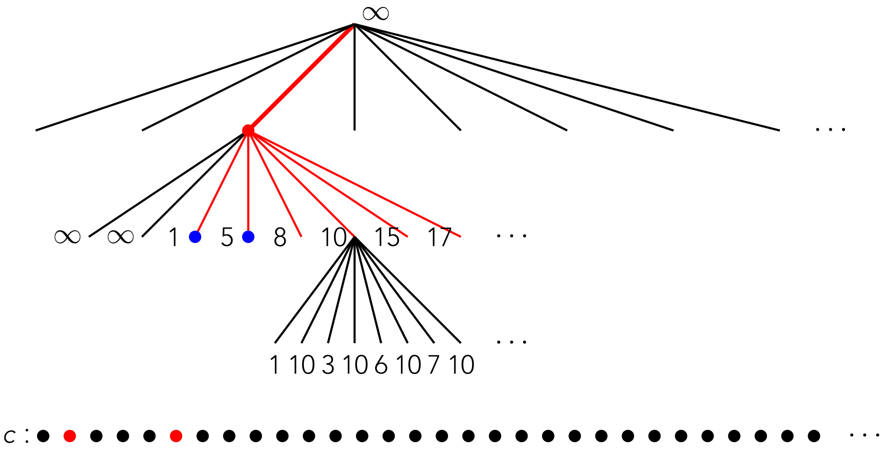

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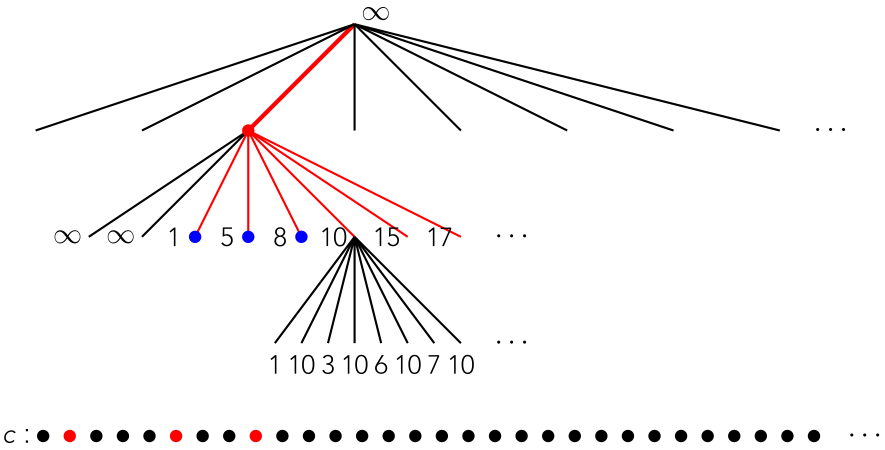

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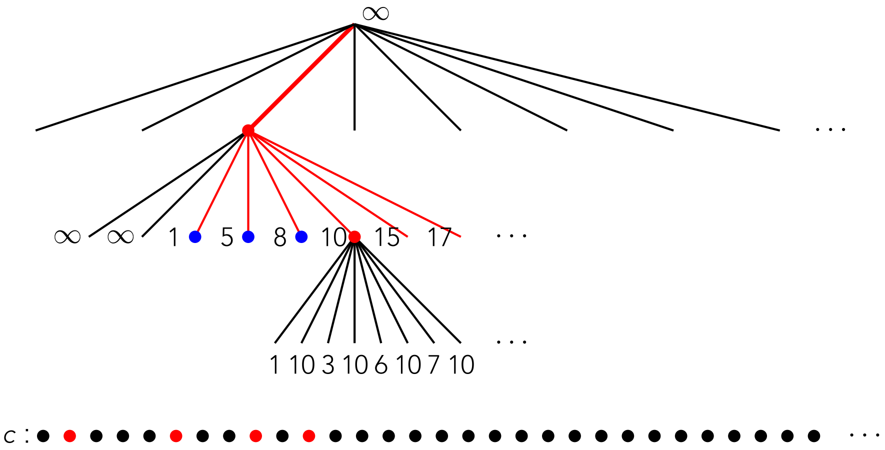

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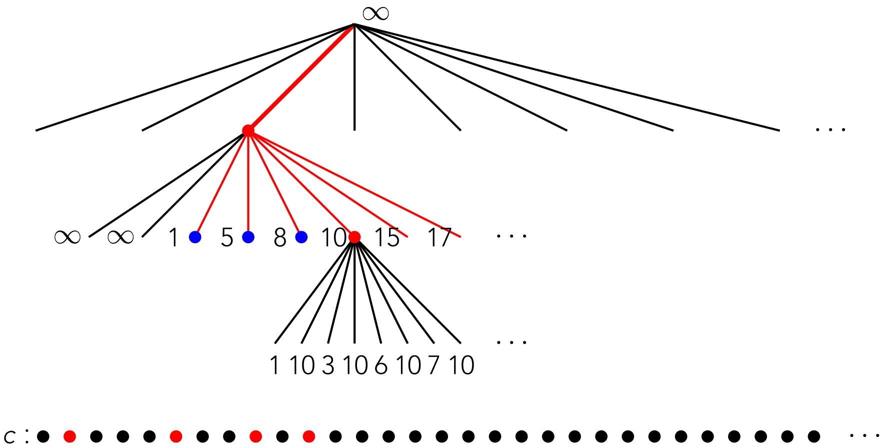
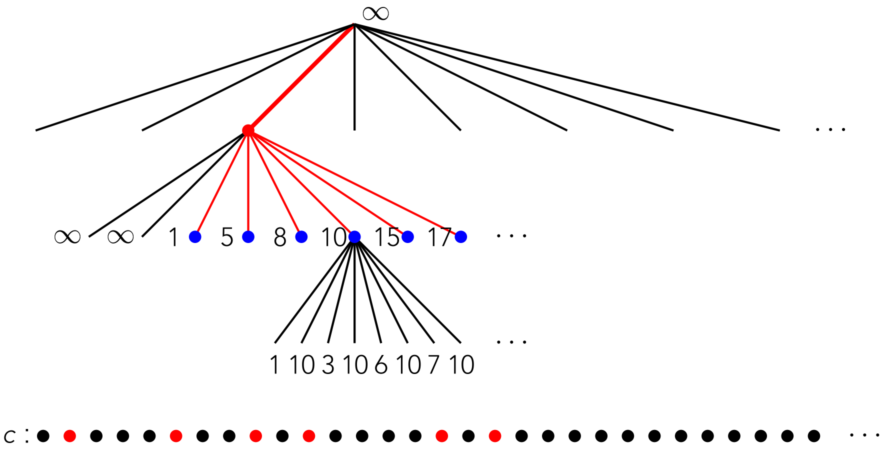


Make progress towards (↓).

The tree labeling construction



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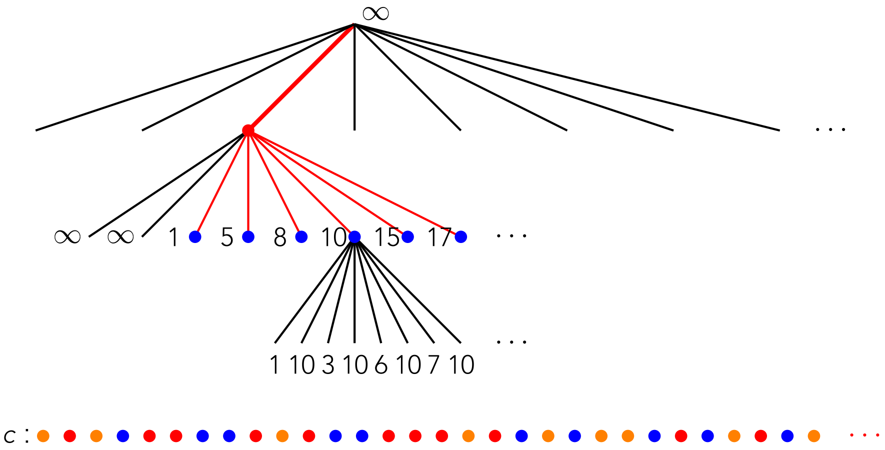

Make c generic relative to the tree. Obtain (↓).

Old applications

Theorem (Dzhafarov). $\mathsf{RT}^1_2 \not\leq_{\mathrm{sc}} \mathsf{SRT}^2_3$.

Theorem (Dzhafarov, Patey, Solomon, Westrick). If $k > j$ then $\mathsf{RT}^1_k \not\leq_{\mathrm{sc}} \mathsf{SRT}^2_j$.

Let P and Q be problems.

P is strongly omnisciently computably reducible to Q, written $\mathsf{P} \leq_{\mathrm{soc}} \mathsf{Q}$, if

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Theorem (Dzhafarov, Patey, Solomon, Westrick). If $k > j$ then $\mathsf{RT}^1_k \not\leq_{\mathrm{soc}} \mathsf{SRT}^2_j$.

Same argument, but now force over a ctbl model of ZFC, apply absoluteness.

New applications

Polarized Ramsey's thoerem (Erdős and Rado).

Fix a coloring $c : [\omega]^2 \to k$. A pair (H_0, H_1) of infinite sets is

- *p*-homogeneous if c is constant on $H_0 \times H_1 \cup H_1 \times H_0$.
- increasing *p*-homogeneous if c is constant on $H_0 \times H_1$.

Analogues of RT^2_k and SRT^2_k: denoted PT^2_k and SPT^2_k, and IPT^2_k and SIPT^2_k.

Theorem (Dzhafarov and Hirst). $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{PT}^2_k \to \mathsf{IPT}^2_k \to \mathsf{SRT}^2_k$.

Theorem (Chong, Lempp, and Yang). $\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \leftrightarrow \mathsf{SPT}^2_k \leftrightarrow \mathsf{SIPT}^2_k$.

Theorem (David Nichols 2019). $\mathsf{SRT}^2_k \not\leq_{sc} \mathsf{SPT}^2_k \not\leq_{sc} \mathsf{SIPT}^2_k$.

New applications

Chain/antichain principle (Dilworth; Hirschfeldt and Jockusch).

CAC: Every infinite partial order has an infinite chain or antichain.

SCAC: Every partial order of type $\omega + \omega^*$ has an infinite chain or antichain.

A partial order $\leq_P$ of ω is ordered if $x \leq_P y \to x \leq y$.

$\mathsf{CAC}^{\mathsf{ord}}$ and $\mathsf{SCAC}^{\mathsf{ord}}$: restrictions of CAC and SCAC to ordered partial orders.

Theorem (Towsner). $\mathsf{RCA}_0 \vdash \mathsf{CAC} \leftrightarrow \mathsf{CAC}^{\mathsf{ord}}$ and $\mathsf{RCA}_0 \vdash \mathsf{SCAC} \leftrightarrow \mathsf{SCAC}^{\mathsf{ord}}$.

Theorem (Noah Hughes 2021).

- $\mathsf{CAC} \not\leq_{\mathrm{c}} \mathsf{CAC}^{\mathsf{ord}}$.
- $\mathsf{SCAC} \equiv_{\mathrm{c}} \mathsf{SCAC}^{\mathsf{ord}}$, but $\mathsf{SCAC} \not\leq_{\mathrm{W}} \mathsf{SCAC}^{\mathsf{ord}}$ and $\mathsf{SCAC} \not\leq_{\mathrm{sc}} \mathsf{SCAC}^{\mathsf{ord}}$.

Thank you for your attention!