The Topological μ-Calculus

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Modal logic and Kripke semantics

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$$p \mid \bot \mid \rightarrow \mid \Diamond\varphi$$

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- $\varphi \vee \psi := \neg\varphi \rightarrow \psi$
- $\Box\varphi := \neg\Diamond\neg\varphi$

Kripke semantics:

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- $[\![\varphi \to \psi]\!] = (W \setminus [\![\varphi]\!]) \cup [\![\psi]\!]$
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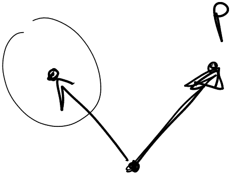
$w \in [\![\Diamond\varphi]\!] \iff \exists v\,(wRv \;\&\; v \in [\![\varphi]\!])$

Models: Triples $\mathcal{M} = (W, R, [\![\cdot]\!])$

A Kripke model



$\Diamond p$

$\neg \Box p$

Axiomatization for modal logic

The basic modal logic is called K.

Axioms

- All classical tautologies
- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ K

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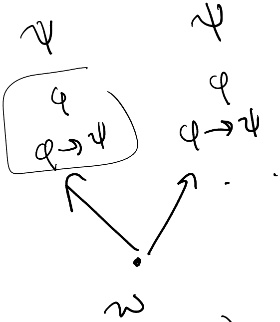
Theorem

A formula is valid over the class of Kripke models iff it is derivable in K.

Proof (Soundness)

$$\underline{\Box(\varphi \to \psi)} \to (\underline{\Box\varphi} \to \Box\psi)$$



$$\frac{\begin{array}{c}\Box(\varphi \to \psi) \\ \Box\varphi\end{array}}{\Box\psi}$$

Proof (Completeness)

Canonical model: $\mathcal{M}_c = (\underline{W_c}, \underline{R_c}, \underline{[\![\cdot]\!]_c})$

W_c = set of "theories" := maximal consistent sets of formulas

$T R_c S : \forall \varphi (\underline{\varphi \in S} \Rightarrow \underline{\Diamond \varphi \in T})$

$[\![p]\!]_c = \{ T \in W_c : p \in T \}$

Proof (Completeness)

Lemma (Truth lemma)

If $T \in W_c$ and φ is any formula, $T \in [\![\varphi]\!]_c$ iff $\varphi \in T$.

φ is consistent $\Rightarrow \exists T \; \varphi \in T \in W_c$

$T \in [\![\varphi]\!]_c$

$\Rightarrow (\mathcal{M}_c \models \varphi \Rightarrow K \vdash \varphi)$

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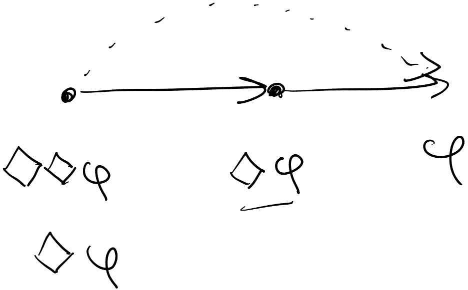
Definition

An extension Λ of K (also called a **normal logic**) is **canonical** if its canonical model is based on a **Λ-frame**.

Example: K4 := K + 4

$$\Box\varphi \to \Box\Box\varphi$$

$$\equiv \quad \Diamond\Diamond\varphi \to \Diamond\varphi$$



The μ-calculus

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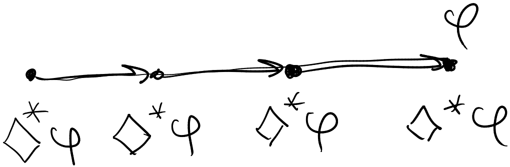
Language $\mathcal{L}_\mu$:

Add expressions $\mu p.\varphi(p)$ to the modal language, where p appears only **positively** in φ.

- $[\![\mu p.\varphi(p)]\!]$ is the **least fixed point** of $X \mapsto [\![\varphi(X)]\!]$.
- $\nu p.\varphi(p) := \neg\mu p.\neg\varphi(\neg p)$ is the **greatest fixed point** of $X \mapsto [\![\varphi(X)]\!]$.

Example: Transitive closure

Define $\Diamond^* \varphi := \mu p.(\varphi \vee \Diamond p)$.



Least fixed point of monotone operators

$F\colon 2^W \to 2^W$ is **monotone** if whenever $A \subseteq B \subseteq W$, it follows that $F(A) \subseteq F(B)$.

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Theorem

Every monotone operator has a least fixed point.

$$F^\alpha: \quad \begin{cases} F^0(A) = A \\ F^{\alpha+1}(A) = F(F^\alpha(A)) \\ F^\lambda(A) = \bigcup_{\alpha<\lambda} F^\alpha(A) \end{cases}$$

$$F(F^{\alpha*}(A)) = F^{\alpha*}(A)$$

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Theorem
Every monotone operator has a least fixed point.

Lemma
If p appears positively in $\varphi(p)$, then $X \mapsto [\![\varphi(X)]\!]$ is a monotone operator.

(Topological) closure semantics of modal logic

If $\mathcal{X} = (X, \mathcal{T})$ is a topological space, we may also define

$$\llbracket \Diamond\varphi \rrbracket := \mathbf{c}\,\llbracket \varphi \rrbracket .$$

$$\llbracket \bot \rrbracket = \emptyset$$

$$\llbracket \varphi \to \psi \rrbracket = (X \setminus \llbracket \varphi \rrbracket) \cup \llbracket \psi \rrbracket$$

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closure

Recall that $\Box := \neg\Diamond\neg$. Then,

$$[\![\varphi]\!] = \boldsymbol{i}\,[\![\varphi]\!]\,.$$

interior

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[Tarski ~ 1940]

Theorem

The logic

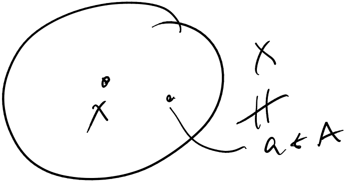
$$\mathsf{S4} := \mathsf{K} + 4 + T$$

is sound and complete for the class of **closure spaces** *(topological spaces equipped with the closure operator).*

Cantor derivative semantics

If X is a topological space and $A \subseteq X$, define the **Cantor derivative** or **set of limit points of** A by

$$dA = \{x \in X : x \in c(A \setminus \{x\})\}.$$



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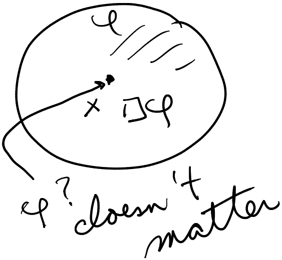
Weak transitivity axiom: $\varphi \wedge \Box\varphi \to \Box\Box\varphi$.



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Theorem

The logic

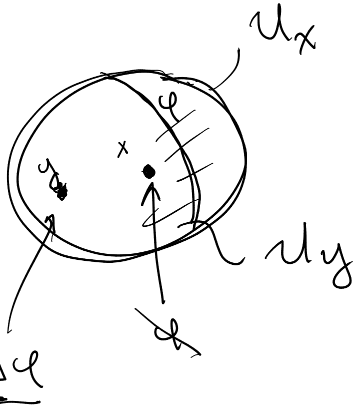
$$\mathrm{wK4} := \mathrm{K} + w4$$

is sound and complete for the class of topological spaces.

Soundness of *w*4

$$\varphi \wedge \Box\varphi \to \Box\Box\varphi$$

$$\Diamond\Diamond\varphi \to \Diamond\varphi \vee \varphi$$



Kripke semantics of $wK4$

A relation $\sqsubset \subseteq W \times W$ is **weakly transitive** if $T \sqsubset S \sqsubset U$ implies that $T \sqsubseteq U$.

$(T \sqsubset U) \vee (T = U)$

Kripke semantics of $wK4$

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Theorem

The logic wK4 *is sound and complete for the class of weakly transitive frames. Moreover,* wK4 *is canonical.*

Unifying Kripke and topological semantics

Definition

A **derivative space** is a pair (X, d) where X is a set and $d\colon 2^X \to 2^X$ satisfies

- $d\varnothing = \varnothing$
- $d(A \cup B) = dA \cup dB$
- $ddA \subseteq dA \cup A$

$A \subseteq B$ $\qquad B = A \cup B$

$d(B) = d(A) \cup d(B) \supseteq d(A)$

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Examples:

- If X is a topological space and d its Cantor derivative, (X, d) is a derivative space.

Unifying Kripke and topological semantics

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Also works for $c =$ closure

Examples:

- If X is a topological space and d its Cantor derivative, (X, d) is a derivative space.
- If $(W, \sqsubset)$ is a wK4 frame, define $d_\sqsubset A := \sqsubset^{-1}(A)$. Then, $(W, d_\sqsubset)$ is a derivative space.

The derivational μ-calculus

If $\mathcal{X} = (X, d)$ is a derivative space, a valuation $[\![\cdot]\!]$ on $\mathcal{X}$ is defined by setting $[\![\Diamond\varphi]\!] := d\,[\![\varphi]\!]$.

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Fact: If p is positive on $\varphi(p)$, then $A \mapsto [\![\varphi(A)]\!]$ is a monotone operator.

Hence the μ-calculus extends to derivative spaces by letting $[\![\mu p.\varphi(p)]\!]$ be the least fixed point of $A \mapsto [\![\varphi(A)]\!]$.

The tangled derivative

Define

$$\Diamond^{\infty}\{\varphi_1, \ldots, \varphi_n\} := \nu p. \bigwedge \Diamond(p \wedge \varphi_i).$$

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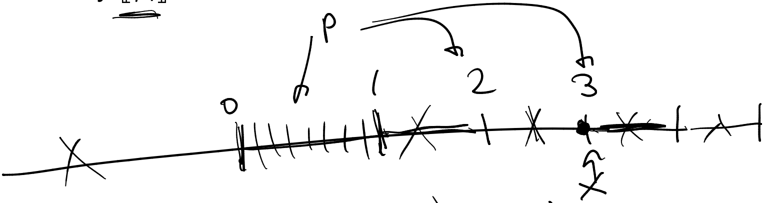
One can check that $[\![\Diamond^\infty\{\varphi_1,\ldots,\varphi_n\}]\!]$ is the largest subspace in which every $\underline{[\![\varphi_i]\!]}$ is dense.



$$[\![p]\!] = \left([0,1] \cap \underline{\mathbb{Q}}\right) \cup \mathbb{N}$$

$$\underline{[\![\Diamond^\infty\{p, \neg p\}]\!]} = \underline{[0,1]}$$

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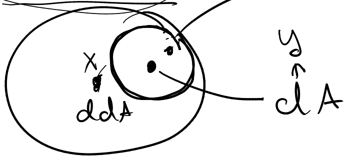
Theorem (Dawar and Otto 2009)

Every formula of the μ-calculus is equivalent to a formula in $\mathcal{L}_{\Diamond\Diamond^{\infty}}$ over the class of T_D spaces: derivative spaces validating

$$ddA \subseteq dA.$$

$z \in A$



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Theorem (Baltag, Bezhanishvili, F-D)

The language $\mathcal{L}_{\Diamond\Diamond^{\infty}}$ is not expressively complete over T_0 spaces.

Axiomatizing the μ-calculus

If Λ is a normal logic, define μ-Λ by adding

- $\varphi(p) \to \varphi(\mu p.\varphi(p))$
- $\dfrac{\varphi(\psi) \to \psi}{\mu p.\varphi(p) \to \psi}$

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Theorem (Walukiewicz, 2000)

μ-K *is sound and complete for the class of Kripke frames.*

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Theorem (Walukiewicz, 2000)
μ-K *is sound and complete for the class of Kripke frames.*

Theorem (Goldblatt, Hodkinson 2018)
μ-S4 *is sound and complete for the class of finite closure spaces, and for any dense-in-itself metric space.*

μ-K4

The final submodel

Let $\mathcal{M}_c = (W_c, \sqsubset_c, [\![\cdot]\!]_c)$ be the canonical model for μ-wK4. This model is based on a wK4 frame, since wK4 is canonical.

The final submodel

Let $\mathcal{M}_c = (W_c, \sqsubset_c, [\![\cdot]\!]_c)$ be the canonical model for μ-wK4. This model is based on a wK4 frame, since wK4 is canonical.

But: The **truth lemma** fails for $\mathcal{M}_c$ over the μ-calculus: it may be that $\mu p.\varphi(p) \in T$ but $T \notin [\![\mu p.\varphi(p)]\!]_c$

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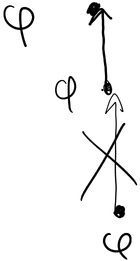
Say that T is **φ-final** if $\varphi \in T$ and whenever $S \sqsupseteq T$ and $\varphi \in S$, it follows that $T \sqsupseteq S$.



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Say that T is φ**-final** if $\varphi \in T$ and whenever $S \sqsupseteq T$ and $\varphi \in S$, it follows that $T \sqsupseteq S$.

Say that T is Σ**-final** if T is φ-final for some $\varphi \in \Sigma$.

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Final submodel: $\mathcal{M}_c^\Sigma = (W_c^\Sigma, \sqsubset_c^\Sigma, [\![\cdot]\!]_c^\Sigma)$ is the submodel of Σ-final theories.

Truth lemma for the final submodel

Lemma (Σ-Final Truth Lemma)

Let Σ be finite and closed under subformulas (and a few other operations, such as single negation). Let

$$\mathcal{M}_c^\Sigma = (W_c^\Sigma, \sqsubset_c^\Sigma, [\![\cdot]\!]_c^\Sigma)$$

be the canonical wK4 *model.*

Then, for $T \in W_c^\Sigma$ and $\varphi \in \Sigma$, $T \in [\![\varphi]\!]$ iff $\varphi \in \overset{T}{\cancel{T}}$.

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Then, for $T \in W_c^\Sigma$ and $\varphi \in \Sigma$, $T \in [\![\varphi]\!]$ iff $\varphi \in W$.

Theorem (Baltag, Bezhanishvili, F-D)

*The logic μ-*wK4 *is sound and complete for the class of* wK4 *frames.*

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Fact: Any fixed point stabilizes after finitely many iterations on a shallow model. This allows us to prove the truth lemma.

$$\cancel{F^{\alpha *}(\emptyset)} \qquad F^{n*}(\emptyset)$$

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Fact: If Σ is finite, $\mathcal{M}_c^\Sigma$ is shallow.

$$\cancel{T_0} \subsetneq T_1 \subsetneq T_2 \subsetneq \cdots$$

$$\underline{\varphi_0} \qquad \varphi_1 \qquad \underline{\underline{\varphi_2}}$$

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Fact: If Σ is finite, $\mathcal{M}_c^\Sigma$ is shallow.

Fact: Shallow frames are bisimilar to finite frames, so we further obtain the following:

Theorem (Baltag, Bezhanishvili, F-D)

The logic μ-wK4 *has the finite model property, hence is decidable.*

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Cofinal subframe logics

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This includes μ-S4, μ-K4, and many other examples.

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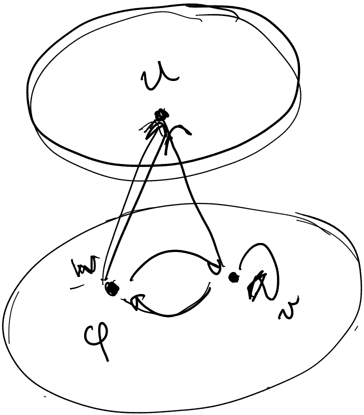
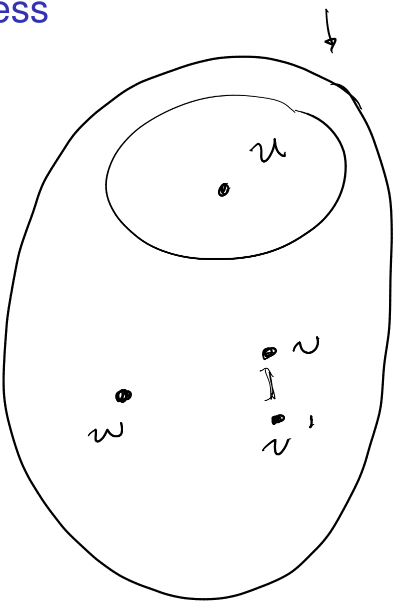
4. *The logic* μ-wK4T$_0$ *(which I won't define here) is sound and complete for the class of* T_D *spaces with topological closure.*

Proof of topological completeness

φ consistent





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S4 $\rightarrow \{A_1, \ldots, A_n\}^*$ $(A_i \subseteq X)$

$\bigcup \{ S \subseteq X \mid \forall i \, (S \subseteq c(S \cap A_i)) \}$

$\rightarrow$ IPC $\hookrightarrow$ S4

Thank you!

(X, τ, R)

f