TECHNISCHE UNIVERSITÄT WIEN
Vienna | Austria

Learning Structures

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TU Wien

joint with Nikolay Bazhenov and Luca San Mauro

Leeds-Ghent Virtual Logic Seminar

January 28, 2021

Learning structures

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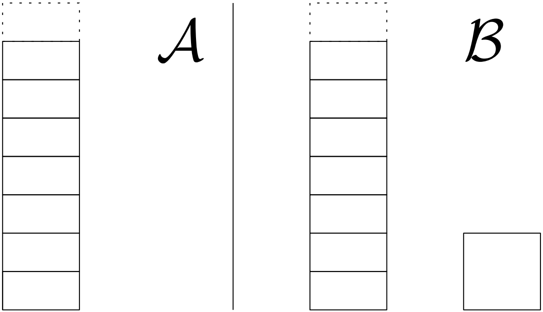
Question

Can we, after finitely many steps, identify the structure (up to an isomorphism or other equivalence relations)?

Example 1



Graphics by Luca San Mauro

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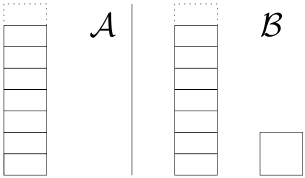

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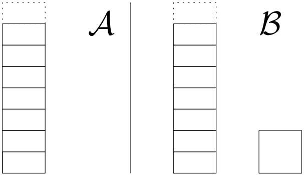

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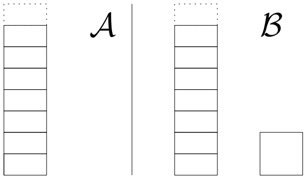

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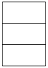

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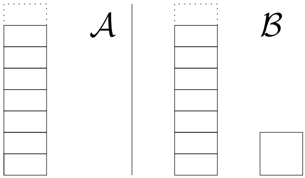

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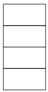

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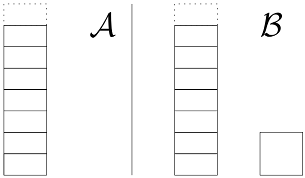

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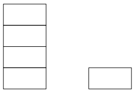

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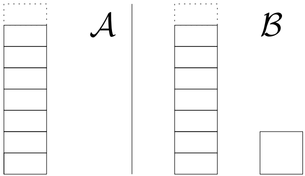

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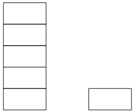

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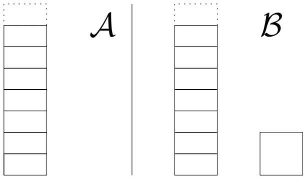

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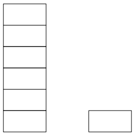

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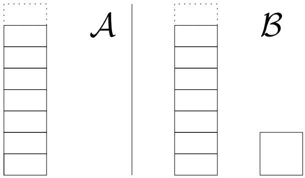

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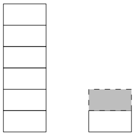

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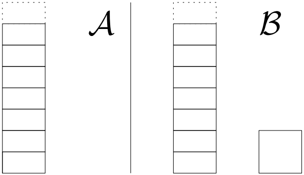

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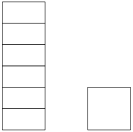

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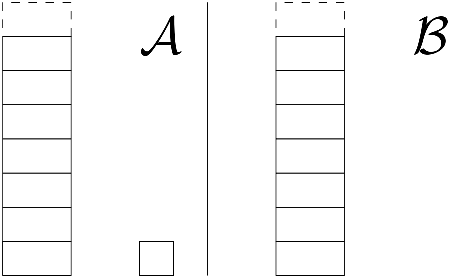

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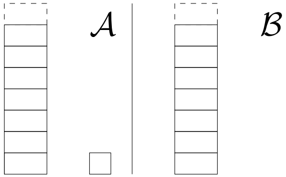

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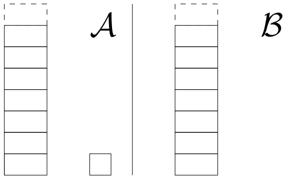

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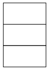

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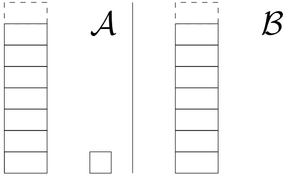

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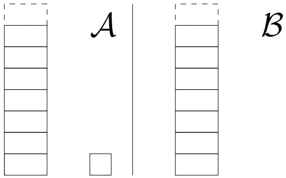

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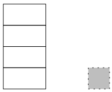

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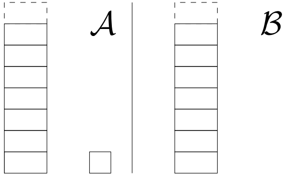

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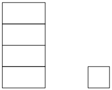

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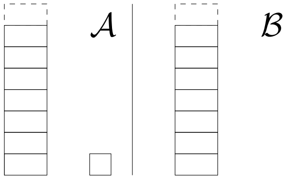

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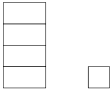

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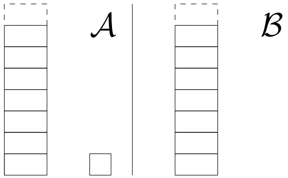

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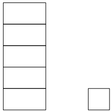

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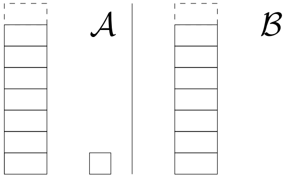

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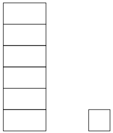

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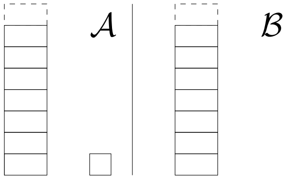

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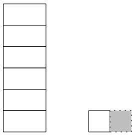

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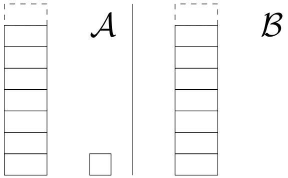

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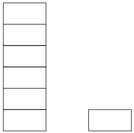

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- Another approach: only the positive information about the structure is revealed (motivated by c.e. structures).
- What does it mean "to classify", or "to identify" the structure?
- To formalize these and other issues, we use the ideas from computable structure theory and computational learning theory.

Computational Learning Theory

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Most work in CLT concerns

- either learning of total functions (where the order in which the data is received matters)
- or learning of formal languages (where the order does not matter)

These paradigms model the data to be learned as an unstructured flow — but what if one deals with data having some structural content?

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Our goal is: to combine the technology of CLT with notions coming from computable structure theory to develop a general framework for learning the isomorphism type of algebraic structures.

Computable Structures

To learn the isomorphism type of a given structure, one should be able to name such an isomorphism type. This is why we focus on the learning of (copies of) computable structures: the name of the isomorphism type of a computable structure $\mathcal{A}$ will be just the index, w.r.t. to some given effective enumeration, of a Turing machine that computes the atomic diagram of $\mathcal{A}$.

Learning should be independent from the way in which data is presented. So, a successful learning procedure should work for all isomorphic copies of a given structure.

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- A *presentation* of a countable algebraic structure is an arbitrary isomorphic copy $\mathcal{M}' \cong \mathcal{M}$ with the universe a subset of ω. We call a structure $\mathcal{M}$ *computably presentable* if it has a presentation $\mathcal{M}'$ which is computable.

Computable structures

- Any computable structure $\mathcal{A}$ in a relational signature can be presented as an increasing union of its finite substructures

$$\mathcal{A}^0 \subseteq \mathcal{A}^1 \subseteq \ldots \subseteq \mathcal{A}^i \subseteq \ldots,$$

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- We assume that every considered class of L-structures is closed under isomorphisms.

Formal Example I

Consider a family $\mathfrak{C}$ of two infinite, undirected graphs:

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- **The learner**: a function (or an algorithm).
- **The prior knowledge**: the target graph is isomorphic to either G_1 or G_2.
- **The criterion of success**: a learner that, receiving larger and larger pieces of any graph G in $\mathfrak{C}^*$, eventually stabilizes to a correct guess about whether G is isomorphic to G_1 or G_2.

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- for each $i \geq 1$, the graph G_i contains infinitely many $(i+1)$-cycles.
- the learning domain is the family $\mathfrak{D}^*$ of *all* presentations of the graphs in $\mathfrak{D}$;
- each informant I provides both positive and negative information about any given graph in $\mathfrak{D}^*$;

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- the learning domain is the family $\mathfrak{D}^*$ of *all* presentations of the graphs in $\mathfrak{D}$;
- each informant I provides both positive and negative information about any given graph in $\mathfrak{D}^*$;
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The only technical problem is how to specify the hypothesis space or, in other words:

How does one formally define the set of possible conjectures?

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First Solution:
For $m \in \omega$, the conjecture "m" means "$H \cong G_{m+1}$."

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- This solution is similar to the so-called *exact learning*, considered in the setting of c.e. languages, where one assumes that the hypothesis space of the problem is precisely the class being learned with the corresponding indexing.
- Drawback: it can be computationally very hard to enumerate certain familiar families of computable structures, up to isomorphism.
- Goncharov and Knight: for the classes of computable Boolean algebras, linear orders, and abelian p-groups one cannot even hyperarithmetically enumerate their isomorphism types.

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Fix a uniformly computable sequence $(\mathcal{M}_e)_{e\in\omega}$ of *all* computable undirected graphs. The conjecture "m" means "$H \cong \mathcal{M}_m$."

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This solution is similar to the so-called *class-comprising learning*, where one assumes that the hypothesis space of the problem should only contain the class being learned.

General framework: consider an arbitrary superclass $\mathfrak{K} \supseteq \mathfrak{D}$ which is *uniformly enumerable*, i.e. there is a uniformly computable sequence of structures $(\mathcal{N}_e)_{e\in\omega}$ such that:

1. Any structure from $\mathfrak{K}$ is isomorphic to some $\mathcal{N}_e$.
2. For every e, $\mathcal{N}_e$ belongs to $\mathfrak{K}$.

Then for a number $e \in \omega$, the conjecture "e" is interpreted as "the input structure is isomorphic to $\mathcal{N}_e$."

Our framework

Let L be a relational signature.

- Let $\mathfrak{K}_0$ be a class of L-structures. An *effective enumeration* of the class $\mathfrak{K}_0$ is a function $\nu\colon \omega \to \mathfrak{K}_0$ with the following properties:

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  We say that $\mathfrak{K}$ is $\mathbf{InfEx}_{\cong}[\nu]$-*learnable* if there is a learner M with the following property:
  If I is an informant for a structure $\mathcal{A} \in \mathfrak{K}$, then there are e and s_0 such that $\nu(e) \cong \mathcal{A}$ and $M(I[s]) = e$ for all $s \geq s_0$. In other words, in the limit, the learner M learns all isomorphism types from $\mathfrak{K}$.

Infinitary formulas

We are able to fully characterize which families of structures are learnable. To do so, we use the logic $\mathcal{L}_{\omega_1\omega}$, which allows to take conjunctions or disjunctions of infinite sets of formulas.
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(a) $\Sigma_0^{\inf}$ and $\Pi_0^{\inf}$ formulas are quantifier-free first-order L-formulas.

(b) A $\Sigma_\alpha^{\inf}$ formula $\psi(x_0, \ldots, x_m)$ is an countable disjunction

$$\bigvee\!\!\!\bigvee_{i \in I} \exists \bar{y}_i \xi_i(\bar{x}, \bar{y}_i),$$

where each ξ_i is a $\Pi_{\beta_i}^{\inf}$ formula, for some $\beta_i < \alpha$.

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(c) A $\Pi_\alpha^{\inf}$ formula $\psi(\bar{x})$ is a countable conjunction

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The class of X-computable infinitary Σ_α L-formulas:

(a) $\Sigma^c_0(X)$ and $\Pi^c_0(X)$ formulas are quantifier-free first-order L-formulas.

(b) A $\Sigma^c_\alpha(X)$ formula $\psi(x_0, \dots, x_m)$ is an X-computably enumerable (X-c.e.) disjunction

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Today we will only need $\Sigma^c_\alpha(X)$ formulas for $\alpha \leq 2$.

Main Theorem

Suppose that $\mathfrak{K}_0$ is a class of L-structures, and ν is an effective enumeration of the class $\mathfrak{K}_0$.

Theorem

Let $\mathfrak{K} = \{\mathcal{B}_i : i \in \omega\}$ be a family of structures such that $\mathfrak{K} \subseteq \mathfrak{K}_0$, and the structures $\mathcal{B}_i$ are infinite and pairwise non-isomorphic. Then the following conditions are equivalent:

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1. *The class $\mathfrak{K}$ is $\mathbf{InfEx}_{\cong}[\nu]$-learnable;*
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The statement is similar to a result due to Martin and Osherson. Yet, our proof is novel and based on Pullback Theorem in the context of Turing computable embeddings introduced by Knight, S. Miller, and Vanden Boom. This provides us with an upper bound for the Turing complexity of the learners which we apply later.

Bounding the complexity

Corollary

Let $X \subseteq \omega$ be an oracle. Let $\mathfrak{K}_0$ be a class of countably infinite L-structures, and ν be an effective enumeration of $\mathfrak{K}_0$.

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Then the class $\mathfrak{K}$ is $\mathbf{InfEx}_{\cong}[\nu]$-learnable via an X-computable learner.

Application I: distributive lattices

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(a) *The class $\mathfrak{K}_{lat}$ is $\mathbf{InfEx}_{\cong}[\nu]$-learnable. Note that here the complexity of the learner depends only on the complexity of the sequence $(e_i)_{i \in \omega}$ from Corollary.*

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Let $\mathfrak{K}$ be some class of infinite Boolean algebras, and let ν be an effective enumeration of $\mathfrak{K}$. Suppose that $\mathfrak{C}$ is a subclass of $\mathfrak{K}$ such that $\mathfrak{C}$ contains at least two non-isomorphic members. Then the class $\mathfrak{C}$ is not $\mathbf{InfEx}_{\cong}[\nu]$*-learnable.*

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Idea of Proof: Suppose $\mathcal{A} \not\cong \mathcal{B}$ are two BAs from $\mathfrak{C}$. Then

$$\Sigma_2^{\inf}\text{-}Th(\mathcal{A}) \subseteq \Sigma_2^{\inf}\text{-}Th(\mathcal{B}) \text{ or } \Sigma_2^{\inf}\text{-}Th(\mathcal{B}) \subseteq \Sigma_2^{\inf}\text{-}Th(\mathcal{A}).$$

Therefore, the class $\mathfrak{C}$ is not $\mathbf{InfEx}_{\cong}[\nu]$-learnable.

Application IV: linear orders

Proposition

Let $n \geq 2$ be a natural number. Then there is a class of computable infinite linear orders $\mathfrak{C}$ with the following properties:

(a) *$\mathfrak{C}$ contains precisely n isomorphism types.*

(b) *Suppose that $\mathfrak{K}$ is a superclass of $\mathfrak{C}$, and ν is an effective enumeration of $\mathfrak{K}$. Then the class $\mathfrak{C}$ is $\mathbf{InfEx}_{\cong}[\nu]$-learnable.*

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Theorem

Let $\mathfrak{K}$ be some class of infinite linear orders, and let ν be an effective enumeration of $\mathfrak{K}$. Suppose that $\mathfrak{C}$ is a subclass of $\mathfrak{K}$ such that $\mathfrak{C}$ contains infinitely many pairwise non-isomorphic members. Then the class $\mathfrak{C}$ is not $\mathbf{InfEx}_{\cong}[\nu]$-learnable.

Equivalence structures

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Theorem (F., Kötzing, San Mauro, 2018)

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Corollary

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2. *There is $\mathfrak{A} \notin \mathbf{InfEx}_{\cong}[\nu])$ such that, for all $\mathfrak{B} \subseteq \mathfrak{A}$, if $\mathfrak{B}/_{\cong}$ is finite, then $\mathfrak{B} \in \mathbf{InfEx}_{\cong}[\nu]$.*

Equivalence structures

Let $\mathbb{E}$ be the class of equivalence structures, and ν its enumeration.

Theorem (F., Kötzing, San Mauro, 2018)

If $\mathfrak{A}$ is a family of equivalence structures, then

$$\mathfrak{A} \text{ is } \mathbf{InfEx}_{\cong}[\nu]\text{-learnable } \Leftrightarrow \mathfrak{A} \text{ is finitely separable.}$$

Corollary

1. *If $\mathfrak{A} \notin \mathbf{InfEx}_{\cong}[\nu]$ and $\mathfrak{A}/_{\cong}$ is finite, then there exists $(\mathcal{A}, B) \subseteq \mathfrak{A}$ such that $(\mathcal{A}, B) \notin \mathbf{InfEx}_{\cong}[\nu]$.*
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3. $\mathbf{Ex} \mapsto \mathbf{BC}$: in **BC**-learning (short for *behaviourally correct*) the learner is allowed to change its mind infinitely many times as far as almost all its conjectures lie in the same E-class (with E defined as in 2.).
4. Yet another dimension to consider is the complexity of the learner.

Other learnability classes, II

$$
\begin{tikzcd}
\mathbf{InfEx}_{\cong\mathbf{0}} \arrow[r, "\subsetneq"] \arrow[d, "\subsetneq"'] & \mathbf{InfEx}_{\cong} \arrow[d, "\subsetneq"] \\
\mathbf{InfEx}_{\approx\mathbf{0}} \arrow[r, "\subsetneq"] & \mathbf{InfEx}_{\approx}
\end{tikzcd}
$$

Other learnability classes, II

$$\begin{tikzcd}
\mathbf{TxtEx}_{\cong^{\mathbf{0}}} \arrow[rr, "\subsetneq"] \arrow[dd, "\subsetneq"'] & & \mathbf{TxtEx}_{\cong} \arrow[dd, "\subsetneq"] \\
& & \\
\mathbf{TxtEx}_{\approx^{\mathbf{0}}} \arrow[rr, "\subsetneq"] & & \mathbf{TxtEx}_{\approx}
\end{tikzcd}$$

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