Highness in computable structure theory

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Outline

- General concepts
- Randomness
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- Computable structure theory
  - Lowness for isomorphism
  - Highness for isomorphism

Lowness and highness in degree theory

Definition
A set X is low if $X' \equiv_T \emptyset'$ and high if $X' \geq_T (\emptyset')'$.

- X is superlow if $X' \equiv_{tt} \emptyset'$ and superhigh if $X' \equiv_{tt} \emptyset''$.
- X is generalized low if $X' \equiv_T X \oplus \emptyset'$ and generalized high if $X' \equiv_T (X \oplus \emptyset')'$.

A general framework

Let $\mathcal{C}$ be a relativizable class, and let $\mathcal{C}^X$ be the class obtained by using X as an oracle for this class.

- A set X is *low* for $\mathcal{C}$ if $\mathcal{C}^X = \mathcal{C}$.
- A set X is *high* for $\mathcal{C}$ if $\mathcal{C}$ has a maximal element M such that $\mathcal{C}^X = \mathcal{C}^M$.

Degree theory: $\mathcal{C}$ is the Δ^0_2 sets, and $M = \emptyset'$.

Question

Let $\mathcal{N}$ be the class of sets that are high in some context. Is $\mathcal{N}$ robust, either in the sense of having a characterization unrelated to highness or in the sense of being a highness class for multiple notions?

Highness for randomness

Difficulty: There's no maximality concept here.

Definition

If $\mathcal{R}_1 \subseteq \mathcal{R}_2$, then

$$X \in High(\mathcal{R}_2, \mathcal{R}_1) \Leftrightarrow \mathcal{R}_2^X \subseteq \mathcal{R}_1.$$

Theorem (F., Stephan, & Yu)

The following are equivalent:

1. $X \geq_T \emptyset'$.
2. $X \in High(\mathsf{Schnorr}, \mathsf{ML})$ *(that is,* $\mathsf{Schnorr}^X \subseteq \mathsf{ML}$*)*.
3. $X \in High(\mathsf{Schnorr}, \mathsf{Comp})$.
4. $X \in High(\mathsf{Schnorr}, \mathsf{W2R})$.

Lowness in computable structure theory

Definition

A set is *low for isomorphism* if, whenever it can compute an isomorphism between two computably presented structures, there is a computable isomorphism between them.

Examples and counterexamples

Degrees that are low for isomorphism:

- Cohen 2-generic degrees
- Mathias 3-generic degrees

Degrees that are not low for isomorphism:

- Noncomputable Δ_2^0 degrees
- Martin-Löf random degrees

Degrees that could go either way:

- Minimal degrees
- Hyperimmune-free degrees (in fact, computably traceable degrees)
- Cohen 1-generic degrees

Restricted classes of structures

Definition

A set is *low for $\mathcal{C}$-isomorphism* if, whenever it can compute an isomorphism between two computably presented structures in $\mathcal{C}$, there is a computable isomorphism between them.

Theorem (Suggs)

Let $\mathcal{C}$ be the class of equivalence structures with one equivalence class of each finite size. Then $\mathbf{d}$ *is low for $\mathcal{C}$-isomorphism if and only if* $\mathbf{d}$ *doesn't compute any noncomputable Δ^0_2 degree.*

Lowness for paths

Definition

A real A is *low for paths for Baire space* (or *low for paths for Cantor space*) if every Π^0_1 class $\mathcal{P} \subseteq \omega^\omega$ (or $\mathcal{P} \subseteq 2^\omega$) with an A-computable element has a computable element.

Metric structures

Definition

A *metric structure* is a quintuple $\mathcal{M} = (U, d, \mathcal{O}, \mathcal{F}, \mathcal{C})$ such that (U, d) is a complete metric space and

1. For each $T \in \mathcal{O}$, there is a positive integer n so that T is a uniformly continuous n-ary operation on U.
2. For each $f \in \mathcal{F}$, there is a positive integer n so that f is a uniformly continuous n-ary functional on U; i.e., $f : U^n \to \mathbb{F}$ and is uniformly continuous.
3. $\mathcal{C} \subseteq U$.

A presentation of a metric structure $\mathcal{M}$ is a pair $(\mathcal{M}, (p_n))$ such that the p_ns (the *distinguished points*) generate $\mathcal{M}$.

Metric structures

Definition

An *isometric isomorphism* preserves both the algebraic structure and distances in a metric structure.

A Turing degree is *low for isometric isomorphism* if for every computably presented metric structure $\mathcal{M}$ and any two of its computable presentations $\mathcal{M}^*$ and $\mathcal{M}^\#$, whenever it can compute an isometric isomorphism between these presentations, there is a computable isometric isomorphism between them.

A summary

Theorem (F. & Turetsky, F. & McNicholl)

The following classes of degrees are identical:

- *the degrees that are low for isomorphism,*
- *the degrees that are low for paths in Cantor space,*
- *the degrees that are low for paths in Baire space,*
- *the degrees that are low for isometric isomorphism.*

Highness in computable structure theory

Definition (Calvert, F., & Turetsky)

We call a degree **d** *high for isomorphism* if for any two computable structures $\mathcal{M}$ and $\mathcal{N}$ with $\mathcal{M} \cong \mathcal{N}$, there is a **d**-computable isomorphism from $\mathcal{M}$ to $\mathcal{N}$.

Kleene's $\mathcal{O}$

Definition
Kleene's $\mathcal{O}$ is a complete Π^1_1 set.

Observation
$\mathcal{O}$ is high for isomorphism.

$\mathcal{O}$ can compute a path through the tree of partial isomorphisms between any two structures whenever such a path exists.

Δ^1_1 sets

Observation

If **d** *is high for isomorphism, then it computes every Δ^1_1 set.*

If X is Δ^1_1, then $\{X\}$ is a Σ^1_1 class and **d** must compute X.

In the Turing degrees

Proposition

Kleene's $\mathcal{O}$ is arithmetical over any degree high for isomorphism. Furthermore, if $\mathbf{d}$ is high for isomorphism, then $\mathcal{O}$ is $\Pi^0_3(\mathbf{d})$ and thus $\mathbf{d}''' \geq_T \mathcal{O}$.

Proof: $X = \{(i,j) : \mathcal{M}_i \cong \mathcal{M}_j\}$ is a Σ^1_1-complete set, but if $\mathbf{d}$ is high for isomorphism with $D \in \mathbf{d}$, then

$$(i,j) \in X \Leftrightarrow \exists e\,[\{e\}^D : \mathcal{M}_i \cong \mathcal{M}_j]$$

and the matrix of the right-hand side is $\Pi^0_2(D)$.

Then Σ^1_1 sets are $\Sigma^0_3(\mathbf{d})$, making $\mathcal{O}$ $\Pi^0_3(\mathbf{d})$.

Proposition

There is a degree $\mathbf{d}$ *which is high for isomorphism with* $\mathbf{d}''' = \mathcal{O}$.

Proof: Inspired by Jockusch and Simpson's construction of a minimal upper bound for Δ^1_1 with a triple jump computable from $\mathcal{O}$, although we use hyperlow trees rather than their Δ^1_1 trees.

Uniform highness

Definition

We say $\mathbf{d}$ is *uniformly high for isomorphism* if there is a $D \in \mathbf{d}$ and a total computable f such that for any isomorphic computable structures $\mathcal{M}_i$ and $\mathcal{M}_j$, the function $\{f(i,j)\}^D$ is an isomorphism from $\mathcal{M}_i$ to $\mathcal{M}_j$.

Proposition

If $\mathbf{d}$ is uniformly high for isomorphism, then $\mathcal{O}$ is $\Sigma^0_2(\mathbf{d})$ and thus $\mathbf{d}'' \geq_T \mathcal{O}$.

$$(i,j) \in X \iff \{f(i,j)\}^D : \mathcal{M}_i \cong \mathcal{M}_j$$

Structural facts

Proposition

There exist degrees $\mathbf{d}_1$ *and* $\mathbf{d}_2$*, each high for isomorphism, with* $\mathbf{d}_i \lneq_T \mathcal{O}$ *such that* $\mathbf{d}_1 \oplus \mathbf{d}_2 \equiv_T \mathcal{O}$*.*

Proof: Straightforward construction.

Highness for paths

Definition

A degree **d** is *high for paths* if for every nonempty Π^0_1 class of functions $\mathcal{P}$, **d** computes an element of $\mathcal{P}$.

Observation

Σ^1_1 classes are uniformly projections of Π^0_1 classes, so we can replace Π^0_1 with Σ^1_1 above: If **d** *is high for paths, then it computes an element of every nonempty Σ^1_1 class.*

Proposition

A degree is high for isomorphism if and only if it is high for paths.

Harrison orders

Definition

A degree **d** is *high for isomorphism for Harrison orders* if **d** computes an isomorphism between any two computable linear orders of order type $\omega_1^{ck}(1 + \mathbb{Q})$.

Theorem

The degrees which are high for isomorphism for Harrison orders are precisely the degrees which are high for isomorphism.

A summary

Theorem

The following classes of degrees are identical:

- *the degrees that are high for isomorphism,*
- *the degrees that are high for paths,*
- *the degrees that are high for isomorphism for Harrison orders.*

Descending sequences

Definition

A degree $\mathbf{d}$ is *high for descending sequences* if any computable ill-founded linear order $\mathcal{L}$ has a $\mathbf{d}$-computable descending sequence.

Theorem

If X is not arithmetical, there is a degree that is high for descending sequences and does not compute X.

Corollary

There is a degree which is high for descending sequences but not high for isomorphism.

References

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Thank you!