Mathematical incompleteness without extensional invariants

Anton Freund (TU Darmstadt)

Incompleteness via extensional invariants

Π^1_2-statements:

- Theorems of the form $\forall_{X\subseteq\mathbb{N}}\exists_{Y\subseteq\mathbb{N}}\,\varphi$ for arithmetical φ
- Examples: infinite Ramsey theorem, Bolzano-Weierstraß, ...
- Independence via relative (un-)computability

Π^0_2-statements:

- Theorems of the form $\forall_{m\in\mathbb{N}}\exists_{n\in\mathbb{N}}\,\theta$ for decidable θ
- Examples: termination of algorithms, Paris-Harrington
- Independence via growth rates of provably total functions

Incompleteness without extensional invariants?

Π^0_1-statements:

- Theorems of the form $\forall_{n\in\mathbb{N}}\,\theta$ for decidable θ
- Examples: very few results (S. Shelah, H. Friedman) and **no fully satisfactory overall picture**
- **High foundational significance** (Hilbert's program)
- **Mathematical challenge**: independence cannot be shown via uncomputability or provably total functions

In this talk:

- Independent axiom **scheme** with Σ^0_2-instances ($\exists_{m\in\mathbb{N}}\forall_{n\in\mathbb{N}}\,\theta$)
- Same mathematical challenge, less foundational significance

Kruskal's theorem

Let $\mathcal{B}$ be the set of (finite) **binary trees**. For $s, t \in \mathcal{B}$ we write $s \leq_{\mathcal{B}} t$ if there is an **infimum-preserving embedding** of s into t:

Theorem (Kruskal 1960). For any infinite sequence $t_0, t_1, \ldots$ of binary trees there are $i < j$ with $t_i \leq_{\mathcal{B}} t_j$.

In fact, Kruskal's theorem is concerned with arbitrary finite (rather than just binary) trees. We consider binary trees for simplicity.

A finite basis property

Definition. For a formula $\varphi \equiv \varphi(t)$, let $\mathcal{K}\varphi$ be the formula which says that there is a finite set $a \subseteq \mathcal{B}$ with

$$\forall_{s \in a}\, \varphi(s) \wedge \forall_{t \in \mathcal{B}}(\varphi(t) \to \exists_{s \in a}\, s \leq_{\mathcal{B}} t).$$

Corollary. All instances $\mathcal{K}\varphi$ are true.

Proof: If $\mathcal{K}\varphi$ was false, we could recursively construct $t_0, \ldots, t_{n-1}$ with $\varphi(t_i)$ and $t_i \not\leq_{\mathcal{B}} t_j$ for $i < j < n$ (consider $a = \{t_0, \ldots, t_{n-1}\}$ to find t_n). This would result in an infinite sequence that contradicts Kruskal's theorem. □

A limitation of extensional invariants

Proposition. Peano arithmetic (PA) has the same provably total functions as its extension by the axiom schema

$$\mathcal{K}\Sigma_1^- := \{\mathcal{K}\varphi \mid \varphi(s) \text{ a } \Sigma^0_1\text{-formula without further free variables}\}.$$

Proof: Consider an algorithm that computes f and terminates provably in $\mathsf{PA} + \mathcal{K}\varphi$. Since $\mathcal{K}\varphi$ is a true Σ^0_2-formula, it follows from a true Π^0_1-formula $\forall_{n\in\mathbb{N}}\,\theta(n)$. To compute f in PA, we

- run the given algorithm, and output its result when a terminating computation is found;
- simultaneously search for an n with $\neg\theta(n)$, and output 0 if such an n is found before a terminating computation. □

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Independence via Gödel's theorem

Theorem (F. 2020, following D. de Jongh and H. Friedman). Peano arithmetic does not prove all instances of $\mathcal{K}\Sigma_1^-$.

Proof: Gentzen derived the consistency of PA from Π_1^--induction up to $\varepsilon_0 = \min\{\alpha \,|\, \omega^\alpha = \alpha\}$. We show that the minimal element version of induction follows from the finite basis property expressed by $\mathcal{K}\Sigma_1^-$, to conclude by Gödel's theorem. For this purpose, we construct $f : \varepsilon_0 \to \mathcal{B}$ such that $f(\alpha) \leq_{\mathcal{B}} f(\beta)$ implies $\alpha \leq \beta$:

□

On Gentzen's ordinal analysis

Gentzen has labelled proofs by ordinals below ε_0. He has shown that each (hypothetical) proof of a contradiction can be transformed into a proof with smaller ordinal label. One can **deduce consistency in two different ways**:

- argue that a proof of contradiction would lead to a descending sequence of ordinals, which contradicts the **primitive recursive well foundedness** of ε_0;
- use **transfinite Π_1^--induction** over $\alpha < \varepsilon_0$ to show that there is no proof of contradiction with height α.

Primitive recursive well foundedness

Theorem (Gentzen, Kreisel,...; Paris & Harrington 1977).
The following are equivalent over Peano arithmetic:

- the primitive recursive well foundedness of ε_0,
- uniform Π^0_2-reflection, which asserts

  $$\text{“for all } n \in \mathbb{N}\text{, if PA proves } \varphi(\overline{n})\text{, then } \varphi(n) \text{ holds”},$$

  where φ ranges over Π^0_2-formulas,
- the strengthened finite Ramsey theorem
  (also known as Paris-Harrington principle).

Transfinite induction

Theorem (F. 2020, following Kreisel, . . .).

The following are equivalent over Peano arithmetic:

- **parameter-free** Π^0_1-induction up to ε_0,
- **local** Σ^0_2-reflection, which consists of the assertions

  "if PA proves ψ, then ψ holds",

  where ψ ranges over closed Σ^0_2-formulas,
- the schema $\mathcal{K}\Sigma^-_1$, which asserts that each computably enumerable property of binary trees has a finite basis.

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Π^0_1-consequences of the finite basis property

By Goryachev's theorem on parameter free reflection we get:

Corollary. The theory $\mathsf{PA} + \mathcal{K}\Sigma_1^-$ proves the same Π^0_1-sentences as

$$\mathsf{PA} + \mathsf{Con}(\mathsf{PA}) + \mathsf{Con}(\mathsf{PA} + \mathsf{Con}(\mathsf{PA})) + \ldots.$$

Each instance of $\mathcal{K}\Sigma_1^-$ follows from a true Π^0_1-sentence.
However, a result of Kreisel and Lévy yields:

Corollary. There is no **computable** consistent Π^0_1-extension of PA that proves all instances of $\mathcal{K}\Sigma_1^-$.

Thank you for your attention!

Details and further references can be found in

A. Freund: *A mathematical commitment without computational strength*, arXiv:2004.06915.