A Combinatorial Approach to Polynomial Functors

Leeds-Ghent Logic Seminar

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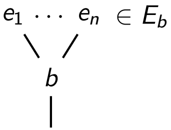
Polynomial functors

function		functor
$p : E \to B$	$\rightsquigarrow$	$\mathbf{Set} \longrightarrow \mathbf{Set}$

$$X \mapsto \sum_{b \in B} X^{E_b} \qquad \text{where } E_b = p^{-1}(b)$$

Idea: an operation $b \in B$ has arity $p^{-1}(b)$



Examples

- $p : \mathbb{N} \to \{*\}$
  $n \mapsto *$

  polynomial functor $X \mapsto X^{\mathbb{N}}$

- $\mathrm{id} : E \to E$

  polynomial functor $X \mapsto E \times X$

- $p : \mathbb{N}' \longrightarrow \mathbb{N}$ where $\mathbb{N}' = \{(n, i) \in \mathbb{N} \times \mathbb{N} \mid i < n\}$
  $(n, i) \mapsto n$

  polynomial functor $X \mapsto \sum_{n \in \mathbb{N}} X^n$

We consider finitary polynomial functors: all fibers are finite and they can be represented as:

$$X \mapsto \sum_{n \in \mathbb{N}} F_n \times X^n$$

Higher types

diagram		functor
$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$	$\rightsquigarrow$	$\mathbf{Set}^I \longrightarrow \mathbf{Set}^J$ $(X_i)_{i \in I} \mapsto \left(\sum_{b \in B_j} \prod_{e \in E_b} X_{s(e)} \right)_{j \in J}$

Morphisms between polynomial functors preserve the arity of operations

pullback squares		cartesian transformations
$\begin{array}{ccccccc} & & E & \xrightarrow{p} & B & & \\ & \overset{s}{\swarrow} & \downarrow \lrcorner & & \downarrow & \overset{t}{\searrow} & \\ I & & & & & & J \\ & \underset{s'}{\nwarrow} & E' & \xrightarrow[p']{} & B' & \underset{t'}{\nearrow} & \end{array}$	$\rightsquigarrow$	$P \Rightarrow P'$

Polynomial functors are not a cartesian closed bicategory

Many applications: quantitative semantics, containers, dependent types, higher categories, implicit complexity, dynamical systems, etc.

But not a cartesian closed bicategory.

- Girard, Hasegawa: normal functors (not cartesian closed unless we quotient 2-cells)
- Taylor: stable functors model by adding an extra structure to objects (but does not model involutive negation)
- Fiore, Gambino, Hyland, Winskel: full model of linear logic with a weaker notion of polynomial
- Finster, Mimram, Lucas, Seiller: polynomial functors over groupoids, homotopy

Analytic Functors: quotients are allowed

Definition

A functor P : **Set** $\to$ **Set** is *analytic* if it is of the form

$$P : X \mapsto \sum_{n \in \mathbb{N}} F_n \times_{\mathfrak{S}_n} X^n$$

- $(F_n)_{n \in \mathbb{N}}$ is a family of sets with a left action of $\mathfrak{S}_n$
- the quotient identifies $(\sigma \cdot p, (x_1, \ldots, x_n)) \sim (p, x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for $\sigma \in \mathfrak{S}_n$.

Example $\quad P : X \mapsto 1 + X + X^2/\mathfrak{S}_2 + \cdots + X^n/\mathfrak{S}_n + \ldots$

Combinatorial Species

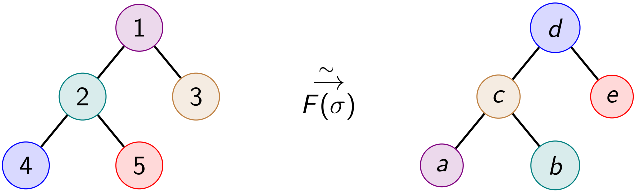
category $\mathbb{B}$ objects: finite sets, morphisms: bijections

Definition (Joyal 1981)

A *species of structure* is a functor $F : \mathbb{B} \to \mathbf{Set}$.

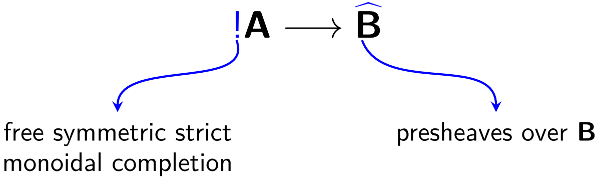
- Given a finite set of *labels* $\mathcal{U} \in \mathbb{B}$, an element $x \in F[\mathcal{U}]$ is called a *F-structure on $\mathcal{U}$*
- Given a bijection $\sigma : \mathcal{U} \xrightarrow{\sim} \mathcal{V} \in \mathbb{B}$, the bijection $F[\sigma] : F[\mathcal{U}] \xrightarrow{\sim} F[\mathcal{V}]$ is called the *transport of F-structures along σ*

$$\mathcal{U} = \{1, 2, 3, 4, 5\} \quad \underset{\sigma}{\xrightarrow{\sim}} \quad \mathcal{V} = \{a, b, c, d, e\}$$



Generalized species

- Fiore, Gambino, Hyland and Winskel 2008: generalized species as a bicategorical model of differential linear logic



- A $(\mathbf{1},\mathbf{1})$-species of structure corresponds to a combinatorial species of structure

$$F : !\,\mathbf{1} \to \widehat{\mathbf{1}} \qquad \Leftrightarrow \qquad F : \mathbb{B} \to \mathbf{Set}$$

Bicategorical model of generalized species

type/formula A	$\mathbf{A}$ groupoid
term/proof π of $\vdash A$	$X \in \widehat{\mathbf{A}}$ presheaf
π of $A \vdash B$	$F : !\mathbf{A} \to \widehat{\mathbf{B}}$ species
reduction $\pi \rightsquigarrow \pi'$	natural transformation

$!\mathbf{A}$ is given by:

- Objects: finite sequences $\langle a_1, \ldots, a_n \rangle$ of objects of $\mathbf{A}$.
- Morphisms: pairs $(\sigma, (f_i)_{i \in \underline{n}}) : \langle a_1, \ldots, a_n \rangle \to \langle b_1, \ldots, b_n \rangle$ of a permutation $\sigma \in \mathfrak{S}_n$ and a finite sequence of morphisms $f_i : a_i \to b_{\sigma(i)}$ in $\mathbf{A}$.

Generalized Species and Analytic Functors

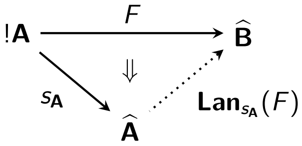
Definition

An *analytic functor* between two small categories $\mathbf{A}$ and $\mathbf{B}$ is a functor

$$P : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$$

that preserves filtered colimits and wide quasi-pullbacks.

Given a generalized species $F : !\mathbf{A} \to\!\!\!\!\!+ \mathbf{B}$, the functor $\mathbf{Lan}_{s_A} F : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ is analytic:



$$\text{where } s_{\mathbf{A}} : \langle a_1, \ldots a_n \rangle \mapsto \sum_{i=1}^{n} \mathbb{y}_{\mathbf{A}}(a_i)$$

Generalized Species and Analytic Functors

Theorem (Fiore 2013)

The bicategory of generalized species (restricted to groupoids) is biequivalent to the 2-category of analytic functors.

bicategory **Esp**		2-category **An**
0-cells: small groupoids $\mathbf{A}$, $\mathbf{B}$	$\longleftrightarrow$	0-cells: small groupoids $\mathbf{A}$,$\mathbf{B}$
1-cells: species $F : !\mathbf{A} \rightarrow\!\!\!\!\!+ \mathbf{B}$		1-cells: analytic functors $P : \widehat{\mathbf{A}} \rightarrow \widehat{\mathbf{B}}$
2-cells: natural transformations		2-cells: quasi cart. natural transformations

Stable Functions

Let D,E be cpo's. A continuous function $h : D \to E$ is *stable* if it satisfies

$$\forall\alpha \in D \;\; \forall\beta \subset h(\alpha),\ \exists\alpha' \subset \alpha \text{ s.t. } \beta \subset h(\alpha') \text{ and } (\forall\alpha'' \subset \alpha,\ \beta \subset h(\alpha'') \Rightarrow \alpha' \subset \alpha'')$$

Definition (Berry 1978)

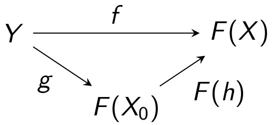
For cpos $(A, \leq_A)$ and $(B, \leq_B)$, a Scott-continuous function $f : A \to B$ is *stable* if for all $y \leq_B f(x)$, there exists $x_0 \in A$ such that:

- $y \leq_B f(x_0)$ and $x_0 \leq_A x$;
- for all $x' \leq_A x$, if $y \leq_B f(x')$ then $x_0 \leq_A x'$.

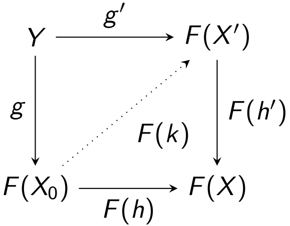
Stability (categorified)

A functor $F : \mathbf{A} \to \mathbf{B}$ admits strict generic factorizations if for every $f : Y \to F(X)$ in $\mathbf{B}$, there exists $X_0 \in \mathbf{A}$ such that

- there exists $g : Y \to F(X_0)$ and $h : X_0 \to X$ such that



- g is strict generic i.e. for every commuting square:



  there exists a unique $k : X_0 \to X'$ such that $h' \circ k = h$ and $F(k) \circ g = g'$.

Extensional characterization

Polynomial functor $P : \mathbf{Set} \to \mathbf{Set}$ $\Leftrightarrow$ P admits strict generic factorizations

Analytic functor $P : \mathbf{Set} \to \mathbf{Set}$ $\Leftrightarrow$ P admits generic factorizations, is finitary and preserves epis

Definition

A functor $P : \mathbf{A} \to \mathbf{B}$ is *stable* if it is finitary, admits strict generic factorizations and preserves epimorphisms.

Our approach

type/formula A	$(\mathbf{A}, \mathscr{A})$ groupoid + extra structure	
term/proof π of $\vdash A$	$X \in \mathcal{S}(\mathbf{A}, \mathscr{A}) \hookrightarrow \widehat{\mathbf{A}}$	presheaf preserving the extra structure
π of $A \vdash B$	$F : !(\mathbf{A}, \mathscr{A}) \mathrel{\rightarrow\!\!\!\!\!\shortmid\;} (\mathbf{B}, \mathscr{B})$	species preserving the extra structure
reduction $\pi \rightsquigarrow \pi'$	natural transformation	

Kits on groups

Definition

A *kit* on a group $(G, \cdot, \mathrm{id})$ is a family $\mathscr{A}$ of subgroups of G that is closed under conjugation i.e.

$$\forall H \leq G, H \in \mathscr{A} \Rightarrow \left(\forall g \in G, gHg^{-1} \in \mathscr{A}\right)$$

We can ask for extra closure properties:

- Downclosed
- Closed under directed unions
- Forms a Heyting algebra
- Saturated $\mathscr{A} = \{H \leq G \mid H \subseteq \bigcup \mathscr{A}\}$ (P. Taylor)
- Forms a Boolean algebra

Boolean kits

For a group $(G, \cdot, \mathrm{id})$, and subgroups $H, K \leq G$, we say that H and K are *orthogonal* if

$$H \perp K \qquad :\Leftrightarrow \qquad H \cap K = \{\mathrm{id}\}$$

For a kit $\mathscr{A}$ on a group G, its *orthogonal* given by

$$\mathscr{A}^{\perp} := \{K \leq G \mid \forall H \in \mathscr{A}, H \perp K\}$$

is a kit on G.

Definition

A kit $\mathscr{A}$ on a group G is called a *boolean kit* if $\mathscr{A} = \mathscr{A}^{\perp\perp}$.

Example

minimal kit structure | maximal kit structure

$$(G, \{\{\mathrm{id}\}\}) \quad \hookrightarrow \quad (G, \mathscr{A}) \quad \hookrightarrow \quad (G, \{H \mid H \leq G\})$$

Consider $G = \langle g \rangle$ with $g^6 = \mathrm{id}$, the cyclic group of order 6, the Boolean kits are:

$$\{\{\mathrm{id}\}, \langle g^3 \rangle, \langle g^2 \rangle, \langle g \rangle\}$$

$$\{\{\mathrm{id}\}, \langle g^2 \rangle\} \qquad \{\{\mathrm{id}\}, \langle g^3 \rangle\}$$

$$\{\{\mathrm{id}\}\}$$

Kits on Groupoids

Definition

A *kit* on a groupoid $\mathbf{A}$ is a family $\mathscr{A} = \{\mathscr{A}(a)\}_{a\in\mathbf{A}}$ of sets $\mathscr{A}(a)$ of subgroups of $\mathbf{A}(a,a)$ that is closed under conjugation i.e.

$$\forall H \leq \mathbf{A}(a,a), H \in \mathscr{A}(a) \Rightarrow \left(\forall f : a \to b, fHf^{-1} \in \mathscr{A}(b)\right)$$

For a kit $\mathscr{A}$ on a groupoid $\mathbf{A}$, its *orthogonal* given by

$$\mathscr{A}^{\perp}(a) := \{K \leq \mathbf{A}^{\mathrm{op}}(a,a) \mid \forall H \in \mathscr{A}(a), H \perp K\} \quad \text{is a kit on } \mathbf{A}^{\mathrm{op}}.$$

Definition

A kit $\mathscr{A}$ on a groupoid $\mathbf{A}$ is called a *boolean kit* if $\mathscr{A} = \mathscr{A}^{\perp\perp}$.

Stabilized presheaves

Fact: If $\mathbf{A}$ is a groupoid, every presheaf $X \in \widehat{\mathbf{A}}$ is isomorphic to a sum of quotiented representables:

$$y(a)/G = \varinjlim(G \hookrightarrow \mathbf{A} \xrightarrow{\mathrm{y}} \widehat{\mathbf{A}}) \qquad \text{where } G \leq \mathbf{A}(a,a).$$

Explicitely $y(a)/G : a' \mapsto \mathbf{A}(a', a)/\sim$ where

$$(f : a' \to a \quad \sim \quad g : a' \to a) \qquad \text{if} \qquad g^{-1}f \in G$$

Definition

For a presheaf $X : \mathbf{A}^{\mathrm{op}} \to \mathbf{Set}$, $a \in \mathbf{A}$ and $x \in X(a)$, define

$$\mathbf{Stab}(x) := \{f : a \to a \mid X(f)(x) = x\}$$

Note: $\forall x \in (y(a)/G)(a), \mathbf{Stab}(x) = G$.

Stabilized presheaves

Definition

For a boolean kit $(\mathbf{A}, \mathscr{A})$, let $\mathcal{S}(\mathbf{A}, \mathscr{A})$ be the full subcategory of $\widehat{\mathbf{A}}$ containing all presheaves X such that for all $a \in \mathbf{A}$, $x \in X(a)$, $\mathbf{Stab}(x) \in \mathscr{A}(a)$.

- Every presheaf in $\mathcal{S}(\mathbf{A}, \mathscr{A})$ has a representation as

  $$\sum_{i \in I} y(a_i)/G_i$$

  for some index set I, $a_i \in \mathbf{A}$ and $G_i \in \mathscr{A}(a_i)$.

- The embedding $\mathcal{S}(\mathbf{A}, \mathscr{A}) \hookrightarrow \widehat{\mathbf{A}}$ creates isomorphisms, coproducts, filtered colimits, epimorphisms and non-empty limits.

Stable Functors

Definition

We define the 2-category **Stable**:

objects: boolean kits $(\mathbf{A}, \mathscr{A}), (\mathbf{B}, \mathscr{B}), \ldots$

1-cells: stable functors $F : \mathcal{S}(\mathbf{A}, \mathscr{A}) \to \mathcal{S}(\mathbf{B}, \mathscr{B})$

2-cells: cartesian natural transformations

- Stable functor **Set** $\to$ **Set**

$$X \mapsto 1 + X + X^2 + \cdots + X^n + \ldots \cong \sum_{n \in \mathbb{N}} X^n$$

- Analytic functor (but not stable) **Set** $\to$ **Set**

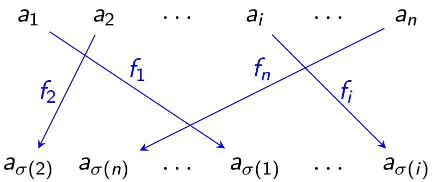
$$X \mapsto 1 + X + X^2/\mathfrak{S}_2 + \cdots + X^n/\mathfrak{S}_n + \ldots \cong \sum_{n \in \mathbb{N}} X^n/\mathfrak{S}_n$$

Recall

type/formula A	$(\mathbf{A}, \mathscr{A})$ groupoid + extra structure	
term/proof π of $\vdash A$	$X \in \mathcal{S}(\mathbf{A}, \mathscr{A}) \hookrightarrow \widehat{\mathbf{A}}$	presheaf preserving the extra structure
π of $A \vdash B$	$F : !(\mathbf{A}, \mathscr{A}) \nrightarrow (\mathbf{B}, \mathscr{B})$	species preserving the extra structure
reduction $\pi \rightsquigarrow \pi'$	natural transformation	

Stable species

Recall, endomorphisms on $\langle a_1, \ldots, a_n \rangle \in !\mathbf{A}$ are of the form $f = (\sigma \in \mathfrak{S}_n, (f_i : a_i \to a_{\sigma(i)}))$:



Definition

For $1 \leq i \leq n$, define

$$\mathrm{loop}(f, i) := a_i \xrightarrow{f_i} a_{\sigma(i)} \xrightarrow{f_{\sigma(i)}} a_{\sigma^2(i)} \ldots \xrightarrow{f_{\sigma^{o(i)-1}}} a_i.$$

where $o(i) :=$ the smallest strictly positive integer such that $\sigma^{o(i)}(i) = i$.

Stable species

Definition

For a kit $(\mathbf{A}, \mathscr{A})$, define $!(\mathbf{A}, \mathscr{A}) := (!\mathbf{A}, !\mathscr{A})$, where for an object $\overline{a} = \langle a_1, \ldots, a_n \rangle \in !\mathbf{A}$,

$$!\mathscr{A}(\overline{a}) := \{H \leq !\mathbf{A}(\overline{a}, \overline{a}) \mid \forall f \in H, \forall 1 \leq i \leq n, \mathrm{loop}(f, i) \in \bigcup \mathscr{A}(a_i)\}^{\perp\perp}.$$

For boolean kits $(\mathbf{A}, \mathscr{A})$ and $(\mathbf{B}, \mathscr{B})$, a *stable species* $P : !(\mathbf{A}, \mathscr{A}) \twoheadrightarrow (\mathbf{B}, \mathscr{B})$ is a species $P : !\mathbf{A} \twoheadrightarrow \mathbf{B}$ such that, for all $\overline{a} \in !\mathbf{A}$, $b \in \mathbf{B}$, $p \in P(\overline{a}, b)$, $f \in !\mathbf{A}(\overline{a}, \overline{a})$, $g \in \mathbf{B}(b, b)$, if $f \cdot p \cdot g = p$ then

$$f \in \bigcup !\mathscr{A}(\overline{a}) \Rightarrow g \in \bigcup \mathscr{B}(b) \text{ and } g \in \bigcup \mathscr{B}^{\perp}(b) \Rightarrow f \in \bigcup (!\mathscr{A})^{\perp}(\overline{a}).$$

$$\begin{array}{ccc} b & \xrightarrow{p} & P(\overline{a}) \\ {\scriptstyle g}\uparrow & & \downarrow{\scriptstyle P(f)} \\ b & \xrightarrow[p]{} & P(\overline{a}) \end{array}$$

Bicategory of stable species

Definition

Define the bicategory **SEsp** as:

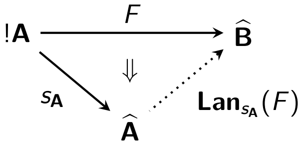
- Objects: boolean kits $(\mathbf{A}, \mathscr{A}), (\mathbf{B}, \mathscr{B}), \ldots$
- 1-cells: stable species $P : !(\mathbf{A}, \mathscr{A}) \nrightarrow (\mathbf{B}, \mathscr{B})$
- 2-cells: natural transformations

Theorem

The bicategory **SEsp** *is cartesian closed.*

Stable Functors

Given a stable species $F : !(\mathbf{A}, \mathscr{A}) \to\!\!\!\!\!\to (\mathbf{B}, \mathscr{B})$, the analytic functor $\mathbf{Lan}_{s_{\mathbf{A}}} F : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$



restricts to a functor $\mathcal{S}(\mathbf{A}, \mathscr{A}) \to \mathcal{S}(\mathbf{B}, \mathscr{B})$ that is stable.

Theorem

The bicategory **SEsp** *is biequivalent to the* 2*-category* **Stable***.*

Corollary: **Stable** is (bi)cartesian closed.

Conclusion and Perspectives

We have a 2-category **Stable** that is cartesian closed (in fact a model of differential linear logic) in which finitary polynomial functors $\mathbf{Set}^I \to \mathbf{Set}^J$ embed.

Future work:

- Study the various possible logical structures on kits.
- Integration/resolution of differential equations in our model
- Categorify the orthogonality construction
- Polymorphism

Thank you