Kripke-Joyal semantics for type theory

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Motivation

Models of HoTT in presheaf categories

- simplicial sets (Voevodsky , ...)
- cubical sets (Coquand , Orton & Pitts , Awodey , ...)

Two descriptions

→ category-theoretic (Awodey , G & Sattler , ...) via diagrams

→ logical (Coquand et al. , Orton & Pitts , ...) via the internal type theory

PROBLEM : HOW DO YOU RELATE THEM ?

Strategy :

1. Fix a presheaf category $\hat{\mathbb{C}} = [\mathbb{C}^{op}, \underline{Set}]$

2. Extract the internal type theory $\underline{Th}_{\hat{\mathbb{C}}}$

3. Find a convenient / mechanical way to test validity of a judgement of $Th_{\hat{\mathbb{C}}}$ in $\hat{\mathbb{C}}$, unfolding it in diagrams.

4. Applications to models of HoTT.

Outline

1. Presheaves

2. The type theory of a presheaf category

3. Kripke-Joyal forcing

4. Applications

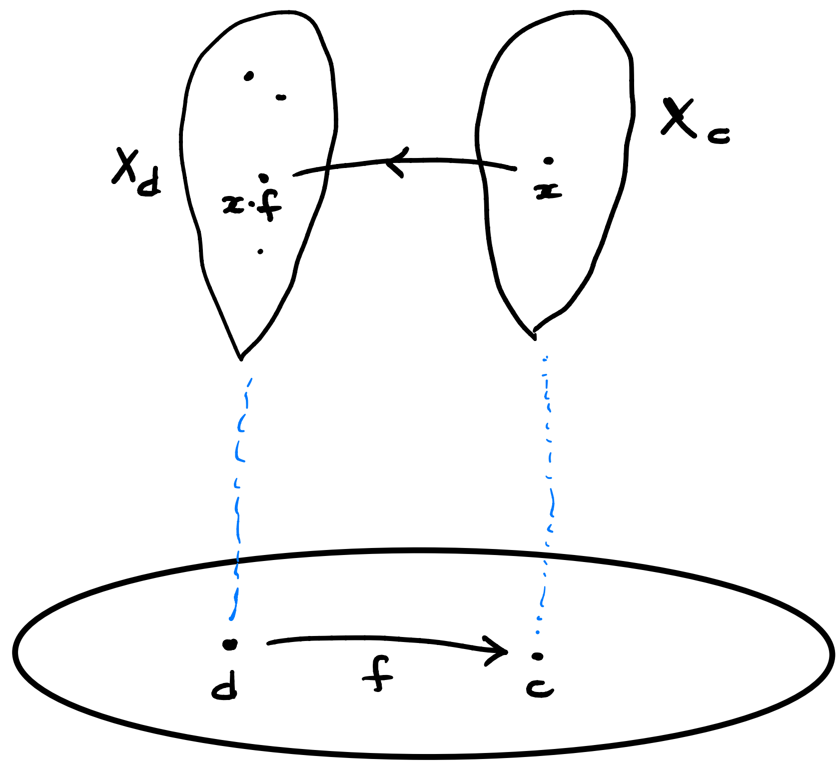
① Presheaves

Fix $\mathbb{C}$ small category, e.g. a poset $(P, \leq)$

Let $\widehat{\mathbb{C}} =_{def} [\mathbb{C}^{op}, \underline{Set}]$

Presheaves = variable sets

$$
\begin{tikzcd}
\mathbb{C} & \mathbb{C}^{op} \arrow[r, "X"] & \underline{Set} \\
c & c \arrow[r, mapsto] \arrow[d] & X_c \arrow[d, "(-)\cdot f"] \\
d \arrow[u, "f"] & d \arrow[r] & X_d
\end{tikzcd}
$$



" $d \leq c$ "

Yoneda embedding

$$\begin{array}{ccccl} \mathbb{C} & \overset{y}{\hookrightarrow} & \widehat{\mathbb{C}} & = & [\mathbb{C}^{op}, \underline{Set}] \\ c & \longmapsto & y(c) & & \end{array}$$

$\rightsquigarrow$ presheaf represented by c.

- In $\underline{Set}$ $\qquad x \in X \iff 1 \xrightarrow{x} X$

generalised element of X

- In $\widehat{\mathbb{C}}$ $\qquad x \in X(c) \iff y(c) \xrightarrow{x} X$

KEY Every presheaf is determined by its generalised elements.

The structure of $\widehat{\mathbb{C}} = [\mathbb{C}^{op}, \underline{Set}]$

$\mathcal{E}$ inherits a lot of structure from Set :

- Limits e.g. 1, $A \times B$, $A \times_X B$, ...
- function spaces, B^A
- dependent products, $\Pi_A(B)$ for $B \longrightarrow A$
- dependent sums, $\Sigma_A(B)$ for $B \longrightarrow A$

KEY We won't need category-theoretic properties, only their logical counterparts.

The subobject classifier

In $\underline{\underline{\mathrm{Set}}}$, $\Omega = \{\text{true}, \text{false}\}$ is a subobject classifier, i.e.

$$
\begin{tikzcd}
S \arrow[r] \arrow[d, tail, "\forall"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\text{true}"] \\
X \arrow[r, dashed, "\exists! \sigma"'] & \Omega
\end{tikzcd}
\qquad S = \{x : X \mid \sigma(x) = \text{true}\}
$$

IDEA : $X \xrightarrow{\sigma} \Omega \iff \sigma(x)$ proposition.

FACT : $\widehat{\mathbb{C}}$ has a subobject classifier.

The small map classifier

fixed inaccessible cardinal

- $A \in \underline{\mathbf{Set}}$ is <u>small</u> if $|A| < \kappa$
- A map $p : A \to X$ is <u>small</u> if $A_x = p^{-1}(x)$ small $\forall x$.

FACT $\underline{\mathbf{Set}}$ has a small map classifier :

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\forall p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\pi"] \\ X \arrow[r, "\exists \alpha"'] & U \end{tikzcd}$$

IDEA $X \xrightarrow{\alpha} U \iff \alpha(x)$ small set $\forall x$

FACT (Hofmann-Streicher) $\widehat{\mathbb{C}}$ has a small map classifier.

② The type theory of $\widehat{\mathbb{C}}$

DEFINITION

- A <u>context</u> Γ is an object of $\widehat{\mathbb{C}}$.
- A <u>type</u> α in context Γ is a map $\alpha : \Gamma \longrightarrow U$.
- An <u>element</u> a of type α in context Γ is a map $a : \Gamma \longrightarrow E$ such that

$$\begin{tikzcd} \Gamma \arrow[r, "a"] \arrow[d, equal] & E \arrow[d, "\pi"] \\ \Gamma \arrow[r, "\alpha"'] & U \end{tikzcd}$$

NOTATION $\quad \Gamma \vdash \alpha : U \quad , \quad \Gamma \vdash a : \alpha .$

Context extension

If α is a type in context Γ, then $\Gamma.\alpha$ is a context $\iff$

$$\begin{tikzcd} \Gamma.\alpha \arrow[r] \arrow[d, "p_\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\pi"] \\ \Gamma \arrow[r, "\alpha"'] & U \end{tikzcd}$$

Substitution

$$\frac{t : \Delta \longrightarrow \Gamma \qquad \Gamma \vdash \alpha : U}{\Delta \vdash \alpha(t) : U} \iff \begin{tikzcd} \Delta \arrow[r, "t"'] & \Gamma \arrow[r, "\alpha"'] & U \end{tikzcd}$$

$$\frac{t : \Delta \longrightarrow \Gamma \qquad \Gamma \vdash a : \alpha}{\Delta \vdash a(t) : \alpha(t)} \iff \begin{tikzcd} \Delta \arrow[r, "t"] \arrow[d, equal] & \Gamma \arrow[r, "a"] \arrow[d, equal] & E \arrow[d, "\pi"] \\ \Delta \arrow[r, "t"'] & \Gamma \arrow[r, "\alpha"'] & U \end{tikzcd}$$

The type theory $\underline{\mathrm{Th}}_{\mathcal{E}}$

- Basic types : $\quad 1 \quad , \quad \alpha \times \beta \quad , \quad \beta^{\alpha}$
- Dependent types $\quad \Sigma_{\alpha}(\beta) \quad , \quad \Pi_{\alpha}(\beta)$
- A type of propositions $\quad \Omega \longleftarrow$ with elements $\top, \bot, \sigma \wedge \tau, \sigma \vee \tau, \sigma \vee \tau, \sigma \Rightarrow \tau, (\forall x : A)\sigma(x), (\exists x : A)\sigma(x)$.
- Subset types $\quad \{ x : \alpha \mid \sigma(x) \}$

NOTE Expressive theory, e.g. there is $\Gamma \vdash \tau : U$ s.t.

$$\Gamma \vdash t : \tau \quad \Longleftrightarrow \quad \begin{tikzcd} S \arrow[rr] \arrow[dr] \arrow[ddr] & & A \arrow[dr] \arrow[ddl] & \\ & T \arrow[rr] \arrow[ur, "t"] \arrow[d] & & B \arrow[dll] \\ & \Gamma & & \end{tikzcd}$$

Example (dependent products)

$$\frac{\Gamma \vdash \alpha : U \qquad \Gamma.\alpha \vdash \beta : U}{\Gamma \vdash \Pi_\alpha(\beta) : U}$$

$$\frac{\Gamma.\alpha \vdash b : \beta}{\Gamma \vdash \lambda(b) : \Pi_\alpha(\beta)} \qquad \frac{\Gamma \vdash t : \Pi_\alpha(\beta) \qquad \Gamma \vdash a : \alpha}{\Gamma \vdash app(t, a) : \beta(a)}$$

Problem

- testing validity may be hard (cf. Boolean-valued models)
- want an alternative (cf. forcing).

3 Kripke-Joyal forcing

DEFINITION Let $X \vdash \alpha : U$ and $x : y(c) \longrightarrow X$.

We say that **c forces $a : \alpha(x)$** if $a : y(c) \longrightarrow E$

is such that

$$\begin{tikzcd} y(c) \arrow[r, "a"] \arrow[d, "x"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "\alpha"'] & U \end{tikzcd}$$

NOTATION $c \Vdash a : \alpha(x)$

NOTE

$$c \Vdash a : \alpha(x) \quad \Longleftrightarrow \quad \begin{tikzcd} y(c) \arrow[r, "a"] \arrow[d, "x"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "\alpha"'] & U \end{tikzcd}$$

$$\Longleftrightarrow \quad \begin{tikzcd} y(c) \arrow[rr, "a"] \arrow[d, equal] & & E \arrow[d, "\pi"] \\ y(c) \arrow[r, "x"'] & X \arrow[r, "\alpha"'] & U \end{tikzcd}$$

$$\Longleftrightarrow \quad y(c) \vdash a : \alpha(x)$$

IDEA We restrict to "representable" contexts

REMARK

$$c \Vdash a : \alpha(x) \iff \begin{tikzcd} & X.\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\pi"] \\ y(c) \arrow[ur, dotted, "{(x,a)}"] \arrow[r, "x"'] & X \arrow[r, "\alpha"'] & U \end{tikzcd}$$

SPECIAL CASE $\quad \sigma : X \longrightarrow \Omega$

$$c \Vdash s : \sigma(x) \iff \begin{tikzcd} & \{x : X \mid \sigma(x)\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\mathrm{true}"] \\ y(c) \arrow[ur, dotted, "s"] \arrow[r] & X \arrow[r] & \Omega \end{tikzcd}$$

$\Longrightarrow$ We get back standard Kripke-Joyal forcing !

PROPOSITION (Monotonicity) If $c \Vdash a : \alpha(x)$ then

$$d \Vdash a(f) : \alpha(xf) \quad \text{for every } f : d \to c .$$

Proof

$$\begin{tikzcd}
y(d) \arrow[r, "f"] \arrow[d, equal] & y(c) \arrow[rr, "a"] \arrow[d, equal] & & E \arrow[d, "\pi"] \\
y(d) \arrow[r, "f"'] & y(c) \arrow[r, "x"'] & X \arrow[r, "\alpha"'] & U
\end{tikzcd}$$

$c \Vdash \alpha(x)$

$d \Vdash a(f) : \alpha(xf)$

PROPOSITION (Uniformity) Let $\alpha : X \to U$. There is a bijection between:

(1) elements $X \vdash a : \alpha$

(2) families $(a_x)_{x : yc \to X}$ such that

(forcing) $c \Vdash a_x : \alpha(x) \quad \forall\, x : yc \to X$

(uniformity) $a_x \cdot f = a_{x \cdot f} \quad \forall\, x : yc \to X$

and $f : d \to c$.

NOTE No uniformity in standard Kripke-Joyal forcing.

Forcing of dependent products

PROPOSITION There is a bijection between

- elements t such that $c \Vdash t : (\Pi_\alpha \beta)(x)$
- families of elements $(b_{f,a})_{f : d \to c,\ d \Vdash a : \alpha(xf)}$

such that

(forcing) $$d \Vdash b_{f,a} : \beta(xf, a)$$

(uniformity) $$b_{f,a}(g) = b_{fg,ag}\,.$$

(4) Applications to HoTT

Presheaf categories with Quillen model structures (W, C, F) ⇝ models of HoTT & Univalent Foundations.

(Awodey-Warren, Voevodsky, ...)

W: weak equivalences

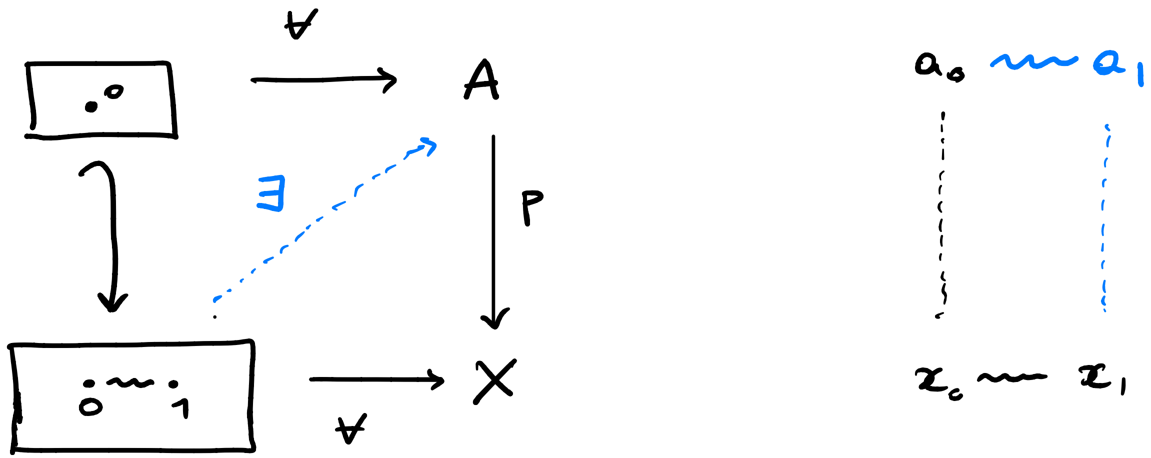
C: cofibrations

F: fibrations

NOTE Axioms for a Quillen model structure involve

- existence of factorisations
- existence of liftings / diagonal filler

EXAMPLE $\underline{\mathrm{Top}}$ admits a Quillen model structure where the fibrations are the maps $p : A \longrightarrow X$ such that



$\forall\, x_0 \sim x_1 \quad \forall\, a_0 \ \ldots \qquad \exists\, a_1 \quad \exists\, a_0 \sim a_1 \ \ldots$

COQUAND ET AL considered **algebraic** Quillen model structures, where one requires additional data for

- functorial factorisations
- explicit **choices** of lifts / diagonal fillers, subject to **uniformity** conditions

PROBLEM Two descriptions of these:

- category-theoretic
- type-theoretic, using $\underline{\mathrm{Th}}_{\hat{\mathbb{C}}}$

NEXT : an example

Fix a class $\underline{\mathrm{Cof}}$ of maps in $\widehat{\mathbb{C}} = [\mathbb{C}^{op}, \underline{\mathrm{Set}}]$.

DEFINITION 1 (Coquand et al, Orton-Pitts, ...)

Let $\alpha : X \longrightarrow U$. A trivial fibration structure on α is an element $t : \mathrm{TFib}(\alpha)$, where

$$\underline{\mathrm{TFib}}(\alpha) = \prod_{\varphi : \Phi} \prod_{v : \alpha^{\{\varphi\}}} \sum_{a : \alpha} v = \lambda(a)$$

classifier of cofibrations

("every partial element of α is extensible")

DEFINITION 2 (G & Sattler, Awodey, ...) Let $p : A \to X$ be a map.

A uniform trivial fibration structure on p is a choice of diagonal fillers $j(m,u,v)$

$$\begin{tikzcd} S \arrow[r, "u"] \arrow[d, tail, "m"'] & A \arrow[d, "p"] \\ T \arrow[r, "v"'] \arrow[ur] & X \end{tikzcd} \qquad \forall m \in \underline{Cof}$$

such that

$$\begin{tikzcd} t^*(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[r, "u"] \arrow[d] & A \arrow[d, "p"] \\ T' \arrow[r, "t"'] \arrow[urr] & T \arrow[r, "v"'] \arrow[ur] & X \end{tikzcd} \qquad \forall m \in \underline{Cof}, \ \forall t : T' \to T.$$

THEOREM Let $\alpha : X \longrightarrow U$. TFAE

- There is a trivial fibration structure on α in the sense of Definition 1.
- There is a uniform fibration structure on $p_\alpha : X.\alpha \longrightarrow X$ in the sense of Definition 2..

KEY Use Kripke-Joyal forcing to relate

- uniformity implicit in Π-type in Def 1
- uniformity explicit in Def 2.

MANY MORE APPLICATIONS

- uniform fibrations
- construction of classifying (trivial) fibrations.
- ...

NOTE

- Kripke-Joyal forcing may be used also for other kinds of algebraic structures in presheaf categories.