# Pseudofiniteness and measurability of the everywhere infinite forest.

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# Pseudofinite structures

## Definition

*An $\mathcal{L}$-structure $M$ is said to be pseudofinite if any of the following equivalent properties holds:*

- *Every $\mathcal{L}$-sentence $\sigma$ that is true in $M$, is also satisfied in some finite $\mathcal{L}$-structure $M_0^\sigma$.* → good to prove something is not pseudofinite
- $M \models \mathrm{FIN}_{\mathcal{L}}$.
- *$M$ is elementarily equivalent to an ultraproduct $\prod_{\mathcal{U}} M_i$ of finite $\mathcal{L}$-structures.* → good to provide examples.

**Observation:** An ultraproduct of finite structures can only be finite or of size $2^{\aleph_0}$. Thus, the last condition allows us to describe structures that are "similar" to ultraproducts of finite structures, but have different cardinalities (for example, can be countable).

# Examples of structures that are not pseudofinite

- The linear orders $(\mathbb{Q}, <), (\mathbb{Z}, <)$ are not pseudofinite.
- The field $(\mathbb{C}, +, \cdot)$ is not pseudofinite: the function $f(x) = x^2$ is definable and surjective, but not injective. Hence $(\mathbb{C}, +, \cdot) \models \underbrace{\forall y \exists x (x^2 = y) \wedge \exists x, y (x \neq y \wedge x^2 = y^2)}_{\sigma}$, but this cannot be true in any finite field.
- $(\mathbb{Z}, +)$ is not pseudofinite: the function $x \mapsto x + x$ is injective, but not surjective.

# Examples of structures that are pseudofinite

- Every ultraproduct of finite $\mathcal{L}$-structures is pseudofinite.
- Pseudofinite fields:

**Theorem (James Ax, 1968)**

An infinite field $K$ is pseudofinite if and only if it satisfies the following conditions:

1. $K$ is perfect.
2. $K$ has a unique extension of degree $n$ for each $n \in \mathbb{N}$.
3. $K$ is *pseudo-algebraically closed* every absolutely irreducible variety over $K$ has a $K$-rational point.

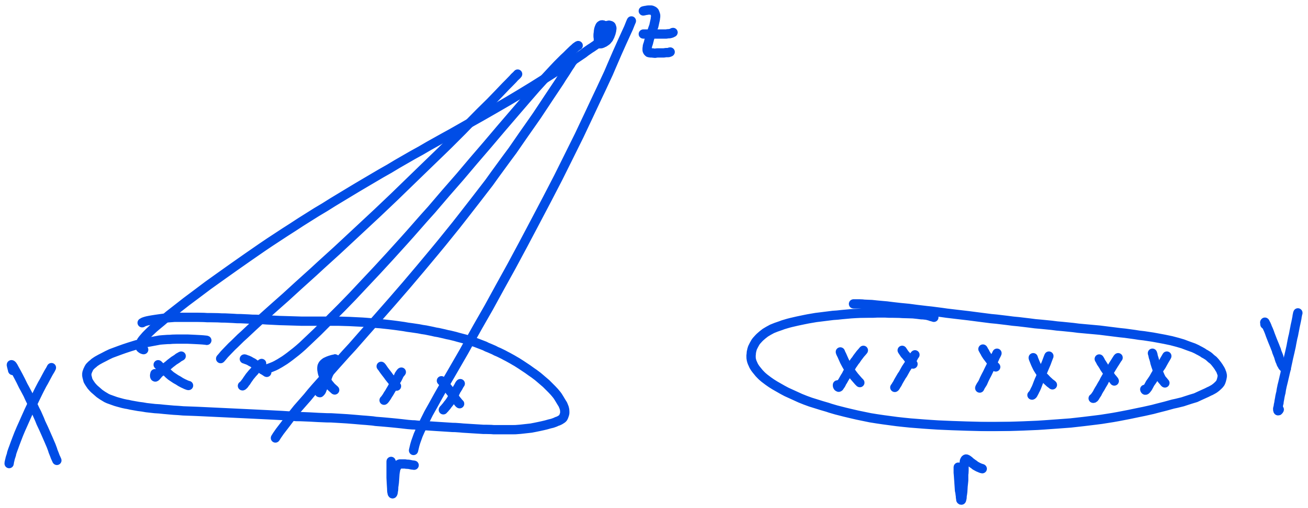
- Vector spaces over $\mathbb{F}_p$ are pseudofinite: we can simply take $\prod_{\mathcal{U}} \mathbb{F}_p^n$.
- The group $(\mathbb{R}, +)$ is isomorphic to $\prod_{\mathcal{U}} (\mathbb{Z}/p\mathbb{Z}, +)$: both are torsion-free divisible abelian groups of cardinality $2^{\aleph_0}$.
- Vector spaces over $\mathbb{Q}$ are pseudofinite in the language $\mathcal{L}_{vs}$.

**Theorem (Erdős, Rényi - 1963)**

Given a fix number $r \geq 1$, $lim_{n\to\infty} \text{Prob}\,(\mathbb{G}(n,p) \models \mathcal{A}_r) = 1$.

Here,

$$\mathcal{A}_r = \forall x_1, \ldots, x_r \forall y_1, \ldots, y_r \left( \bigwedge_{1\leq i,j\leq r} x_i \neq y_j \to \exists z \bigwedge_{i\leq r} zRx_i \wedge \neg zRy_i \right).$$




Rado:

there is a unique countable structure M satisfying RG

Theory of the *random graph*:

$$\mathrm{RG} = \{\forall x(\neg xRx), \forall x, y(xRy \to yRx)\} \cup \{\mathcal{A}_r : r \geq 1\}.$$

# Theories of tree-like graphs

## Definition

*A tree is a (simple) graph without cycles. This property can be axiomatized in the language of graphs $\mathcal{L} = \{R\}$ by the theory:*

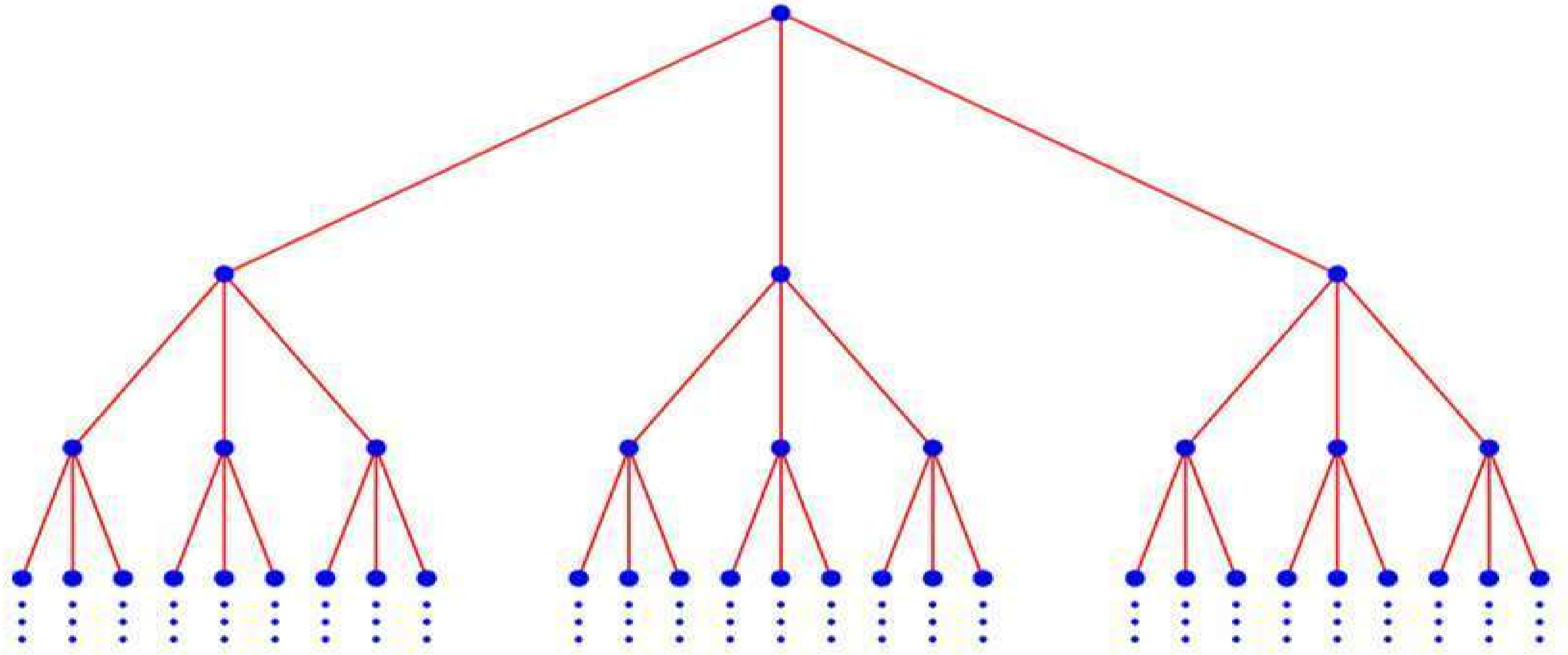
$$\text{Tree} = \{\forall x(\neg xRx), \forall x, y(xRy \to yRx)\}$$
$$\cup \left\{ \neg\exists x_1, \ldots, x_n \left( \bigwedge_{1 \le i < j \le n} (x_i \neq x_j) \wedge \bigwedge_{i=1}^{n-1} (x_i R x_{i+1}) \wedge x_n R x_1 \right) : n \geq 3 \right\}.$$

## Question

- *(May be too wide) Which kind of infinite trees are pseudofinite?*
- *(perhaps less wide) Is every infinite tree of bounded diameter pseudofinite?*

# Pseudofiniteness in countable trees

Example of a countable tree that is not pseudofinite.



$$\sigma_{(1;3,4)} := \exists x\,[\deg(x) = 3 \wedge \forall y(y \neq x \rightarrow \deg(y) = 4)]$$

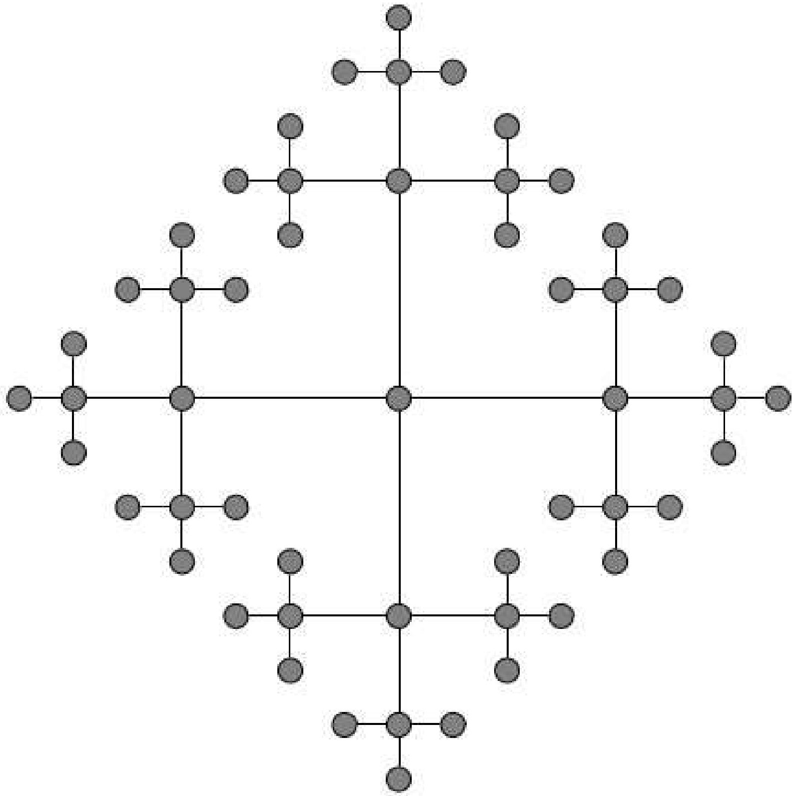
This sentence does not have finite models, due to the *Handshaking lemma*:

odd

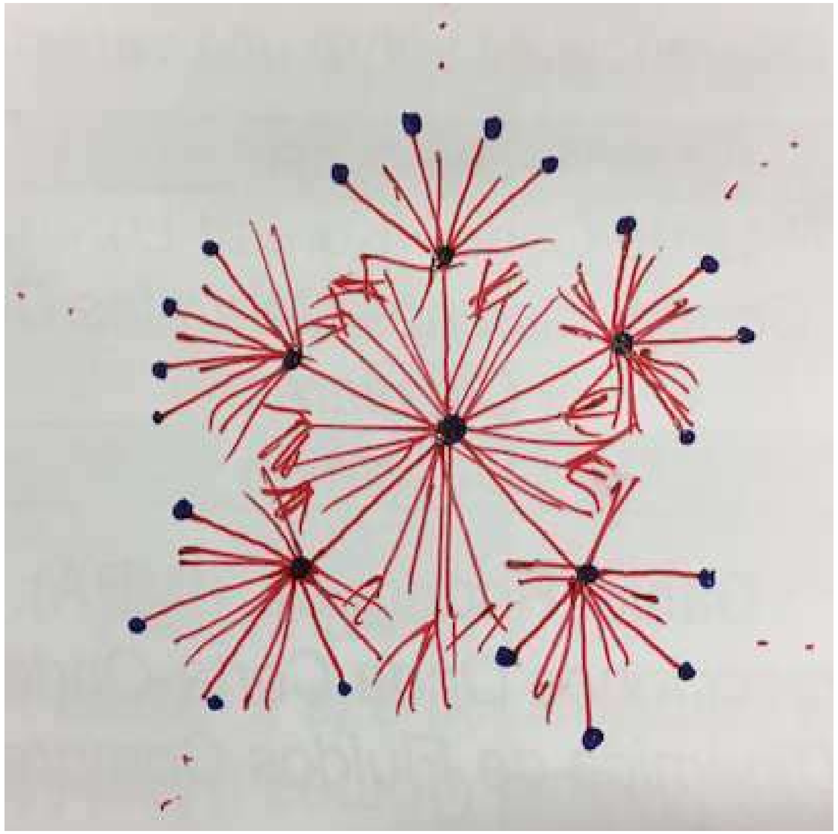
$$3 + 4(n-1) = \sum_{v \in V} \deg(v) = 2|E(G)|.$$ even.

# The $r$-regular and the everywhere infinite forest

The theory $T_r$ is the theory of an infinite tree such that every vertex has degree $r$. The theory $T_\infty$ (also known as the theory of the *everywhere infinite forest*) is the theory of an infinite tree in which every vertex has infinite degree.



$T_4$



$T_\infty$ (artistic representation)

- Both $T_r$ and $T_\infty = \text{Tree} \cup \{\forall x\, \exists^{\geq n} y(xRy) : n \geq 1\}$ are complete theories, and have quantifier elimination in the language $\mathcal{L}' = \{D_n : n \geq 0\}$, where $D_n(x, y) \Leftrightarrow \text{dist}(x, y) = n$.
- The theory $T_r$ is strongly minimal. Moreover, for every $M \models T_r$ and $A \subseteq M$,

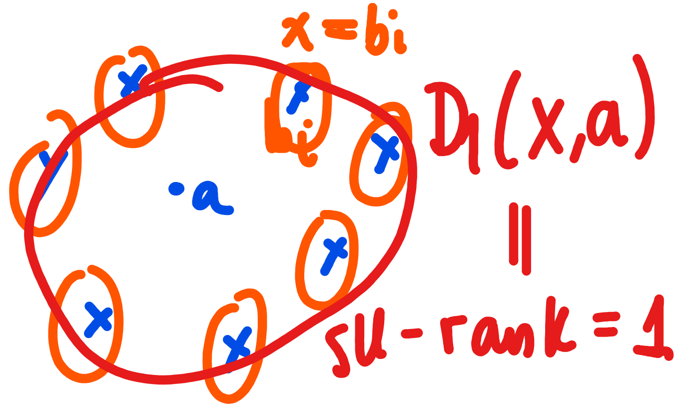
trivial.

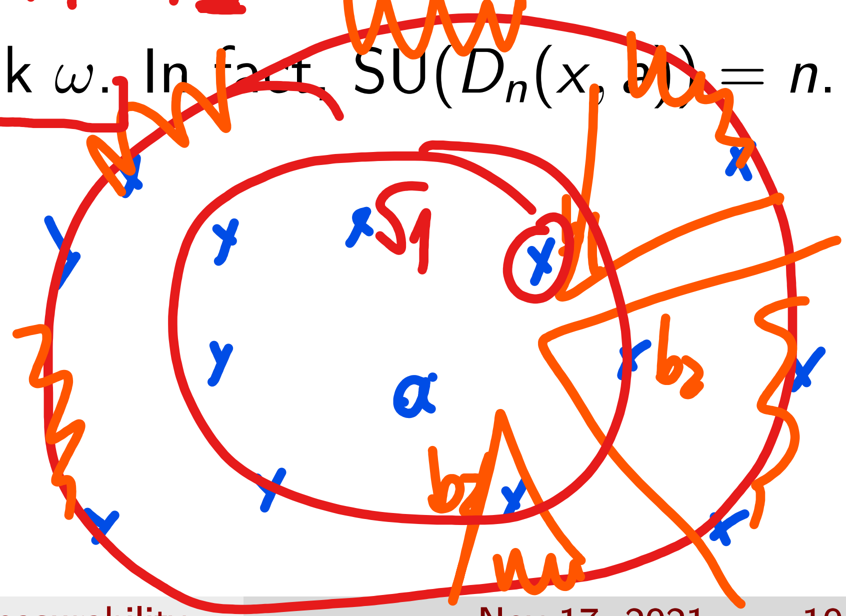
$$\text{acl}_M(A) = \bigcup_{a \in A} \text{acl}_M(a) = \text{connected components of A}$$

$\wedge\, D_2(x,a) \wedge D_1(x,b_i)$  $D_2(x,a)$

- The theory $T_\infty$ is $\omega$-stable of SU-rank $\omega$. In fact, SU$(D_n(x, a)) = n$.

$x = y$



**Theorem (G., Robles)**

The theories $T_r$ and $T_\infty$ are both pseudofinite.

**Proposition**

Let $\mathcal{C} = \{G_n : n \in \mathbb{N}\}$ be a class of finite graphs such that:

(a) Each graph $G_n$ is $r$-regular (resp. $d_n$-regular)

(b) $\mathrm{girth}(G_n) \to \infty$

Then, every infinite ultraproduct $M$ of graphs in $\mathcal{C}$ is a model of $T_r$ (resp. a model of $T_\infty$ if $d_n \to \infty$.

$G \longrightarrow L[G] :=$ lifting of $G$

r-regular $\longrightarrow$ r-regular

$\mathrm{girth}(G) = k \longrightarrow \mathrm{girth}(L[G]) = 2k$

$K_{r+1}$

$V(L(G)) = V(G) \times \{0,1\}^{E(G)}$

# Why study pseudofinite structures?

- If $M = \prod_{\mathcal{U}} M_i$ is an ultraproduct of finite structures, every definable set $\varphi(M^n; \overline{b})$ has a *non-standard cardinality*

$$\boxed{|\varphi(M^n; \overline{b})| = [|\varphi(M_i^n; \overline{b}_i)|]_{\mathcal{U}} \in \mathbb{R}^{\mathcal{U}}.}$$

- The counting measure on a class of finite structures can be lifted using Łoś' theorem to give notions of dimension and measure on their ultraproduct.

→ probability measure

$$\mu_D(X) = st\left(\frac{|X \cap D|}{|D|}\right)$$

$|D| \neq 0$

Di Nasso, Goldbring, Lupini

(BOOK)

- This kind of finite/infinite connection can sometimes be used to prove qualitative properties of large finite structures.
  - Szemerédi's Regularity (Goldbring, Towsner)
  - Freiman conjecture for non-abelian groups (Hrushovski)
  - Expanders maps in finite fields (Tao)
  - Stable graphs and Erdős-Hajnal conjecture (Malliaris, Shelah / Chernikov, Starchenko)

# Strongly minimal ultraproducts of finite structures

**Theorem (Pillay, 2015)**

Let $M = \prod_{\mathcal{U}} M_i$ be a **strongly minimal** ultraproduct of finite structures, and let $\alpha \in \mathbb{N}^*$ be the pseudofinite cardinality of $M$ ($\alpha = \underbrace{|M|}$). Then,

1. For any definable (with parameters) set $X \subseteq M^n$, there is a polynomial $P_X(x) \in \mathbb{Z}[x]$ with positive leading coefficient such that $|X| = P_X(\alpha)$. Moreover, $RM(X) = \text{degree}(P_X)$.
2. For any $L$-formula $\varphi(\overline{x}, \overline{y})$ there is a finite number of polynomials $P_1, \ldots, P_k \in \mathbb{Z}[x]$ and $L$-formulas $\psi_1(\overline{y}), \ldots, \psi_k(\overline{y})$ such that:
   (a) $\{\psi_i(\overline{y}) : i \leq k\}$ is a partition of the $\overline{y}$-space.
   (b) For any $\overline{a}$, $|\varphi(M^{|x|}; \overline{a})| = P_i(\alpha)$ if and only if $M \models \psi_i(\overline{a})$.

For instance, if $M = (\mathbb{R}, +) = \prod_{\mathcal{U}}(\mathbb{Z}/p\mathbb{Z}, +)$, and we consider the formula $\varphi(x_1; y_1, y_2) : \ x = y_1 \vee x \neq y_2$, we have

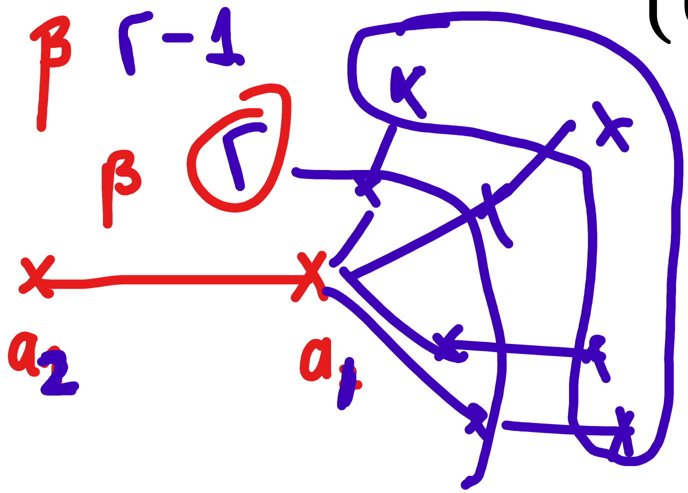
$$|\varphi(M; a_1, a_2)| = \begin{cases} \alpha & \text{if } a_1 = a_2, \\ \underbrace{\alpha - 1} & \text{if } a_1 \neq a_2. \end{cases}$$

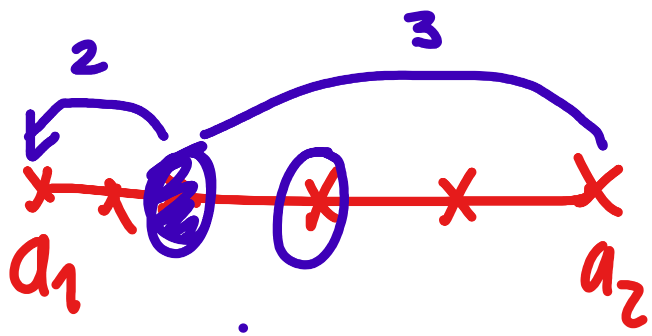
Consider the theory $T_r$ and an ultraproduct of finite graphs $M \models T_r$,

- For the formula $\varphi_1(x; y_1, y_2) := D_2(x, y_1) \wedge D_3(x, y_2)$

$$|\varphi_2(M, a_1, a_2)| = \begin{cases} r(r-1) & \text{if } M \models D_1(a_1, a_2) \\ 1 & \text{if } M \models D_5(y_1, y_2) \\ 0 & \text{if } M \models \neg D_1(a_1, a_2) \wedge D_5(a_1, a_2) \end{cases}$$

$\to \psi_i(\bar{y})|$




- For the formula $\varphi_3(x; y_1, y_2) := \neg D_2(x, y_1) \wedge \neg D_3(x, y_2)$

$$|\varphi_3(M, a_1, a_2)| = \alpha - |\varphi_1(M, a_1, a_2)|$$

# Asymptotic classes

**Definition (Macpherson, Steinhorn)** → 2008

*Let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures. We say that $\mathcal{C}$ is a 1-dimensional asymptotic class if for every $\mathcal{L}$-formula $\varphi(x, \bar{y})$ there is a positive constant $C_\varphi > 0$ and a finite set $E_\varphi \subseteq \mathbb{R}^{>0}$ such that the following hold:*

(a) *For every $M \in \mathcal{C}$ and $\bar{a} \in M^{\bar{y}}$, either $\underbrace{|\varphi(M; \bar{a})| \leq C}$ or there is $\mu \in E$ such that*

$$\underbrace{||\varphi(M, \bar{a})| - \mu|M|| \leq C|M|^{1/2}}. \quad (*)$$

$o(|M|)$

$$\frac{|\varphi|}{|M|} \approx \mu$$

(b) *For every $\mu \in E$ there is a formula $\psi_\mu(\bar{y})$ such that for every $M \in \mathcal{C}$, and $\bar{a} \in M^{|y|}$, $M \models \psi_\mu(\bar{a}) \Leftrightarrow (*)$ holds.*

Elwes (2007): notion of $N$-dimensional asymptotic classes, with dimensions $0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N}{N}$ for formulas in one variable.

# Examples of asymptotic classes

$\{ (\mathbb{Z}/n\mathbb{Z}, +) \}$

- Finite fields. (Chatzidakis, van den Dries, Macyntire)
- Finite cyclic groups. (Macpherson, Steinhorn, based on Szmielew)
- Finite simple groups of fixed Lie type. (Elwes)
- Finite fields with a Frobenius automorphism. (Ryten)
- **Paley graphs**: $V = \mathbb{F}_q$ $\quad xRy \quad x - y$ is a non-zero square

**Theorem (Bollobás, Thomason - 1985)**

Let $U, W$ be disjoint subsets of $\mathbb{F}_q$ ($q \equiv 1 \pmod 4$), such that $|U \cup W| = m$, and let $S$ be the set of non-zero squares in $\mathbb{F}_q$. Let $v(U, W)$ be the set of elements $x \in \mathbb{F}_q$ such that $x - U \subseteq S$ and $x - W \subseteq \mathbb{F}_q \setminus S$.

Then, $\left||v(U,W)| - \frac{q}{2^m}\right| \leq \frac{1}{2}(m - 2 + 2^{-m+1})q^{\frac{1}{2}} + \frac{m}{2}$.

- Some classes of residue rings, e.g. $\{(\mathbb{Z}/p^d\mathbb{Z}, +, \cdot, 0, 1) : p \text{ prime}\}$. (Bello-Aguirre)

d-dimensional asymptotic class

# Ultraproducts of asymptotic classes

**Theorem (Macpherson, Steinhorn)**

1. If every ultraproduct of a class $\mathcal{C}$ is strongly minimal, then $\mathcal{C}$ is a 1-dimensional asymptotic class.
2. Every infinite ultraproduct of structures in a 1-dimensional asymptotic class is supersimple of SU-rank 1.

Similarly, every ultraproduct of an $N$-dimensional asymptotic class is supersimple of finite rank ($\leq N$).

**Idea:** Each instance of dividing for formulas in one variable is witnessed by a drop of dimension.

In ultraproducts of asymptotic classes, there are only finitely many possible dimensions.

In general, the infinite ultraproducts of asymptotic classes are examples of **measurable structures**: structures of finite SU-rank where there is a well-defined notion of dimension and measure for definable sets, satisfying definibility and additivity properties.

# Multidimensional asymptotic classes

**Definition (Anscombe, Macpherson, Steinhorn, Wolf)**

*Let $\mathcal{C}$ be a class of finite structures and let $R$ be any set of functions $\mathcal{C} \to \mathbb{R}^{\geq 0}$. We say that $\mathcal{C}$ is an $R$-**multidimensional asymptotic class** (or an $R$-**m.a.c.** if for every formula $\varphi(\overline{x}, \overline{y})$ there is a finite $\emptyset$-definable partition $\varphi$ of $(\mathcal{C}, \overline{y})$ and a set $H_\varphi := \{h_P \in R : P \in \varphi\}$ of functions such that*

$$||\varphi(M^{|\overline{x}|}, \overline{a}) - h_P(M)| = o(h_P(M))$$

*for $(M, \overline{a}) \in P$, as $|M| \to \infty$.*
*In addition, we say that $\mathcal{C}$ is an $R$-m.e.c (multidimensional **exact** class) if in the condition above we have $|\varphi(M^{|\overline{x}|}|, \overline{a})| = h_P(M)$.*

There is a corresponding notion for **generalized measurable structures**, and it turns out that every ultraproduct of structures in an $R$-mac is a generalized measurable structure.

# Examples of macs

1. (G., Macpherson, Steinhorn) The class of 2-sorted structures $(V, \mathbb{F}_q)$ with $V$ a finite-dimensional vector space over $\mathbb{F}_q$. Given a formula $\varphi(\overline{x}, \overline{y})$ there is a finite set $E_\varphi$ of polynomials $g(\boldsymbol{V}, \boldsymbol{F})$ with coefficients in $\mathbb{Q}$ such that if $M = (V, F)$, then $h_P(M)$ has the form $g(|V|, |F|)$ for some $g \in E_\varphi$.
   The ultraproducts of structures in this class are supersimple, but the $V$-sort may have rank $\omega$.
2. (Bello Aguirre) In the language of rings, for a fixed $d \in \mathbb{N}$ we can consider the class $C_d$ of all residue rings $\mathbb{Z}/n\mathbb{Z}$ where $n$ is the product of powers of at most $d$ primes, each with exponent at most $d$. Then $\mathcal{C}_d$ is a m.a.c. (after an appropriate expansion by unary predicates).

Recall here that the class $\{(\mathbb{Z}/p^d\mathbb{Z} : p \text{ prime}\}$ is a $d$-dimensional asymptotic class.

## Theorem (G., Robles)

Let $\mathcal{C} = \{G_n : n \in \mathbb{N}\}$ be a class of finite graphs such that each graph $G_n$ is $d_n$-regular and $d_n, \mathrm{girth}(G_n) \to \infty$.

$\alpha = |M|$ $\beta = D_1(M, a)$

Let $M$ be an infinite ultraproduct of graphs in $\mathcal{C}$ (a model of $T_\infty$) and fix the non-standard integers $\alpha = |M|$ and $\beta = [d_n]_{\mathcal{U}}$. Then for every formula $\varphi(\overline{x}, \overline{y})$ in the language of graphs there is a finite number of polynomials $p_1(X, Y), \ldots, p_k(X, Y) \in \mathbb{Z}[X, Y]$ such that:

1. For every $\overline{a} \in M^{|\overline{y}|}$, $|\varphi(M^{|\overline{x}|}, \overline{a})| = p_i(\alpha, \beta)$ for some $i \leq k$.
2. Moreover, there are formulas $\psi_1(\overline{y}), \ldots, \psi_k(\overline{y})$ such that for every $\overline{a} \in M^{|\overline{y}|}$,
$$M \models \psi_i(\overline{a}) \Leftrightarrow |\varphi(M^{|\overline{x}|}, \overline{a})| = p_i(\alpha, \beta).$$

This is enough to show that any class of graphs with the properties above is a multidimensional exact class.

# Final remarks/questions

- Which nice classes of graphs satisfy the conditions described to obtain models of $T_r$, $T_\infty$ as ultraproducts? (Ramanujan graphs, expanders, etc.) [work in progress with Melissa Robles]
- In famous examples of pseudofinite structures, what can we say about the classes of finite structures approximating them?
  For instance, is $\mathcal{M}_\alpha$ (the generic limit of the class of graphs with predimension $\delta_\alpha(X) = |X| - \alpha|R(X)|$) a generalized measurable structure?
- What kind of measurability properties are preserved when we apply different constructions (H-structures, lovely pairs) to ultraproducts of finite structures? [some progress here with A. Berenstein and T. Zou]

**Example (Anscombe)**

*If $M$ is the Fraïssé limit of a free amalgamation class then $M$ is generalized measurable. (note for example that the generic triangle-free graph is an example, that has a TP1 and TP2 theory)*

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