Logical Inference in Constructive Theories

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1 Introduction

A formal theory often consists of two things:

a logic	*the logical (schematic) principles of reasoning*
axioms (and rules)	*the content, "what the theory is about"*

Ex

Peano Arithmetic PA:

classical first-order logic CQC & axioms describing $+, \times, S, 0$:

$$x + 0 = x$$
$$x \times 0 = 0$$
$$\vdots$$

Heyting Arithmetic HA:

intuitionistic first-order logic IQC & the nonlogical axioms of PA.

Constructive Zermelo-Fraenkel Set Theory CZF:

intuitionistic first-order logic IQC & the set-theoretic axioms of CZF: extensionality, pairing, union, ...

Ideally, nonlogical axioms do not affect the logic of a theory: no new logical principles (propositional or predicate) become valid by adding the nonlogical axioms to the theory.

Counterexample (Diaconescu 1975)
If T consists of the set-theoretic axioms of union, pairing, separation and extensionality and LEM denotes the law of excluded middle ($\varphi \vee \neg\varphi$), then

$$\mathrm{IQC} + \mathrm{T} \nvdash \mathrm{LEM} \qquad \mathrm{IQC} + \mathrm{T} + \text{Axiom of Choice} \vdash \mathrm{LEM}.$$

Consider $x = \{0\} \cup \{1 \mid \varphi\}$ and $y = \{1\} \cup \{0 \mid \varphi\}$. The values of a choice function on x, y decide whether $x = y$ and thus whether φ.

Question What *is* the logic of a theory?

Def (In this talk) a *constructive (classical) theory* T consists of intuitionistic (classical) predicate logic IQC (CQC) plus nonlogical axioms. $\mathcal{F}_{\mathrm{T}}$ denotes the set of formulas in the language of T.

IPC (CPC) denotes intuitionistic (classical) propositional logic. The set of propositional formulas is denoted $\mathcal{F}_{\mathrm{prop}}$.

2 The Logical Principles of Theories

Def The *propositional logic* of a theory T:

$$\mathrm{PropL}(\mathrm{T}) \equiv_{df} \{\varphi \in \mathcal{F}_{\mathrm{prop}} \mid \forall\sigma : \mathrm{T} \vdash \sigma\varphi\}.$$

($\mathcal{F}_{\mathrm{prop}}$ is the set of propositional formulas, σ ranges over substitutions, i.e. $\sigma : \mathcal{F}_{\mathrm{prop}} \to \mathcal{F}_{\mathrm{T}}$ is a map that commutes with the connectives.)

Ex If $\mathrm{T} = \mathrm{PA}$ and $\sigma(p)$ is the sentence $0 < 1$:

$$\sigma(\neg p) = \neg(0 < 1) \quad \neg p \notin \mathrm{PropL}(\mathrm{PA}) \quad (\neg\neg p \to p) \in \mathrm{PropL}(\mathrm{PA}).$$

Fact For classical theories T: $\mathrm{PropL}(\mathrm{T}) = \mathrm{CPC}$.

Fact For constructive theories T: $\mathrm{IPC} \subseteq \mathrm{PropL}(\mathrm{T})$.
Not always $\mathrm{PropL}(\mathrm{T}) = \mathrm{IPC}$.

Question Given a constructive theory T, does $\mathrm{PropL}(\mathrm{T}) = \mathrm{IPC}$ hold?

Thm (de Jongh 1970) $\mathrm{PropL}(\mathrm{HA}) = \mathrm{IPC}$.

Thm (Rose 1953) $\mathrm{PropL}(\mathrm{HA} + \mathrm{MP} + \mathrm{ECT}_0) \neq \mathrm{IPC}$.
(MP is Markov's Principle, ECT_0 is Extended Church Thesis)

Previous results are from 1950-1980s, and for arithmetical theories.
Rest of the talk: recent results for constructive set theories.

Previous results are about the logical *principles* of a theory.
Rest of the talk: extension to the logical *inferences* of a theory.

This talk:

1 Introduction
2 The logical priciples of a theory
3 Constructive set theories
3 The logical inferences of a theory
5 Final thoughts and open problems

3 Constructive Set Theories

Constructive set theories are set theories based on intuitionistic predicate logic IQC.

Def IZF,CZF and IKP: the axioms of extensionality, empty set, pairing, union, set induction and further

IZF	CZF	IKP
separation	bounded separation	bounded separation
collection	strong collection	bounded collection
strong infinity	strong infinity	infinity
power set	subset collection	

set induction	$\big(\forall a(\forall x \in a \varphi(x) \to \varphi(a))\big) \to \forall a \varphi(a)$
bounded collection	$\forall x \in a \exists y \varphi(x,y) \to \exists b \forall x \in a \exists y \in b \varphi(x,y)$ (φ is bounded)

Aim: For constructive set theories T determine their propositional logic, i.e.

$$\mathrm{PropL}(\mathrm{T}) = \{\varphi \in \mathcal{F}_{\mathrm{prop}} \mid \forall \sigma : \mathrm{T} \vdash \sigma\varphi\}.$$

($\sigma : \mathcal{F}_{\mathrm{prop}} \to \mathcal{F}_{\mathrm{T}}$ is a map that commutes with the connectives.)

Note For two constructive theories $\mathrm{T}_1 \subseteq \mathrm{T}_2$:

$$\mathrm{PropL}(\mathrm{T}_2) = \mathrm{IPC} \quad \Rightarrow \quad \mathrm{PropL}(\mathrm{T}_1) = \mathrm{IPC}.$$

Thm (Paßmann 2018)
$\mathrm{PropL}(\mathrm{IZF}) = \mathrm{PropL}(\mathrm{CZF}) = \mathrm{IPC}.$

Thm (Paßmann 2019)
For any intermediate logic L characterized by a class of finite trees:

$$\mathrm{PropL}(\mathrm{IZF} + \mathrm{L}) = \mathrm{PropL}(\mathrm{CZF} + \mathrm{L}) = \mathrm{L}.$$

Thm (I. and Paßmann 2020)
For any Kripke-complete intermediate logic L:

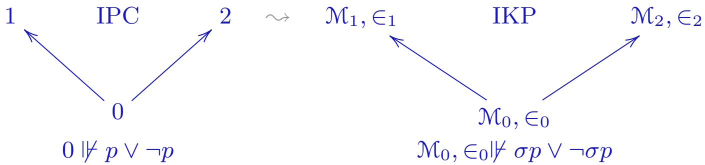
$$\mathrm{PropL}(\mathrm{IKP} + \mathrm{L}) = \mathrm{L}.$$

In particular, $\mathrm{PropL}(\mathrm{IKP}) = \mathrm{IPC}$.

Thm $\mathrm{PropL(IKP)} = \mathrm{IPC}$.

Prf Given $\mathrm{IPC} \nvdash \varphi$, find σ such that $\mathrm{IKP} \nvdash \sigma\varphi$.

From an IPC model that refutes φ, construct an IKP model that refutes $\sigma\varphi$ for some σ. E.g.



Observation Δ_0-formulas are evaluated locally.

Thm (I. 2010) A Kripke model with classical domains (the $\mathcal{M}_i$ are transitive models of ZF) is a model of IKP.

Thm (I. and Paßmann 2020)

- $\mathrm{PropL}(\mathrm{IKP}) = \mathrm{IPC}$.
- $\mathrm{PredL}(\mathrm{IKP}) = \mathrm{IQC}$
- $\mathrm{PredL}(\mathrm{IKP} + \mathrm{equality})$ is stronger than IQC with equality.

Thm (Friedman & Scedrov 1986)
If a set theory T contains the axioms of extensionality, separation, pairing and union, then the predicate logic of T ($\mathrm{PredL}(\mathrm{T})$) is a proper extension of IQC.

Cor $\mathrm{PredL}(\mathrm{IZF})$ is a proper extension of IQC.

Open: $\mathrm{PredL}(\mathrm{CZF}) = \mathrm{IQC}$?

4 The Logical Inferences of Theories

Def A propositional rule Γ/φ is *admissible* in a theory or logic T ($\Gamma \mathrel{|\!\sim}_{\mathrm{T}} \varphi$) if for all substitutions $\sigma : \mathcal{F}_{\text{prop}} \to \mathcal{F}_{\mathrm{T}}$:

$$\mathrm{T} \vdash \sigma(\bigwedge \Gamma) \ \Rightarrow \ \mathrm{T} \vdash \sigma\varphi.$$

The *propositional admissible rules* or *logical inferences* of T are

$$\mathrm{AR(T)} \equiv_{df} \{\Gamma/\varphi \mid \Gamma \mathrel{|\!\sim}_{\mathrm{T}} \varphi\}.$$

Ex $\neg\neg\varphi \mathrel{|\!\sim}_{\mathrm{ZF}} \varphi$ and not $\neg\neg\varphi \mathrel{|\!\sim}_{\mathrm{CZF}} \varphi$.

Fact

$$\varphi \in \mathrm{PropL(T)} \text{ if and only if } \top/\varphi \in \mathrm{AR(T)}.$$

$\varphi \to \psi \in \mathrm{PropL(T)}$ implies $\varphi/\psi \in \mathrm{AR(T)}$ (*derivable rule*), but not vice versa.

Are there constructive theories with nonderivable admissible rules?

Ex Yes: $\mathrm{IKP} \nvdash (\neg\varphi \to \psi \vee \neg\psi') \to (\neg\varphi \to \psi) \vee (\neg\varphi \to \psi')$

$$\neg\varphi \to \psi \vee \neg\psi' \mathrel{|\!\sim}_{\mathrm{IKP}} (\neg\varphi \to \psi) \vee (\neg\varphi \to \psi') \qquad \text{(Harrop Rule)}$$

Intuition $\Gamma \mathrel{|\!\sim}_{\mathrm{T}} \varphi$: Adding the rule Γ/φ to T does not change the theorems of T.
The theorems of T and $\mathrm{T} + \Gamma/\varphi$ are equal.

$$\mathrm{PropL}(\mathrm{T}) \equiv_{df} \{\varphi \mid \forall \sigma : \vdash_{\mathrm{T}} \sigma\varphi\}.$$

$$\mathrm{AR}(\mathrm{T}) \equiv_{df} \{\Gamma/\varphi \mid \Gamma \mathrel{|\!\sim}_{\mathrm{T}} \varphi\} = \{\Gamma/\varphi \mid \forall \sigma : \mathrm{T} \vdash \sigma(\bigwedge \Gamma) \Rightarrow \mathrm{T} \vdash \sigma\varphi\}.$$

Aim For constructive set theories T, describe $\mathrm{AR}(\mathrm{T})$, and thereby $\mathrm{PropL}(\mathrm{T})$.

Sub aim Establish whether $\mathrm{PropL}(\mathrm{T}) = \mathrm{IPC}$ and $\mathrm{AR}(\mathrm{T}) = \mathrm{AR}(\mathrm{IPC})$.
(The latter implies the former.)

Also logics can have nondrivable admissible rules.

Thm (Harrop 1960) IPC has nonderivable admissible rules.

Prf IPC: $\nvdash_{\mathrm{IPC}} (\neg\varphi \to \psi \vee \neg\psi') \to (\neg\varphi \to \psi) \vee (\neg\varphi \to \psi')$

$$\neg\varphi \to \psi \vee \neg\psi' \mathrel{|\!\sim}_{\mathrm{IPC}} (\neg\varphi \to \psi) \vee (\neg\varphi \to \psi') \qquad \text{(Harrop Rule)}$$

Thm (Visser '99) $\mathrel{|\!\sim}_{\mathrm{HA}} = \mathrel{|\!\sim}_{\mathrm{IPC}}$. $\mathrm{AR(HA)=AR(IPC)}$

Thm (Carl, Galeotti, and Paßmann 2020) $\mathrel{|\!\sim}_{\mathrm{IKP}} = \mathrel{|\!\sim}_{\mathrm{IPC}}$.

Thm (I. and Paßmann 2019) $\mathrel{|\!\sim}_{\mathrm{IPC}} = \mathrel{|\!\sim}_{\mathrm{IZF}_R}$.
(IZF_R is IZF in which Replacement replaces Collection)

Open: $\mathrel{|\!\sim}_{\mathrm{HA+MP}} = \mathrel{|\!\sim}_{\mathrm{IPC}}$? $\mathrel{|\!\sim}_{\mathrm{CZF}} = \mathrel{|\!\sim}_{\mathrm{IPC}}$?

Thm $\mathrel{|\!\sim}_{\mathrm{HA+MP+ECT_0}} \neq \mathrel{|\!\sim}_{\mathrm{IPC}}$.

Thm (Carl, Galeotti, and Paßmann 2020) $\mid\!\sim_{\mathrm{IKP}} = \mid\!\sim_{\mathrm{IPC}}$.

Thm (I. and Paßmann 2019) $\mid\!\sim_{\mathrm{IPC}} = \mid\!\sim_{\mathrm{IZF}_R}$.
(IZF_R is IZF in which Replacement replaces Collection)

Def The *Visser Rules* V form an infinite collection of rules, generalizing the Harrop Rule. A theory T has the *disjunction property* DP if for any φ, ψ:

$$\mathrm{T} \vdash \varphi \vee \psi \ \Rightarrow \ \mathrm{T} \vdash \varphi \text{ or } \mathrm{T} \vdash \psi.$$

Thm (I. and Roziére independently)
The Visser Rules axiomatize the admissible rules of IPC.

Thm (I. 2005)
If a theory T has DP, all rules in V are admissible and $\mathrm{PropL}(\mathrm{T}) = \mathrm{IPC}$, then $\mid\!\sim_{\mathrm{IPC}} = \mid\!\sim_{\mathrm{T}}$.

Cor If the rules in V are admissible in IKP and IZF_R, then

$$\mid\!\sim_{\mathrm{IPC}} = \mid\!\sim_{\mathrm{IKP}} = \mid\!\sim_{\mathrm{IZF}_R}.$$

Thm (Carl, Galeotti, and Paßmann 2020) $\mathrel{|\!\!\sim}_{\mathrm{IKP}} = \mathrel{|\!\!\sim}_{\mathrm{IPC}}$.

Thm (I. and Paßmann 2019) $\mathrel{|\!\!\sim}_{\mathrm{IPC}} = \mathrel{|\!\!\sim}_{\mathrm{IZF}_R}$.

Prf In both cases it suffices to show that the rules in V are admissible.

For IZF_R, use semantical methods as before.

For IKP, use realizability.

$\dashv$

Final thoughts and (some) open problems

- Are $\mathrel|\joinrel\sim_{\mathrm{IZF}}$ and $\mathrel|\joinrel\sim_{\mathrm{CZF}}$ equal to $\mathrel|\joinrel\sim_{\mathrm{IPC}}$?
- What are the predicate admissible rules of set theories?
  (Visser proved that those of HA are Π_2-complete)
- How important is it that $\mathrm{PropL(T)} = \mathrm{IPC}$ or $\mathrel|\joinrel\sim_{\mathrm{T}} = \mathrel|\joinrel\sim_{\mathrm{IPC}}$ for constructive theories T?

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