Variants of Ramsey's theorem over RCA_0^*

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Second order arithmetic

Language $\mathcal{L}_2$: first-order variables: $x, y, z, \ldots$, second-order variables: $X, Y, Z, \ldots$, non-logical symbols: $0, 1, +, \cdot, \exp, <, \in$.

Models: $(M, \mathcal{X})$, where $\mathcal{X} \subseteq \mathcal{P}(M)$.

Arithmetical hierarchy: Σ^0_n, Π^0_n allow set parameters; Σ_n, Π_n are purely first-order; $\Sigma_n(A), \Pi_n(A)$ contain only one distinguished set parameter A.

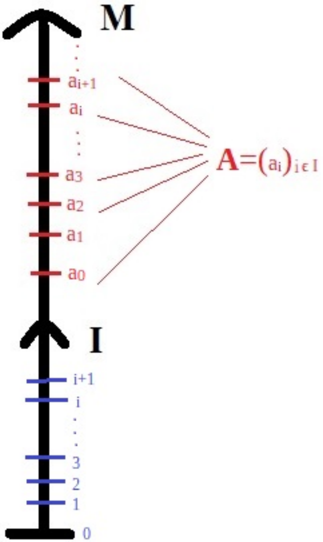
The traditional base theory RCA_0:

1. basic properties of $+$, $\cdot$, exp, $<$ etc.,
2. comprehension scheme for Δ^0_1-formulas,
3. induction scheme for Σ^0_1-formulas ($\mathrm{I}\Sigma^0_1$).

RCA^*_0 is obtained from RCA_0 by replacing Σ^0_1-induction with Δ^0_1-induction + exp.

exp = „2^x is a total function"

Failure of Σ^0_1-induction



I is a Σ^0_1-definable proper cut.

A is an unbounded set enumerated in increasing order by the cut I. Its cardinality is strictly smaller then $\mathbb{N}$.

Two notions of an infinite set:

- A set S is *unbounded* if for every x there exists $y \in S$ with $y \geq x$.
- A set S is *of cardinality* $\mathbb{N}$ if there exists a bijection from $\mathbb{N}$ to S.

Ramsey-theoretic principles

$\mathsf{RT}^2_2 =$ for every $c\colon [\mathbb{N}]^2 \to 2$ there exists an infinite set $S \subseteq \mathbb{N}$ such that c is constant on $[S]^2$.

$\mathsf{CAC} =$ For every partial order $(\mathbb{N}, \preceq)$ there exists an infinite set $S \subseteq \mathbb{N}$ which is a $\preceq$-chain or $\preceq$-antichain.

$\mathsf{ADS} =$ For every linear order $(\mathbb{N}, \preceq)$ there exists an infinite set $S \subseteq \mathbb{N}$ which is an $\preceq$-ascending or $\preceq$-descending sequence.

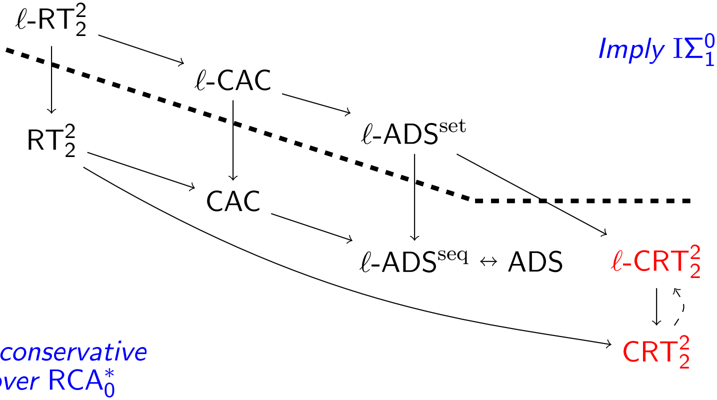
$\mathsf{CRT}^2_2 =$ for every $c\colon [\mathbb{N}]^2 \to 2$ there exists an infinite $S \subseteq \mathbb{N}$ such that $c \restriction S$ is stable, i.e. for every $x \in S$ there exists $y \in S$ such that for all $z \in S$ if $z \geq y$, then $c(x, y) = c(x, z)$.

$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \Rightarrow \mathsf{CAC} \Rightarrow \mathsf{ADS} \Rightarrow \mathsf{CRT}^2_2$

More beasts in the reverse-mathematical zoo

Normal versions: *infinite* = *unbounded*
RT^2_2, CAC, ADS, CRT^2_2

Long versions: *infinite* = *of cardinality* $\mathbb{N}$
ℓ-RT^2_2, ℓ-CAC, ℓ-$\mathsf{ADS}^{\mathrm{set}}$, ℓ-$\mathsf{ADS}^{\mathrm{seq}}$, ℓ-CRT^2_2



Normal versions

$$\mathrm{Cod}(M/I) = \{X \subseteq I \colon \exists s \in M \ (s)_{\mathsf{Ack}} \cap I = X\}$$

Theorem

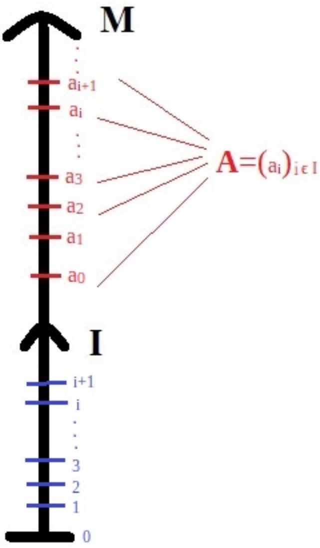
Let P be one of $\mathsf{RT}^2_2, \mathsf{CAC}, \mathsf{ADS}, \mathsf{CRT}^2_2$. For every $(M, \mathcal{X}) \vDash \mathsf{RCA}^*_0$ and every proper Σ^0_1-definable cut $I \subseteq M$, it holds that

$$(M, \mathcal{X}) \vDash P \quad \textit{iff} \quad (I, \mathrm{Cod}(M/I)) \vDash P.$$

Consequences

- Each of the principles $\mathsf{RT}^2_2, \mathsf{CAC}, \mathsf{ADS}, \mathsf{CRT}^2_2$ can be satisfied in a model of the form $(M, \Delta_1\text{-}\mathrm{Def}(M))$.
- $\mathsf{RT}^2_2, \mathsf{CAC}, \mathsf{ADS}$ are not Π_4- and CRT^2_2 is not Π_5-conservative over RCA^*_0.
- Each of the principles $\mathsf{RT}^2_2, \mathsf{CAC}, \mathsf{ADS}, \mathsf{CRT}^2_2$ is Π^0_3-conservative over RCA^*_0.

Idea of the proof for $(M,\mathcal{X}) \models \mathsf{CRT}^2_2 \Rightarrow (I, \mathrm{Cod}(M/I)) \models \mathsf{CRT}^2_2$



Let $(M,\mathcal{X}) \models \mathsf{CRT}^2_2$, I be a Σ^0_1-definable cut and $A = \{a_i\}_{i\in I}$ a cofinal set indexed by I. Let $f\colon [I]^2 \to 2$ be a colouring in $\mathrm{Cod}(M/I)$. Define a colouring f' on A by $f'(a_i, a_j) = f(i,j)$ and extend it on the whole M by looking at closest elements of A. Use CRT^2_2 in $(M,\mathcal{X})$ to get an unbounded set S on which f' is stable. Now $S' = \{i \in I\colon S \cap [a_i, a_{i+1}) \neq \emptyset\}$ is in $\mathrm{Cod}(M/I)$ by [Chong-Mourad 1990].

Long versions

One of two different behaviours:

- ℓ-RT^2_2, ℓ-CAC and ℓ-$\mathsf{ADS}^{\mathrm{set}}$ imply $\mathrm{I}\Sigma^0_1$
- ℓ-$\mathsf{ADS}^{\mathrm{seq}}$ and ℓ-CRT^2_2 are Π^0_3-conservative over RCA^*_0.

$\mathsf{RCA}^*_0 \vdash \ell$-$\mathsf{RT}^2_2 \Rightarrow \mathrm{I}\Sigma^0_1$ was observed by Yokoyama in 2013.

Theorem

$\mathsf{RCA}^*_0 \vdash \ell$-$\mathsf{ADS}^{\mathrm{seq}} \Leftrightarrow \mathsf{ADS}$ *and* $\mathsf{WKL}^*_0 \vdash \ell$-$\mathsf{CRT}^2_2 \Leftrightarrow \mathsf{CRT}^2_2$.

$\mathsf{WKL}^*_0 = \mathsf{RCA}^*_0 + \mathsf{WKL}_0$

Growing grouping principle

The **growing grouping principle** GGP^2_2 states that for every colouring $c\colon [\mathbb{N}]^2 \to 2$ there exists a sequence of finite sets $(G_i)_{i\in I}$ such that

1. for every $i < j \in I$ and every $x \in G_i$, $y \in G_j$ it holds that $x < y$,
2. for every $i < j \in I$, the colouring $c \restriction (G_i \times G_j)$ is constant,
3. for every $i \in I$, $|G_i| \leq |G_{i+1}|$ and $\sup_{i\in I} |G_i| = \mathbb{N}$.

Lemma

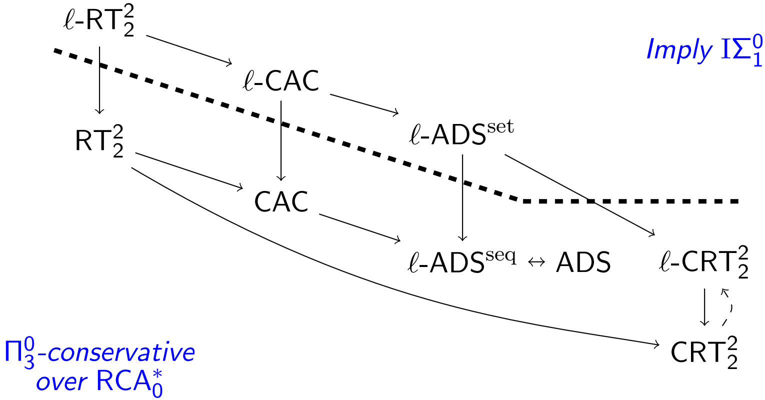
$\mathrm{WKL}^*_0 + \neg \mathrm{I}\Sigma^0_1 \vdash \mathrm{GGP}^2_2$.

GGP^2_2 *restricted to transitive colourings is provable in* $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$.

Proof of $\mathrm{WKL}_0^* \vdash \ell\text{-}\mathrm{CRT}_2^2 \Leftrightarrow \mathrm{CRT}_2^2$

We only have to prove $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma_1^0 \vdash \mathrm{CRT}_2^2 \Rightarrow \ell\text{-}\mathrm{CRT}_2^2$.
Take any $c\colon [\mathbb{N}]^2 \to 2$. Apply GGP_2^2 to obtain a sequence of finite sets $G_0 < G_1 < \ldots < G_i < \ldots$ indexed by some Σ_1^0-cut I. Let $D = \{\min(G_i)\colon i \in I\}$.
Apply CRT_2^2 to $c \restriction D$ and get an unbounded set $S \subseteq D$ on which c is stable. S has the form $\{\min(G_{i_j})\colon j \in J\}$ for some cut $J \subseteq I$.
Now c is also stable on the set $\bigcup_{j \in J} G_{i_j}$, which has cardinality $\mathbb{N}$: $\sup_{j\in J} |G_{i_j}| = \sup_{i \in I} |G_i| = \mathbb{N}$.

Summary



ℓ-RT^2_2
ℓ-CAC
ℓ-$\mathsf{ADS}^{\mathrm{set}}$
Imply $\mathrm{I}\Sigma^0_1$
RT^2_2
CAC
ℓ-$\mathsf{ADS}^{\mathrm{seq}} \leftrightarrow$ ADS
ℓ-CRT^2_2
CRT^2_2
Π^0_3-*conservative*
over RCA^*_0

Cohesiveness Principle

COH: For each sequence $(R_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$, there exists an unbounded set C which is cohesive for $(R_n)_{n\in\mathbb{N}}$ (i.e. for every $i \in \mathbb{N}$ either $C \subseteq^* R_i$ or $C \subseteq^* \overline{R}_i$).

- $\mathsf{RCA}_0^* \vdash \mathsf{COH} \Rightarrow \mathsf{CRT}^2_2$: given a colouring $c\colon [\mathbb{N}]^2 \to 2$ take a cohesive set S for the sequence $\{x \in \mathbb{N}\colon c(n,x) = 0\}_{n\in\mathbb{N}}$. Then the colouring c is stable on S.
- $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \Rightarrow \mathsf{COH}$ (Cholak, Jockusch, Slaman 2001, Mileti 2004).
- COH is Π^1_1-conservative over RCA_0 (Cholak, Jockusch, Slaman 2001).
- $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2 \vdash \mathsf{CRT}^2_2 \Leftrightarrow \mathsf{COH}$ (Hirschfeldt, Shore 2007).

Σ^0_2-separation: For every two disjoint Σ^0_2-definable sets A_0, A_1 there exists a Δ^0_2-definable set B such that $A_0 \subseteq B$ and $A_1 \subseteq \overline{B}$.

Lemma

$\mathrm{RCA}_0^* \vdash \mathrm{COH} \Rightarrow \Sigma^0_2$*-separation.*

$\mathrm{RCA}_0 \vdash \mathrm{COH} \Rightarrow \Sigma^0_2$-separation was proved by Belanger in 2015.

Proof sketch.
Given two Π^0_2-sets A_0, A_1 such that $A_0 \cup A_1 = \mathbb{N}$ we look for a Δ^0_2-set B such that $B \subseteq A_0$ and $\overline{B} \subseteq A_1$.
One can define a computable function $f\colon \mathbb{N} \times \mathbb{N} \to 2$ such that

$$\{s\colon f(x,s) = i\} \text{ is unbounded } \implies x \in A_i.$$

Define a computable sequence of sets $R_x = \{s\colon f(x,s) = 0\}$ and let C be cohesive for this sequence. Put $n \in B$ iff $C \subseteq^* R_n$.

Lemma

$\mathrm{B}\Sigma_1 + \exp$ *proves that there exist two disjoint* Σ_2*-sets that cannot be separated by a* Δ_2*-set.*

Take $A_0 = \{e \in \mathbb{N} : \Phi_e^{0'}(e) = 0\}$ and $A_1 = \{e \in \mathbb{N} : \Phi_e^{0'}(e) = 1\}$ and check that with a careful formalisation of basic computability theory it goes through in $\mathrm{B}\Sigma_1 + \exp$. (Cf. Chong and Yang *The jump of a* Σ_n*-cut*.)

COH is never computably true over RCA_0^* :

Corollary

Every model of the form $(M, \Delta_1\text{-}\mathrm{Def}(M))$ *satisfying* RCA_0^* *does not satisfy* COH.

Theorem

$\mathsf{RCA}_0^* \nvdash \mathsf{RT}_2^2 \Rightarrow \mathsf{COH}$.

There exist models of $\mathsf{RCA}_0^* + \mathsf{RT}_2^2$ of the form $(M, \Delta_1\text{-}\mathrm{Def}(M))$.

Questions

- Does ADS or CAC imply CRT^2_2 over RCA^*_0?
- Does $\mathsf{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ imply GGP^2_2? Is GGP^2_2 equivalent to WKL^*_0 over $\mathsf{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$?
- Are ℓ-CRT^2_2 and CRT^2_2 equivalent over RCA^*_0? Does ℓ-CRT^2_2 follow from RT^2_2?
- Does COH imply $\mathrm{I}\Sigma^0_1$ over RCA^*_0? Is COH Π^0_3-conservative over RCA^*_0?

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Thank you!