Simplicity of the automorphism groups of countable structures

Aleksandra Kwiatkowska

joint work with Filippo Calderoni and Katrin Tent

March 18, 2021

Let G be a Polish group.

Proposition

The following conditions are equivalent:

1. *G is a closed subgroup of $S_\infty = \mathrm{Sym}(X)$ – topological group of all bijections of a countable set X, equipped with the pointwise convergence topology;*
2. *G has a neighbourhood basis of the identity that consists of open subgroups;*
3. *G is an automorphism group of a countable first-order structure;*
4. *G is an automorphism group of a countable homogeneous relational first-order structure.*

Homogeneous structures

Definition

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Example

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- the random graph
- the random poset
- the rational Urysohn metric space

How do we obtain countable homogeneous structures?

A countable family $\mathcal{F}$ of finite structures is a Fraïssé family if:

1. (F1) (hereditary property: HP) if $A \in \mathcal{F}$ and $B \subseteq A$ then $B \in \mathcal{F}$;
2. (F2) (joint embedding property: JEP) for any $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ and embeddings from A to C and from B to C;
3. (F3) (amalgamation property: AP) for $A, B_1, B_2 \in \mathcal{F}$ and embeddings $\varphi_1 \colon A \to B_1$ and $\varphi_2 \colon A \to B_2$, there exist C, and embeddings $\psi_1 \colon B_1 \to C$ and $\psi_2 \colon B_2 \to C$ such that $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$.

Fraïssé's Theorem

Theorem (Fraïssé)

For every Fraïssé family $\mathcal{F}$ there is a unique countable homogeneous structure $\mathbf{M}$ *(called Fraïssé limit), such that the set of finite substructures of* $\mathbf{M}$ *is equal to* $\mathcal{F}$*.*

Examples

Example

- $\mathcal{F}$ = the family of finite linear orders
  Fraïssé limit = rationals with the ordering
- $\mathcal{F}$ = the family of finite graphs
  Fraïssé limit = the random graph
- $\mathcal{F}$ = the family of finite metric spaces with rational distances
  Fraïssé limit = the rational Urysohn metric space

Some history 1

Theorem (Higman, 1955)

The following is the complete list of proper normal subgroups of $\mathrm{Aut}(\mathbb{Q})$*:*

1. $H_1 = \{f : \exists_a \ f \restriction (-\infty, a) = \mathrm{Id}\}$,
2. $H_2 = \{f : \exists_b \ f \restriction (b, \infty) = \mathrm{Id}\}$,
3. $H_1 \cap H_2$.

Some history 2

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Theorem (Macpherson-Tent, 2011)

Let $\mathbf{M}$ *be the Fraïssé limit of a free, transitive and nontrivial amalgamation class, or a random tournament. Then the automorphism group of* $\mathbf{M}$ *is simple.*

Some history 3

Theorem (Tent - Ziegler, 2012, 2013)

Let $\mathbf{M}$ *be the Fraïssé limit of a free, transitive and nontrivial amalgamation class, or the bounded countable Urysohn space. Then the automorphism group of* $\mathbf{M}$ *is simple.*

To prove their theorems, Tent-Ziegler introduced a stationary independence relation.

Free fusion of Fraïssé families

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Fraïssé classes. Let $\mathcal{F}_i$ consist of L_i-structures. Suppose $L_1 \cap L_2 = \emptyset$.

Definition

The free fusion of $\mathcal{F}_1$ and $\mathcal{F}_2$ is

$$\mathcal{F}_1 * \mathcal{F}_2 = \{A\colon\ A \text{ is an } L_1 \cup L_2\text{-structure } \ A \upharpoonright L_i \in \mathcal{F}_i,\ i = 1, 2\}.$$

Let $\mathcal{F}$ be a Fraïssé class.

Definition (Order Expansion)

It is $\mathcal{F} * LO$, where LO is the Fraïssé class of finite linear orders.

Order and Tournament Expansions

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Definition (Tournament Expansion)

It is $\mathcal{F} * \mathcal{T}$, where $\mathcal{T}$ is the Fraïssé class of finite tournaments.

Main Theorem

Theorem

Assume that $\boldsymbol{M}$ is one of the following:

1. *the Fraïssé limit of a free, transitive and nontrivial amalgamation class, or*
2. *the bounded countable Urysohn space.*
3. *the random poset.*

If $\boldsymbol{M}^*$ is an order expansion of $\boldsymbol{M}$, then $G := Aut(\boldsymbol{M}^*)$ is simple. The same holds if $\mathbf{M}^*$ is a tournament expansion of (1) or (2).

Stationary Independence Relation

Definition

Let $\boldsymbol{M}$ be a countable structure with universe M and let $\downfree$ be a ternary relation between finite subset of M. We say that $\downfree$ is a stationary independence relation on **M** if for all finite sets $A, B, C, D \subseteq M$ the following hold:

- (Invariance) Whether A and B are independent over C depends only on the type of ABC.
- (Monotonicity) $A \downfree_B CD$ implies that $A \downfree_B C$ and $A \downfree_{BC} D$.
- (Transitivity)

$$A \underset{B}{\downfree} C \text{ and } A \underset{BC}{\downfree} D \text{ implies } A \underset{B}{\downfree} D.$$

Stationary Independence Relation

Definition (Continuation of the definition)

- (Symmetry) $A \mathop{\smile\!\!\!\!\!\mid}_B C$ if and only if $C \mathop{\smile\!\!\!\!\!\mid}_B A$.
- (Existence) If p is a type over B and C is a finite set, there is some a realizing p such that $a \mathop{\smile\!\!\!\!\!\mid}_B C$.
- (Stationarity) If the tuples x and y have the same type over B and are both independent from C over B, then x and y have the same type over BC.

Examples

Example (Free amalgamation classes)

We put $A \mathop{\smile\!\!\!\!\!\!\mid}_B C$ if and only if ABC is isomorphic to the free amalgam of A and C over B, i.e. if and only if $A \cap C = B$ and for every n-ary relation R in L, if $d_1, \ldots, d_n$ is an n-tuple in $A \cup B \cup C$ with some $d_i \in A \setminus B$ and $d_j \in C \setminus B$, then $R(d_1, \ldots, d_n)$ does not hold.

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Example (Metric spaces)

We put $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$ if and only if for all $a \in A, b \in B$ there is some $c \in C$ such that $d(a,b) = d(a,c) + d(c,b)$, and $A \mathop{\smile\!\!\!\!\!\!\;|} B$ if and only if for all $a \in A, b \in B$ the distance $d(a,c)$ is maximal.

Order-homogeneity and Tournament-homogeneity

Definition (Order-homogeneity)

If $\mathbf{M}_<$ is an order expansion of $\mathbf{M}$, then $g \in G$ is $<$-homogeneous if and only if g is unboundedly increasing or unboundedly decreasing.

Order-homogeneity and Tournament-homogeneity

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Definition (Tournament-homogeneity)

If $\mathbf{M}_\rightarrow$ is a tournament expansion of $\mathbf{M}$, then $g \in G$ is $\rightarrow$-homogeneous if and only if $a \rightarrow g(a)$ for all $a \in M$ or $g(a) \rightarrow a$ for all $a \in M$.

Notation

Definition

$A \downarrow_{B;C} D$ means $AB \downarrow_C D$ and $A \downarrow_B CD$

Moving maximally

Definition

Let $\mathbf{M}^*$ be the Fraïssé limit of $\mathcal{F} * LO$ or $\mathcal{F} * \mathcal{T}$, where $\mathcal{F}$ carries a stationary independence relation and let $G = \mathrm{Aut}(\mathbf{M}^*)$. We say that $g \in G$ moves maximally if

1. g is $<$-homogeneous (respectively, $\to$-homogeneous); and
2. every type over a finite set X has a realization a such that

$$a \underset{X;g(X)}{\downarrow} g(a).$$

Step 1

Theorem

Let $\mathbf{M}^*$ be the Fraïssé limit of $\mathcal{F} * LO$ or $\mathcal{F} * \mathcal{T}$, where $\mathcal{F}$ carries a stationary independence relation. If $g \in G$ moves maximally, then any element of G is the product of at most eight conjugates of g and g^{-1}.

Step 1-continuation

Proposition

If $\mathbf{M}^*$ *is a tournament expansion of a homogeneous* L_1*-structure* $\mathbf{M}_1$ *carrying a stationary independence relation, any* $g \in G = Aut(\mathbf{M}^*)$ *that moves maximally is compatible.*

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- In particular, every finite set X_0 has a finite extension X depending only on X_0 and g, which is full for g, i.e. the following holds: for $Y = g(X)$ and all tuples x, y such that $g(\mathrm{tp}(x/X)) = \mathrm{tp}(y/Y)$, $\mathrm{tp}_{L_2}(y^i/x^i) = \mathrm{tp}_{L_2}(g(x^i)/x^i)$ and $x \perp_{X;Y} y$ there is some $a \in \mathrm{Fix}(XY)$ such that $g^a(x) = y$

Step 1-continuation

Proposition

Let $g_1, \ldots, g_4 \in G$ move maximally, and assume that g_2 is conjugate to g_3^{-1}. Then, for any open non-empty set $U \subseteq G^4$, there is some open non-empty set $W \subseteq G$ such that the image $\phi(U)$ under the map

$$\phi\colon G^4 \to G : (h_1, \ldots, h_4) \mapsto g_4^{h_4} g_3^{h_3} g_2^{h_2} g_1^{h_1}.$$

is dense in W.

Step 2

Theorem

If $\mathbf{M}^*$ is an order or tournament expansion of a structure $\mathbf{M}$ as in the Main Theorem and $h \in G = Aut(\mathbf{M}^*)$, $h \neq \mathrm{Id}$, then there is some $g \in \langle h \rangle^G$ that moves maximally.

Step 2 for $\mathbf{M}_<$

We construct an unboundedly increasing automorphism $g \in \langle h \rangle^G$ moving maximally starting from an arbitrary $h \in G$. This is done in four steps.

1. construct a fixed point free $h_1 = [h, f_1] \in \langle h \rangle^G$;
2. construct a strictly increasing $h_2 = [h_1, f_2] \in \langle h_1 \rangle^G$;
3. construct an unboundedly increasing $h_3 \in \langle h_2 \rangle^G$;
4. construct an unboundedly increasing $g = [h_3, f_3] \in \langle h_3 \rangle^G$ moving maximally.

Questions

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Question

- *Do automorphims groups of structures studied above have the automatic continuity property?*
- *In particular, does the automorphism group of the linearly ordered random graph have the automatic continuity property?*