On normal forms for ill-founded proofs

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In a World without Cut …

Virtues of the cut-free proof

- Consistency
- Analyticity
- Herbrand's Theorem
- (Constructive) Interpolation
- Characterisations of Provability
- Proof-search
- …

But

- Arithmetic creates infinite proofs …
- Modal & Temporal logics need ad hoc changes to sequent calculus …

… breaking many virtues of cut-free proofs

Overview of Talk

1. Modal Logic with Fixed Points
2. Ill-founded Proofs
3. A Normal Form for Ill-Founded Proofs

Modal Logic with Fixed Points

Syntax & Semantics of Modal Logic

- Language of modal logic with actions

$$\varphi = \top \mid \varphi \wedge \psi \mid \neg\varphi \mid [a]\varphi$$

  for $a \in \mathrm{Act}$.

- Evaluated over *labelled transition systems* $\langle S, \{\to_a\}_{a \in \mathrm{Act}} \rangle$ where
  - S non-empty set of *states*.
  - $\langle S, \to_a \rangle$ is a *directed graph* for each $a \in \mathrm{Act}$

- Define $\| \varphi \| \subseteq S$:

$$\| \top \| = S \qquad \| \varphi \wedge \psi \| = \| \varphi \| \cap \| \psi \|$$

$$\| \neg\varphi \| = S \setminus \| \varphi \| \qquad \| [a]\varphi \| = \{ s \in S \mid \forall t (s \to_a t \Rightarrow t \in \| \varphi \| \}$$

- We require at most two primitive actions: $\square\varphi = [0]\varphi$ and $\blacksquare\varphi = [1]\varphi$

Equations

Basic Modal Logic K is axiomatised by

- Classical Propositional Logic
- $[a]\varphi \wedge [a](\varphi \rightarrow \psi) \rightarrow [a]\psi$
- if $\vdash \varphi$ then $\vdash [a]\varphi$

And can be extended by additional operators. E.g.

- Reachability: $\mathsf{R}\varphi \leftrightarrow \varphi \vee \Diamond \mathsf{R}\varphi$
- Common knowledge: $\mathsf{C}\varphi \leftrightarrow \varphi \wedge \Box \mathsf{C}\varphi$
- Path quantifiers: $\mathsf{I} \leftrightarrow \blacklozenge(\mathsf{R}\mathsf{I})$

With induction rules:

$$\frac{(\varphi \vee \Diamond\psi) \rightarrow \psi}{\mathsf{R}\varphi \rightarrow \psi} \qquad \frac{\psi \rightarrow (\varphi \wedge \Box\psi)}{\psi \rightarrow \mathsf{C}\varphi} \qquad \frac{\psi \rightarrow \blacklozenge \mathsf{R}\psi}{\psi \rightarrow \mathsf{I}}$$

$f: X \to X$ is *monotone* if $x \leq y$ implies $fx \leq fy$.

Knaster–Tarski Theorem

Let L be a complete lattice and $f: L \to L$ a monotone function on L. The set of fixed points of f,

$$\{x \in L \mid fx = x\}$$

forms a complete lattice.

Note, the power set lattice, $\langle \mathcal{P}(S), \subseteq \rangle$, is complete,

Corollary

Every monotone function on $\mathcal{P}(S)$ attains (unique) least and greatest fixed points.

Fixed Point Semantics

- $f : x \mapsto \| \varphi \vee \Diamond x \|$ is monotone. The *least fixed point* is the set

$$\begin{aligned} \mathrm{lfp} f &= \{ s \in S \mid \exists t (s \to_a^* t \text{ and } t \in \| \varphi \|)\} \\ &= \bigcap \{ U \subseteq S \mid \| \varphi \wedge \Diamond U \| \subseteq U \} \\ &= \| \mathrm{R}\varphi \| \end{aligned}$$

- $f : x \mapsto \| \varphi \wedge \Box x \|$ is monotone. The *greatest fixed point* is the intended semantics of $\mathrm{C}\varphi$:

$$\mathrm{gfp} f = \| \mathrm{C}\varphi \| = \bigcup \{ U \subseteq S \mid U \subseteq \| \varphi \wedge \Box U \| \}$$

- $f : x \mapsto \| \blacklozenge \mathrm{R} x \|$ is monotone. The *greatest fixed point* is the intended semantics of I:

$$\begin{aligned} \| \mathrm{I} \| &= \bigcup \{ U \subseteq S \mid U \subseteq \| \blacklozenge \mathrm{R} U \| \} \\ &= \mathrm{gfp}(x \mapsto \blacklozenge \mathrm{lfp}(y \mapsto (x \vee \Diamond y))) \end{aligned}$$

Hierarchical Equational Systems

An HES is a tuple $H = \langle \mathsf{V}, E, \trianglelefteq, G \rangle$ where

- V is a finite set of propositional *variables*
- $E\colon \mathsf{V} \to \mathrm{ML}_{\mathsf{V}}$ assigns to each variable v an *equation* $E(v)$ in the language ML_{V}:

$$\varphi = w \in \mathsf{V} \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid [a]\varphi \mid \langle a \rangle \varphi$$

- $\trianglelefteq$ is a partial order on V which is total on the sub-formula relation:

  If $w \in E(v)$ then either $v \trianglelefteq w$ or $w \trianglelefteq v$

- $G \subseteq \mathsf{V}$

Semantics

- For $v \in G$, $\| v \|$ is the *greatest fpt* of function $v \mapsto E(v)$ with $\{w \mid w \triangleleft v\}$ as parameters.
- For $v \notin G$, $\| v \|$ is the *least fpt* of function $v \mapsto E(v)$ with $\{w \mid w \triangleleft v\}$ as parameters.

Hierarchical Equations

Henceforth, we write $\begin{Bmatrix} \nu =_\mu \varphi \\ \nu =_\nu \varphi \end{Bmatrix}$ to mean $E(\nu) = \varphi$ and $\begin{Bmatrix} \nu \notin G \\ \nu \in G \end{Bmatrix}$

- Reachability: $r =_\mu \varphi \vee \Diamond r$
- Even/odd: $\begin{Bmatrix} r =_\mu e \vee o \\ e =_\mu \varphi \vee \Diamond o \\ o =_\mu \Diamond e \end{Bmatrix}$ with (say) $r \trianglelefteq o \trianglelefteq e$
- Common knowledge: $c =_\nu \varphi \wedge \Box c$
- Infinite paths: $\begin{Bmatrix} i =_\nu \blacklozenge r \\ r =_\mu i \vee \Diamond r \end{Bmatrix}$ with $i \trianglelefteq r$

$i \to_1 r \to_0 i \to_1 r \to_0 i \to \ldots$

$\|i\|$: there is an inf. path in which a 1-transition occurs i.o.

HESs cover all finite fragments of the *modal μ-calculus*.

Ill-founded Proofs

Proofs For Equations

Fix an HES $H = \langle \mathsf{V}, E, \trianglelefteq, G \rangle$ and a sequent of ML_{V} formulas:

$$A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n$$

Inference rules (selection):

$\wedge$:

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\,\mathsf{L}\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\,\mathsf{R}\wedge$$

Act:

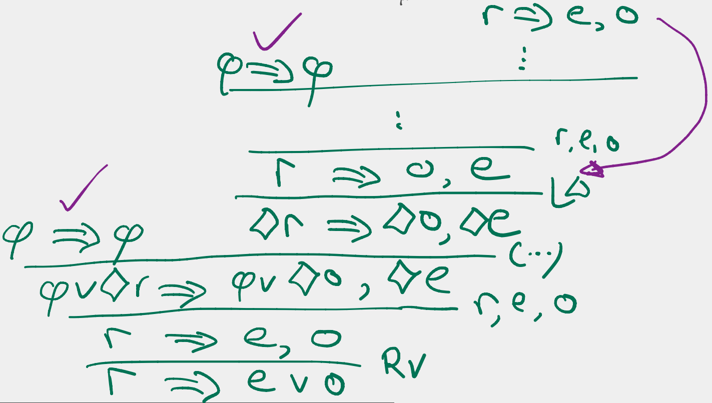
$$\frac{\Gamma, A \Rightarrow \Delta}{\Lambda, [a]\Gamma, \langle a \rangle A \Rightarrow \langle a \rangle \Delta, \Sigma}\,\mathsf{L}a \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Lambda, [a]\Gamma \Rightarrow \langle a \rangle \Delta, [a]A, \Sigma}\,\mathsf{R}a$$

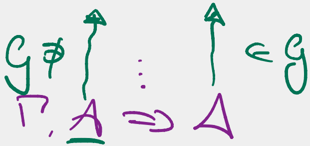
ν:

$$\frac{\Gamma, E(\nu) \Rightarrow \Delta}{\Gamma, \nu \Rightarrow \Delta}\,\mathsf{L}\nu \qquad \frac{\Gamma \Rightarrow \Delta, E(\nu)}{\Gamma \Rightarrow \Delta, \nu}\,\mathsf{R}\nu$$

Three equations:

$$r =_\mu \varphi \vee \Diamond r \qquad e =_\mu \varphi \vee \Diamond o$$
$$o =_\mu \Diamond e$$





Fix an HES $H = \langle V, E, \trianglelefteq, G \rangle$.

Definition

A pre-proof (over H) is a *∞-proof* if every infinite path contains an infinite *ancestor trace* $\alpha = (A_i)_{i \in \omega}$ such that the *$\trianglelefteq$-minimal variable occurring i.o. in α* is:

- $\in G$ and α resides on the right of $\Rightarrow$; or
- $\notin G$ and α resides on the left of $\Rightarrow$.

Theorem (Niwinski, Walukiewicz)

$\Gamma \Rightarrow \Delta$ (over H) is *valid* iff there exists an ∞-proof with end-sequent $\Gamma \Rightarrow \Delta$.

A tree is *regular* if it is the unravelling of a finite (directed) graph.

Corollary

There exists a ∞-proof of $\Gamma \Rightarrow \Delta$ iff there exists a regular ∞-proof of $\Gamma \Rightarrow \Delta$.

Proof.

The class of ∞-proofs is an ω-regular tree language. By Rabin's Tree Theorem every ω-regular language contains a regular tree. □

Interlocking Traces

$$
\dfrac{
\dfrac{\dfrac{\begin{array}{c}\dagger\\ \Rightarrow f, g, h\end{array}}{\Rightarrow f \vee h, g}}{\Rightarrow \Diamond(f \vee h), \Box g, \blacksquare h}
\qquad
\dfrac{\dfrac{\begin{array}{c}\ddagger\\ \Rightarrow f, g, h\end{array}}{\Rightarrow f \vee g, h}}{\Rightarrow \blacklozenge(f \vee g), \Box g, \blacksquare h}
}{\Rightarrow f, g, h \quad (\dagger, \ddagger)}
$$

$g \to f \to \cdots$

$h \to h \to h \cdots$

Equations: $\begin{cases} f =_\mu \Diamond(f \vee h) \wedge \blacklozenge(f \vee g) \\ g =_\nu \Box g \\ h =_\nu \blacksquare h \end{cases}$ where $f \trianglelefteq g, h$.

Interlocking Traces

Not an ∞-proof:

$$
\dfrac{\dfrac{\dfrac{\dagger}{\Rightarrow f, g, h}}{\Rightarrow f \vee h, g}}{\Rightarrow \lozenge(f \vee h), \square g, \blacksquare h}
\qquad
\dfrac{\dfrac{\dfrac{\ddagger}{\Rightarrow f, g, h}}{\Rightarrow f \vee g, h}}{\Rightarrow \blacklozenge(f \vee g), \square g, \blacksquare h}
$$

$$
\Rightarrow f, g, h \quad (\dagger, \ddagger)
$$

Equations: $\left\{ \begin{array}{l} f =_\mu \lozenge(f \vee h) \wedge \blacklozenge(f \vee g) \\ g =_\nu \square g \\ h =_\nu \blacksquare h \end{array} \right\}$ where $f \trianglelefteq g, h$.

A Normal Form for Ill-Founded Proofs

Annotated Sequents

We present a seq. calculus for Modal Logic with equations. For simplicity, we assume a one-sided calculus. Fix an HES $H = \langle V, E, \trianglelefteq, G\rangle$.

Names

- Each variable $v \in G$ is associated two sets of *names*, N_v and N_v^+.
- Each name $n \in N_v$ has associated a *promotion* $n^+ \in N_v^+$.

A *sequent* is an expression $a_0 : A_1^{a_1}, \ldots, A_k^{a_k}$ where each a_i is a finite non-repeating sequence of names.

$b \trianglelefteq v$ means b contains only names for variables $\trianglelefteq v$.

Annotations control/record variable unfoldings:

provided that $b \trianglelefteq v$: $\dfrac{a : \Gamma, E(v)^b}{a : \Gamma, v^b}\, v$ and if $n \in N_v$: $\dfrac{a : \Gamma, E(v)^{bn^+}}{a : \Gamma, v^{bn}}\, v^+$

A Calculus of ω-Proofs

Rules for equations:

$$(b \trianglelefteq v)\ \frac{a : \Gamma, E(v)^{b}}{a : \Gamma, v^{b}}\ v \qquad\qquad (b \trianglelefteq v \ \&\ n \in N_v)\ \frac{a : \Gamma, E(v)^{bn^+}}{a : \Gamma, v^{bn}}\ v^+$$

Rules for annotation

$$\frac{a : \Gamma, A^{bnb'}}{a : \Gamma, A^{bn^+b'}}\ +$$

Discharing assumptions: for $v \in G$ and $n \in N_v$

$$\begin{array}{c} \left[ana' : \Gamma, A_0^{a_0 n^+ a_0'}, \ldots, A_k^{a_k n^+ a_k'}\right]^n \\ \vdots \\ \dfrac{an : \Gamma, A_0^{a_0 n}, \ldots, A_k^{a_k n}}{a : \Gamma, A_0^{a_0}, \ldots, A_k^{a_k}}\ n \end{array}$$

A *ω-proof* is a finite 'proof'-tree in which all leaves are either *axiomatic* or *discharged*.

An ∞-proof:

$$\dfrac{\dfrac{f,g,h,i,j}{\vdots}\quad \dfrac{\dfrac{f,g,h,i,j}{\vdots}}{\Diamond(f\vee h),g,h,i,j} \qquad \dfrac{\blacklozenge(f\vee g),g,h,i,k}{\blacklozenge(f\vee g),g,h,i,j}\,j}{\underline{f,g,h,i,j}}\,f,\wedge$$

Equations: $\begin{cases} f =_\mu \Diamond(f\vee h)\wedge\blacklozenge(f\vee g) & i =_\mu \Diamond(i\vee j) \\ g =_\nu \Box g & j =_\nu k \\ h =_\nu \blacksquare h & k =_\mu \blacklozenge(i\vee j) \end{cases}$ where $f \trianglelefteq g, h$ and $j \trianglelefteq k \trianglelefteq i$.

An ω-proof:

$$
\dfrac{
  \dfrac{
    \dfrac{
      \begin{array}{c}[mn : f, g^{n^+}, h, i^m, j^m]^n \\ \vdots \\ mn : \lozenge(f \vee h), g^n, h, i^m, j^m\end{array}
      \qquad
      \dfrac{\begin{array}{c}[mn : f, g, h, i^{m^+}, j^{m^+}]^m \\ \vdots \\ mn : \blacklozenge(f \vee g), g^n, h, i^m, k^{m^+}\end{array}}{mn : \blacklozenge(f \vee g), g^n, h, i^m, j^m}\, j^+
    }{mn : f, g^n, h, i^m, j^m}\, f, \wedge
  }{m : f, g, h, i^m, j^m}\, n \in N_g
}{: f, g, h, i, j}\, m \in N_j
$$

Equations: $\left\{\begin{array}{ll} f =_\mu \lozenge(f \vee h) \wedge \blacklozenge(f \vee g) & i =_\mu \lozenge(i \vee j) \\ g =_\nu \square g & j =_\nu k \\ h =_\nu \blacksquare h & k =_\mu \blacklozenge(i \vee j) \end{array}\right\}$ where $f \trianglelefteq g, h$ and $j \trianglelefteq k \trianglelefteq i$.

Theorem (Afshari, L.)

A sequent $\Gamma = A_0, \dots, A_k$ is *valid* iff there exists an ω-proof of $: \Gamma$.

Proof.

Completeness:

- Use a labelled sequent calculus for obtaining regular ∞-proofs due to N. Jungteerapanich (2009) & C. Stirling (2014).
- Witness ω-proofs as ∞-proofs in 'annotation normal form'.

Soundness:

- By transfinite induction on assignments of ordinals to names where $o(m^+) < o(m)$.
- Interpret ν^{an} as $o(n)$-th approximation of $\mathrm{gfp}(\nu \mapsto E(\nu)^a)$. □

Normal Forms for ∞-Proofs

For Modal Logic extended by hierarchical equational systems ω-proofs provide a robust normal form which:

- can be constructed as 'regularisation' of ∞-proofs.
- admit a direct soundness argument.
- are cut-free and analytic (sub-formula property).
- can be used to extract interpolants.
- can be embedded into finitary calculi with induction axioms.

Open questions:

- Give general completeness proofs of Hilbert calculi for temporal logics.
- Generalise methods to first-order logic with (co-)inductive definitions.

Thank you!