Broad Infinity and Generation Principles

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November 12, 2020

Outline

Grothendieck universes

A Grothendieck universe is a set $\mathfrak{U}$ such that

- Any element of a set in $\mathfrak{U}$ is in $\mathfrak{U}$.
- $\emptyset \in \mathfrak{U}$.
- If $x, y \in \mathfrak{U}$ then $\{x, y\} \in \mathfrak{U}$.
- If I is a set in $\mathfrak{U}$ and $(A_i)_{i \in I}$ is a family of sets in $\mathfrak{U}$ then $\bigcup_{i \in I} A_i \in \mathfrak{U}$.
- If A is a set in $\mathfrak{U}$ then $\mathcal{P}A \in \mathfrak{U}$.

The Grothendieck-Verdier universe axiom says that everything belongs to a universe.

Mahlo's principle

- "Every normal function defined for all ordinals has at least one inaccessible number in its range." [Lévy, 1960]
- "Given a function f and an ordinal β there is a regular ordinal α greater than β such that $\gamma < \alpha$ implies $f(\gamma) < \alpha$." [Jorgensen, 1970]
- Axiom F: Every normal function has a regular fixed point. [Drake, 1974]
- "Mahlo's principle says that every closed unbounded class of ordinals contains a regular cardinal." [Wang, 1977]
- "Mahlo's principle: Every ordinal valued ordinal function has arbitrarily large inaccessible points." [Mayberry, 1977]
- "For every function f, mapping families of sets in V to families of sets in V, there exists a universe closed under f." [Setzer, 2000]
- "Ord is Mahlo [...] also sometimes referred to as the Lévy scheme." [Hamkins, 2003]
- "There are several interesting connections between Mahlo notions and induction-recursion." [Dybjer and Setzer, 2003]

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But I also have secondary goals:

1. To highlight the analogy between Broad Infinity and the ordinary axiom of Infinity.
2. To track the use of the Axiom of Choice (AC) and the Law of Excluded Middle (LEM).
3. To make clear that everything still works if urelements and/or non-well-founded sets are admitted.

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For the sake of these secondary goals, let's exclude Infinity, AC and LEM, and allow urelements and non-well-founded sets.

Preliminary Intuitionistic IZF

Intuitionistic first order theory with equality, using isSet(a) and $a \in b$.

Extensionality: Any two sets with the same elements are equal.

Inhabitation: Anything that has an element is a set.

Separation, Empty Set, Pairing, Union Set, Powerset, Replacement.

Not included

Infinity, LEM, AC, $\in$-induction, Collection.

Purity: Everything is a set.

Element Set: For every thing a, there's a set with the same elements.

Classes

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Tuples of classes

- An ordered pair of classes $\langle C, D \rangle$ is represented as the class $C + D$.
- For a class I, a tuple $(C_i)_{i \in I}$ of classes is represented as the class $\sum_{i \in I} C_i$.

Axiom of Infinity: my favourite version

Zermelo natural numbers: $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots$

Define constuctors, which are injective and disjoint:

$$\begin{array}{rcl} \mathsf{Zero} & \stackrel{\text{def}}{=} & \emptyset \\ \mathsf{Succ}(x) & \stackrel{\text{def}}{=} & \{x\} \end{array}$$

A set of all natural numbers is a minimal set X such that

- $\mathsf{Zero} \in X$
- for any $x \in X$, we have $\mathsf{Succ}(x) \in X$.

Infinity: there's a set $\mathbb{N}$ of all natural numbers.

Axiom of W-types

A signature $S = (K_i)_{i \in I}$ is a family of sets.

I is a set of symbols, and K_i is the arity of i.

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A set of all S-terms is a minimal set X such that

- for any symbol $i \in I$ and K_i-tuple $(a_k)_{k \in K_i}$ of X-elements, we have $\langle i, (a_k)_{k \in K_i} \rangle \in X$.

W-types: For every signature S, there's a set $\mathsf{W}(S)$ of all S-terms.

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Theorem

W-types is equivalent to Infinity.

Axiom scheme of Broad Infinity

Define constuctors, which are injective and disjoint:

$$\begin{array}{rcl} \mathsf{Start} & \stackrel{\mathrm{def}}{=} & \emptyset \\ \mathsf{Build}(x,i,g) & \stackrel{\mathrm{def}}{=} & \{\{x\},\{x,\{\{i\},\{i,g\}\}\}\} \end{array}$$

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A broad signature G is a function sending each x to a signature Gx.

A set of all G-broad numbers is a minimal set X such that

- Start $\in X$
- for any $x \in X$ with $Gx = (K_i)_{i \in I}$, and any $i \in I$ and K_i-tuple $(a_k)_{k \in K_i}$ of X-elements, we have Build$(x, i, (a_k)_{k \in K_i}) \in X$.

Broad Infinity: For any broad signature G, there's a set Broad(G) of all G-broad numbers.

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Broad Infinity: For any broad signature G, there's a set Broad(G) of all G-broad numbers.

This implies Infinity.

Axiom scheme of Reduced Broad Infinity

Assuming LEM, Broad Infinity can be reduced to a simpler scheme.

Define constuctors, which are injective and disjoint:

$$\begin{aligned} \mathsf{Begin} &\stackrel{\text{def}}{=} \emptyset \\ \mathsf{Make}(x, g) &\stackrel{\text{def}}{=} \{\{x\}, \{x, g\}\} \end{aligned}$$

A reduced broad signature F is a function sending each x to a set Fx, its arity.

A set of all F-broad numbers is a minimal set X such that

- Begin $\in X$
- for any $x \in X$ and Fx-tuple $(a_k)_{k \in Fx}$ of X-elements, we have Make$(x, (a_k)_{k \in Fx}) \in X$.

Reduced Broad Infinity: For any reduced broad signature F, there's a set rBroad(F) of all F-broad numbers.

Generation of sets and families

Given a class C, we consider

- Subset of C generated by a rubric
- Family of C-elements generated by a rubric
- Subset of C generated by a broad rubric
- Family of C-elements generated by a broad rubric.

Example: rubric $\mathcal{R}$ on $\mathbb{N}$

Intuition the rubric tells you when to accept an element of $\mathbb{N}$.

- Rule 0. Arity $= 2$. $(m_0, m_1) \mapsto (m_0 + m_1 + p)_{p \geqslant 2m_0}$.
- Rule 1. Arity $= 0$. $() \mapsto (2p)_{p \geqslant 50}$.

Elements accepted by the rubric

- 100 has derivation $\langle 1, (), 50 \rangle$.
- 102 has derivation $\langle 1, (), 51 \rangle$.
- 402 has derivations $\langle 0, (\langle 1, (), 50 \rangle, \langle 1, (), 50 \rangle), 202 \rangle$ and $\langle 0, (\langle 1, (), 50 \rangle, \langle 1, (), 51 \rangle), 200 \rangle$.

Rubric on a Class C

A rule $\langle K, R \rangle$ on C consists of

- a set K—the arity
- a function R sending each K-tuple $(a_k)_{k \in K}$ of C-elements to a family $(y_j)_{j \in J}$.

A rubric on C is a family of rules $(\langle K_i, R_i \rangle)_{i \in I}$, indexed by a set.

Rubric generating a set or family

Let $\mathcal{R} = (\langle K_i, R_i \rangle)_{i \in I}$ be a rubric on a class C.

Set generated by $\mathcal{R}$

A minimal subset X of C that is $\mathcal{R}$-closed:

- for $i \in I$ and tuple $(a_k)_{k \in K_i}$ of X-elements, with $R(a_k)_{k \in K_i} = (y_j)_{j \in J}$, and $j \in J$, we have $y_j \in X$.

Intuition X consists of all the elements obtained from $\mathcal{R}$.

Family generated by $\mathcal{R}$

A minimal family $(x_m)_{m \in M}$ such that

- for $i \in I$ and $g : K_i \to M$, with $R_i(x_{gk})_{k \in K_i} = (y_j)_{j \in J}$, and any $j \in J$, we have $\langle i, g, j \rangle \in M$ and $x_{\langle i,g,j \rangle} = y_j$.

Intuition M is the set of derivations, and $m \in M$ is derivation of x_m.

Example: broad rubric $\mathcal{S}$ on $\mathbb{N}$

Intuition An accepted element triggers an advanced rubric.
The basic rubric of $\mathcal{S}$ is as follows.

- Rule 0. Arity $= 2$. $(m_0, m_1) \mapsto (m_0 + m_1 + p)_{p \geqslant 2m_0}$.
- Rule 1. Arity $= 0$. $() \mapsto (2p)_{p \geqslant 50}$.

The advanced rubric triggered by 7 is as follows.

- Rule 0. Arity $= 2$. $(m_0, m_1) \mapsto (m_0 + m_1 + 500p)_{p \geqslant 9}$.

The advanced rubric triggered by 100 is as follows.

- Rule 0. Arity $= 3$. $(m_0, m_1, m_2) \mapsto (m_0 + m_1 m_2 + p)_{p \geqslant 17}$.
- Rule 1. Arity $= 0$. $() \mapsto (p)_{p \geqslant 1000}$.
- Rule 2. Arity $= 1$. $(m_0) \mapsto (m_0 + p)_{p \geqslant 4}$.

The advanced rubric triggered by any other natural number is empty.

Elements accepted by $\mathcal{S}$

Define constructors, which are injective and disjoint:

$$\begin{aligned}\mathsf{Basic}(i,g,j) &\stackrel{\text{def}}{=} \mathsf{inl}\ \langle i,g,j\rangle \\ \mathsf{Advanced}(m,i,g,j) &\stackrel{\text{def}}{=} \mathsf{inr}\ \langle m,i,g,j\rangle\end{aligned}$$

- $\mathsf{Basic}(1,(),50)$ is a derivation of 100.
- $\mathsf{Basic}(1,(),51)$ is a derivation of 102.
- $\mathsf{Advanced}(\mathsf{Basic}(1,(),50),2,(\mathsf{Basic}(1,(),51)),5)$ is a derivation of 107.

Broad rubric $\mathcal{S}$ on a class C

The broad rubric $\mathcal{S} = (\mathcal{R}_0, \mathcal{R}_1)$ consists of

- the basic rubric $\mathcal{R}_0$
- for each $x \in C$, an advanced rubric $\mathcal{R}_1(x)$.

Broad rubric generating a set or family

Let $\mathcal{S} = (\mathcal{R}_0, \mathcal{R}_1)$ be a broad rubric on a class C.

Set generated by $\mathcal{S}$

A minimal subset X of C such that

- is $\mathcal{R}_0$-closed
- for $x \in X$, is $\mathcal{R}_1(x)$-closed.

Family generated by $\mathcal{S}$

A minimal family $(x_m)_{m \in M}$ such that

- [...] then $\mathsf{Basic}(i, (a_k)_{k \in K_i}, j) \in M$ and $x_{\mathsf{Basic}(i,g,j)} = y_j$.
- [...] then $\mathsf{Advanced}(m, i, g, j) \in M$ and $x_{\mathsf{Advanced}(m,i,g,j)} = y_j$.

Generation principles

Set Generation principle

Every rubric on a class generates a set.

Family Generation principle

Every rubric on a class generates a family.

Broad Set Generation principle

Every broad rubric on a class generates a set.

Implies the universe axiom.

Broad Family generation principle

Every broad rubric on a class generates a family.

Theorems

- Infinity $\Leftrightarrow$ Family Generation.
- Assuming AC Infinity $\Leftrightarrow$ Set Generation.
- Broad Infinity $\Leftrightarrow$ Broad Family Generation.
- Assuming AC Broad Infinity $\Leftrightarrow$ Broad Set Generation.

Family Generation $\Rightarrow$ Set Generation, using AC

Given a rubric $\mathcal{R} = (\langle R_i, K_i \rangle)_{i \in I}$ on C.

If $(x_m)_{m \in M}$ is the family generated by $\mathcal{R}$,

then $\{x_m \mid m \in M\}$ is the set generated by $\mathcal{R}$.

Given a K_i-tuple of elements, we choose a derivation for each one.

Weaker than AC suffices

The WISC axiom says that for every set X there is a weakly initial of covers. [Moerdijk, Palmgren, Rathjen, van den Berg]

A WISC operator is a function symbol that sends every set X to a weakly injective set of covers.

This suffices for (Broad) Family Generation $\Rightarrow$ (Broad) Set Generation.

Diagram of subsystems (starting from PIZF)

$$
\begin{array}{ccc}
\text{PIZF + Infinity} & \longrightarrow & \text{PIZF + Broad Infinity} \\
\downarrow & & \downarrow \\
\text{PIZF + Set Generation} & \longrightarrow & \text{PIZF + Broad Set Generation} \\
\downarrow & & \downarrow \\
\text{PIZF + Infinity + WISCOP} & \longrightarrow & \text{PIZF + Broad Infinity + WISCOP} \\
\downarrow & & \downarrow \\
\text{PIZF + Infinity + AC} & \longrightarrow & \text{PIZF + Broad Infinity + AC}
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Ordinals

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Hartogs of $\mathcal{P}A$

1. A does not have a strictly increasing $H(\mathcal{P}A)$-chain of subsets.
2. Any A-indexed family of ordinals has range with order type $< H(\mathcal{P}A)$.

Generation by transfinite induction

Let $\mathcal{R}$ be a rubric or broad rubric on a class C.

We define an increasing chain $(X_\gamma)_{\gamma \in \mathsf{Ord}}$ of subsets of C.

At zero, take the empty set.
At limit ordinals, take the union.
At successor $\alpha + 1$, take those elements obtained from applying a rule once to elements are in X_γ.

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Stabilization

- If $X_\gamma = X_{\gamma+1}$, then the chain stabilizes at γ, and X_γ is the set generated by $\mathcal{R}$.
- Conversely, if $\mathcal{R}$ generates a set A, then the chain stabilizes before $H(\mathcal{P}A)$, by Hartogs property (1).

Blass and Mahlo, usual formulation

Blass's principle

The class of regular cardinals is unbounded.

Mahlo's principle

The class of regular cardinals is stationary,
i.e. meets every closed unbounded class of ordinals.

Ordinal generation principles

Note that ordinal = lower subset of Ord.

Blass's principle, instance of Set Generation

For any α, there is least β such that

- $\gamma < \alpha$ and $\forall i < \gamma.\, x_i < \beta$ implies $\text{ssup}_{i<\gamma}\, x_i < \beta$

Mahlo's principle, instance of Broad Set Generation

For any $F :$ Ord $\rightarrow$ Ord there is least β such that

- $\beta > 0$
- $x < \beta$ implies $f(x) < \beta$
- $\gamma < \beta$ and $\forall i < \gamma.\, x_i < \beta$ implies $\text{ssup}_{i<\gamma}\, x_i < \beta$

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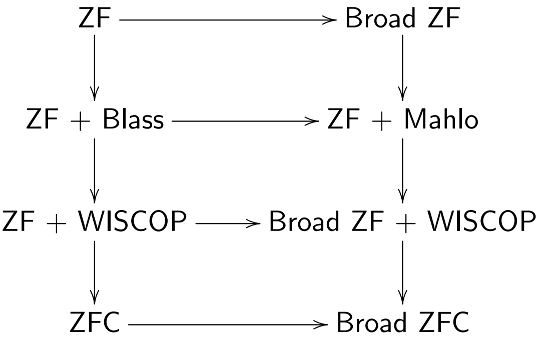
Because of Hartogs property (2):

- Blass is equivalent to Set Generation.
- Mahlo is equivalent to Broad Set Generation.

Diagram of subsystems (starting from ZF)



Conclusion

- Broad Infinity is a new axiom scheme. Hopefully you find it intuitive.
- It is equivalent to Broad Family Generation.
- It is equivalent to many other schemes if LEM, WISCOP or AC is assumed. (AC implies LEM and WISCOP.)
- Everything works in the presence of urelements and/or non-well-founded sets.
- The Infinity story and the Broad Infinity story are somewhat analogous.