Distributive Laws for Relative Monads

Ghentrire–Leeds Virtual Logic Seminar

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University of Leeds

11 June 2020

Outline of the talk

① **Monads** ⇝ ③ **Relative Monads**

② **Distributive Laws** ⇝ ④ ???

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(1) **Monads** $\rightsquigarrow$ (3) **Relative Monads**

(2) **Distributive Laws** $\rightsquigarrow$ (4) ???

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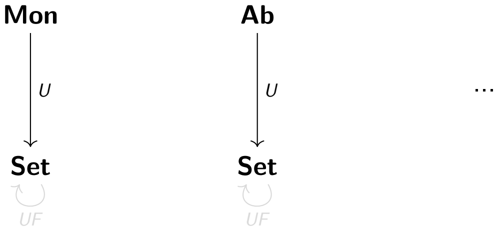
Why Monads?

When dealing with *algebraic structures* we usually have the following picture:



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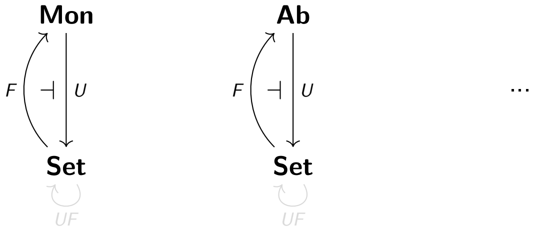

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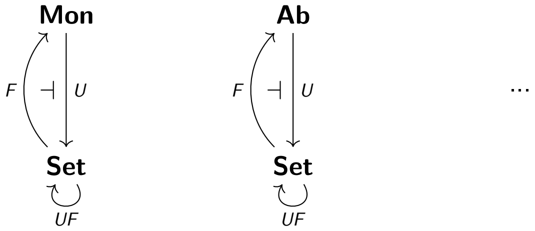

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Looking at the properties that all the UF satisfy we can generalise this notion.

Theorem

Mon *is complete and cocomplete.*

Theorem

Ab *is complete and cocomplete.*

Thanks to monads we can prove these theorems all at once with:

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If $U : \mathbb{D} \to \mathbb{C}$ *is "monadic", then:*

- *If* $\mathbb{C}$ *is complete, then* $\mathbb{D}$ *is complete;*
- *If* $\mathbb{C}$ *is cocomplete and* $\mathbb{D}$ *has reflexive coequalizers, then* $\mathbb{D}$ *is cocomplete.*

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Definition

Let $\mathbb{C}$ be a category. A **monad** (S, m, s) **on** $\mathbb{C}$ *consists of:*

- *A functor* $S : \mathbb{C} \to \mathbb{C}$;
- *Natural transformations* $m : S^2 \to S$ *and* $s : 1_{\mathbb{C}} \to S$ *s.t.*

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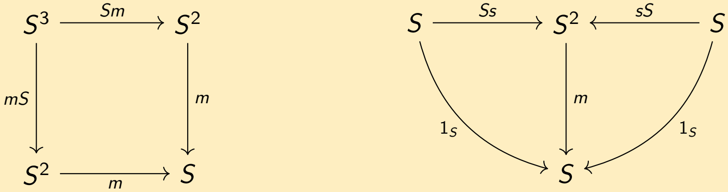

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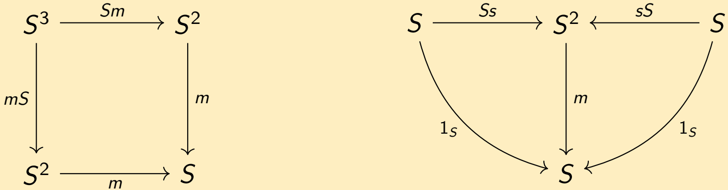

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Algebras for a Monad

$S : \mathbb{C} \to \mathbb{C}$ monad, an S-**algebra** is an object $A \in \mathbb{C}$ with a structural map $a : SA \to A$ s.t.

" $s_2 \cdot (s_1 \cdot x) = s_2 s_1 \cdot x$ "

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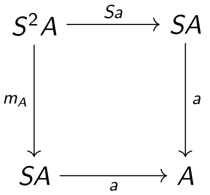

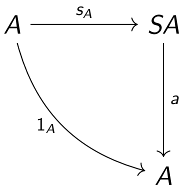

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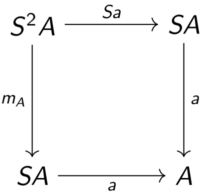

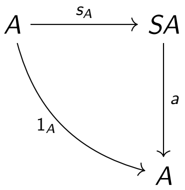

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Let $a : SA \to A$ and $b : SB \to B$ be two S-algebras. An **algebra morphism** is a map $f : A \to B$ s.t.

$$\begin{array}{ccc} SA & \xrightarrow{Sf} & SB \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

$\Rightarrow$ we get a category of S-algebras $S\text{-}\mathrm{Alg}$.

Theorem (Eilenberg-Moore)

For any monad $S : \mathbb{C} \to \mathbb{C}$, there is an adjunction $\mathbb{C} \underset{}{\overset{}{\rightleftarrows}}_{\perp} S\text{-}Alg$ that induces exactly S as monad.

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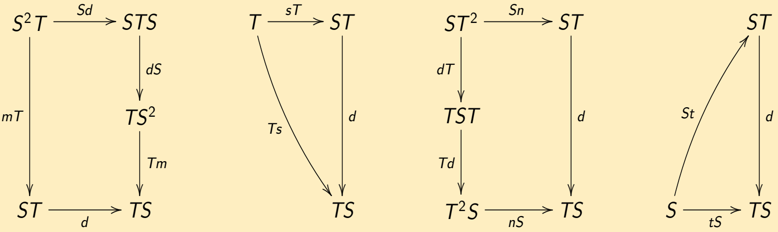
Examples

	$S(X)$	Algebras
$\mathbf{Set} \underset{\perp}{\rightleftarrows} \mathbf{Mon}$	Underlying set of the free monoid generated by X	Monoids
$\mathbf{Set} \underset{\perp}{\rightleftarrows} \mathbf{Ab}$	Underlying set of the free abelian group generated by X	Abelian Groups
Power set	$\mathcal{P}(X)$	Suplattices

Distributive Laws

Definition (Beck)

Let (S, m, s) and (T, n, t) be monads on $\mathbb{C}$. A **distributive law** *of T over S consists of a natural transformation $d : ST \to TS$ such that:*



▸ Relative Distributive Laws

Distributive Laws

Definition (Beck)

Let (S, m, s) and (T, n, t) be monads on $\mathbb{C}$. A **distributive law** *of T over S consists of a natural transformation $d : ST \to TS$ making four diagrams commutative.*

Lemma (Beck)

Given a distributive law d of T on S, then there is a monad structure on TS given by

$$TSTS \xrightarrow{TdS} T^2S^2 \xrightarrow{nS} TS^2 \xrightarrow{Tm} TS \qquad 1_{\mathbb{C}} \xrightarrow{s} S \xrightarrow{tS} TS$$

Example

T = power set monad, S = monad of monoids.

For $X \in \mathbf{Set}$

$$SX = \{x_1 \cdots x_n \mid x_i \in X,\ n \in \mathbb{N}\}$$
$$TX = P(X) = \{A \mid A \subseteq X\}$$

$\Rightarrow$ a distributive law $d : ST \to TS$ of T on S:

$$d_X : STX \longrightarrow TSX$$
$$A_1 \dots A_n \longmapsto \{a_1 \dots a_n \mid a_i \in A_i\}$$

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Theorem (Beck)

Let S and T be two monads on $\mathbb{C}$. TFAE

(i) *a* **distributive law** $d : ST \Longrightarrow TS$;

(ii) *a* **lifting** *of T to S-algebras* $\hat{T} : S\text{-}Alg \to S\text{-}Alg$;

(iii) *an* **extension** $\tilde{S} : Kl(T) \to Kl(T)$ *of S to the Kleisli category $Kl(T)$;*

(iv) *A monad structure on TS that is* **compatible** *with S and T.*

Corollary

- *There exists a lifting* $\hat{P}$: **Mon** $\to$ **Mon** *of the power set monad P;*
- *There exists an extension* $\tilde{S}$: **Rel** $\to$ **Rel** *of the monoid monad S to the category* **Rel** *of sets and relations;*
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Why **Relative** Monads?

Problem: Let $\mathbb{C}$ be a small category, then $P(\mathbb{C}) := \mathbf{Cat}(\mathbb{C}^{op}, \mathbf{Set})$ is just locally small.

$$P : \mathbf{Cat} \longrightarrow \mathbf{CAT}$$

Relative Monads generalise the concept of monad to functors defined on a subcategory.

Aim: Have a new version of Beck's Theorem explaining Day's convolution product.

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An Example

P = power set monad.

- $\mathrm{Kl}(P)$ = category of sets and relations;
- P-Alg= sup-semilatices.

What if we want to consider relations/sup-semilatices with an *upper bound* on cardinality of sets? Or even a set theory where PX is a class?

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- $TX \in \mathbb{C}$, *for every* $X \in \mathbb{C}_0$;
- *functions* $(-)^{\dagger}_{X,Y} : \mathbb{C}(IX, TY) \to \mathbb{C}(TX, TY)$ *for* $X, Y \in \mathbb{C}_0$;
- *morphisms* $t_X : IX \to TX$ *in* $\mathbb{C}$ *for* $X \in \mathbb{C}_0$;

such that:

Associativity: $(g^{\dagger} \cdot f)^{\dagger} = g^{\dagger} \cdot f^{\dagger}$ *(for* $f : IX \to TY$, $g : IY \to TZ$*);*

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Examples

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$$\begin{array}{rccc} t_X : & IX & \longrightarrow & PX \\ & x & \longmapsto & \{x\} \end{array} \qquad \frac{f : \quad IX \longrightarrow PY}{\begin{array}{rccc} f^\dagger : & PX & \longrightarrow & PY \\ & J & \longmapsto & \bigcup_{j \in J} f(j) \end{array}}$$

2. $I : \mathbf{Fin} \hookrightarrow \mathbf{Set}$ inclusion, $Tn := \mathbf{Set}(In, R)$ with R ring,

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$$t_X : \begin{array}{ccc} IX & \longrightarrow & PX \\ x & \longmapsto & \{x\} \end{array} \qquad \frac{f : IX \longrightarrow PY}{f^\dagger : \begin{array}{ccc} PX & \longrightarrow & PY \\ J & \longmapsto & \bigcup_{j\in J} f(j) \end{array}}$$

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Relative Monads generalise Monads

Relative Monads with $I = 1$	Monads
$(-)^{\dagger}_{X,Y} : \mathbb{C}(X, SY) \to \mathbb{C}(SX, SY)$	$m : S^2 \to S$
$(g^{\dagger} \cdot f)^{\dagger} = g^{\dagger} \cdot f^{\dagger}$	Associativity
$f = f^{\dagger} \cdot s_X$ and $s_X^{\dagger} = 1_{SX}$	Left/Right Unit Law

Proof.

($\Leftarrow$) For any $f : X \to SY$, we define $f^{\dagger}$ as $m_Y \cdot Sf : SX \to SY$;

($\Rightarrow$) Given an extension $(-)^{\dagger}$ we define m_X as $(1_{SX})^{\dagger} : S^2X \to SX$;

Using unity, and axioms for a relative monad we can prove that these constructions are inverse of each other. □

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Relative Algebras

Definition

I, $T : \mathbb{C}_0 \to \mathbb{C}$ relative monad. A **relative T-algebra** *consists of $A \in \mathbb{C}$*

$$\text{with maps } (-)^A_X : \mathbb{C}(IX, A) \to \mathbb{C}(TX, A)$$

satisfying the following axioms for $h : IX \to A$ and $k : IX' \to TX$:

$$\begin{array}{ccc} IX & \xrightarrow{t_X} & TX \\ & \underset{h}{\searrow} & \downarrow{\scriptstyle h^A} \\ & & A \end{array} \qquad \begin{array}{ccc} TX' & \xrightarrow{k^\dagger} & TX \\ & \underset{(h^A \cdot k)^A}{\searrow} & \downarrow{\scriptstyle h^A} \\ & & A \end{array}$$

Theorem (Altenkirch, Chapman and Uustalu)

Relative monads $\Leftrightarrow$ Relative adjuctions

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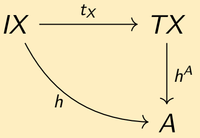

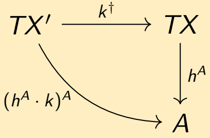

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Relative monads $\Leftrightarrow$ Relative adjuctions

When can we talk about **relative** distributive laws?

We want a relative monad I, $T : \mathbb{C}_0 \to \mathbb{C}$ and a monad $S : \mathbb{C} \to \mathbb{C}$ that *restrict nicely* to $\mathbb{C}_0$, i.e.

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Let $I : \mathbb{C}_0 \to \mathbb{C}$ be a functor. We define a **compatible monad with** *I as a pair of monads $S_0 : \mathbb{C}_0 \to \mathbb{C}_0$ and $S : \mathbb{C} \to \mathbb{C}$ such that $SI = IS_0$, $mI = Im_0$ and $sI = Is_0$.*

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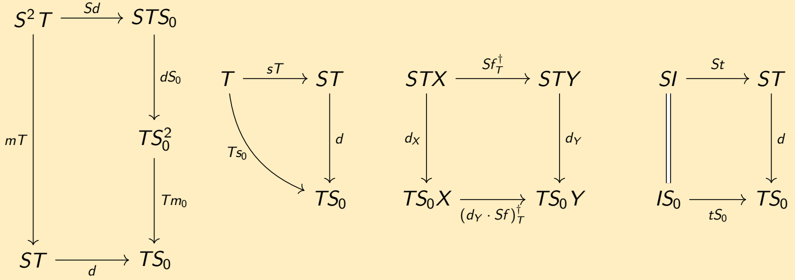
Relative Distributive Laws

▸ Distributive Laws

Definition

$I, T : \mathbb{C}_0 \to \mathbb{C}$ relative monad, (S, S_0) compatible with I. A **relative distributive law** *of T over (S, S_0) is a transformation $d : ST \to TS_0$ satisfying four axioms (for any $f : IX \to TY$):*



Example

T = power set relative monad, S = monad of monoids, $S_\kappa = S \restriction \mathbf{Set}_{\leq k}$.

For $X \in \mathbf{Set}_{\leq k}$ and $Y \in \mathbf{Set}$

$$SY = \{y_1 \cdots y_n \mid y_i \in Y,\, n \in \mathbb{N}\}$$

$$S_\kappa X = \{x_1 \cdots x_n \mid x_i \in X,\, n \in \mathbb{N}\}$$

$$TX = P(X) = \{A \mid A \subseteq X\}$$

$\Rightarrow$ a *relative* distributive law $d : ST \to TS$ of T on (S, S_k):

$$\begin{aligned} d_X : STX &\longrightarrow TS_kX \\ A_1 \ldots A_n &\longmapsto \{a_1 \ldots a_n \mid a_i \in A_i\} \end{aligned}.$$

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Theorem (Lobbia)

Given a relative monad $I, T : \mathbb{C}_0 \to \mathbb{C}$ and a compatible monad (S, S_0) with I, TFAE:

(1) *A* **relative distributive law** $d : ST \to TS_0$;

(2) *A* **lifting** $\hat{T} : S_0\text{-}\mathcal{A}lg \to S\text{-}\mathcal{A}lg$ *of T to the algebras of S_0 and S;*

(3) *An* **extension** $\tilde{S} : \mathcal{K}l(T) \to \mathcal{K}l(T)$ *of S to the Kleisli of T.*

Corollary

There is a lifting of the power set relative monad to $\mathbf{Mon}_{\leq\kappa} \hookrightarrow \mathbf{Mon}$.

There exists an extension of the free monoid monad to the category of relations over sets with cardinality $\leq \kappa$.

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Future Work

- Prove that a relative distributive law of T over (S, S_0) is equivalent to a relative monad structure on TS_0 *compatible* with T and (S, S_0);
- Extend this work to relative **pseudo**monads;
- Possible connection with Lawvere Theories,
  MEMO: Lawvere Theories are equivalent to finitary monads.

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