Surreal numbers and Simplicity

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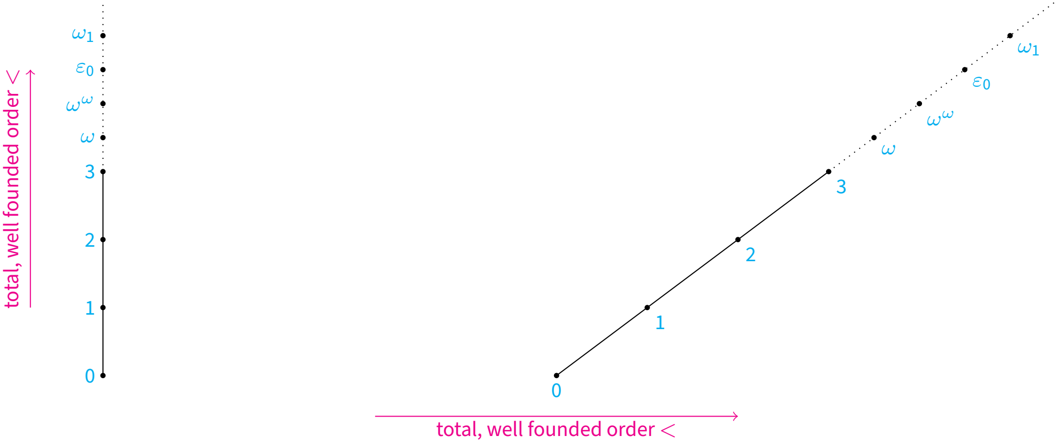
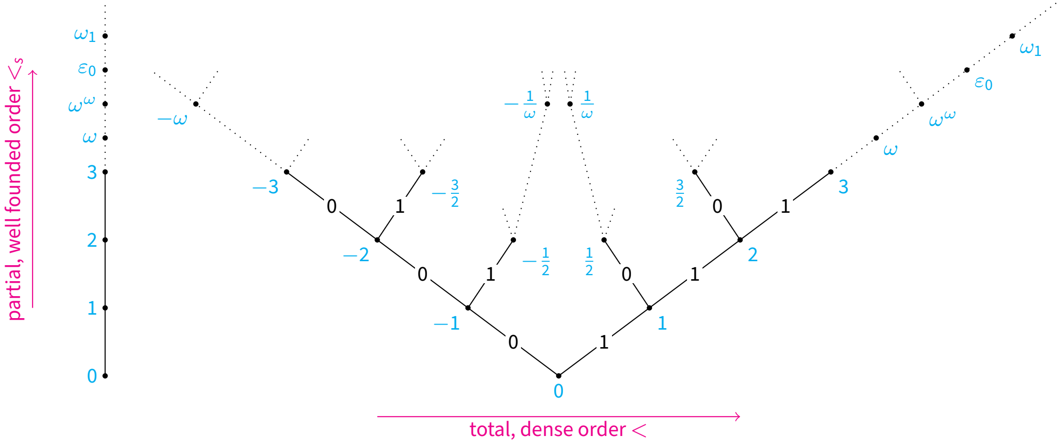
Surreal numbers



Surreal numbers

- **No** $:= 2^{<\mathbf{On}} = \bigcup_{\alpha \in \mathbf{On}} 2^{\alpha}$ (a surreal number is a function $\alpha \to 2 = \{0, 1\}$)
- $x <_s y$ if $x \subsetneq y$ (we shall say "x is simpler than y")
- $x < y$ if $x(\beta) < y(\beta)$ on the minimum β such that $x(\beta) \neq y(\beta)$
  (where $0 < \perp < 1$, and "$x(\beta) = \perp$" when $\beta \notin \mathrm{dom}(x)$)



Arithmetic operations – refresher on ordinals

Classical ordinal arithmetic

$S(\alpha) := \min\{\beta > \alpha\}$ $\quad$ $\alpha \odot \beta := \sup\{S((\alpha \odot \beta') \oplus \alpha') : \beta' < \beta, \alpha' < \alpha\}$

$\alpha \oplus \beta := \sup\{\alpha, S(\alpha \oplus \beta') : \beta' < \beta\}$ $\quad$ $\alpha^\beta := \sup\{1, S(\alpha^{\beta'} \odot \alpha' \oplus \gamma) : \beta' < \beta, \alpha' < \alpha, \gamma < \alpha^{\beta'}\}$

These are all increasing in the second argument, but not commutative, e.g. $1 \oplus \omega = \omega < \omega \oplus 1$.

Hessenberg natural arithmetic

$\alpha + \beta := \sup\{S(\alpha + \beta'), S(\alpha' + \beta) : \beta' < \beta, \alpha' < \alpha\}$

$\alpha \cdot \beta = \alpha\beta := \min\{\gamma : \gamma + \alpha'\beta' > \alpha'\beta + \alpha\beta' \text{ for } \alpha' < \alpha, \beta' < \beta\}$

These are all increasing in both arguments, and are commutative $1 + \omega = \omega + 1 = \omega \oplus 1 = S(\omega)$.
Moreover, $\omega^{\alpha+\beta} = \omega^\alpha\omega^\beta$.

Degree and Cantor Normal Form

$\deg(\alpha) := \sup\{\beta : \omega^\beta \leq \alpha\}$ (and $\deg(0) := -\infty$). That's always a maximum.

$\alpha = \omega^{\beta_1}k_1 + \omega^{\beta_2}k_2 + \ldots + \omega^{\beta_n}k_n$ (Cantor Normal Form)
for unique $\beta_1 > \beta_2 > \ldots > \beta_n$ and $k_1, \ldots, k_n \in \mathbb{N}^{\neq 0}$ (where $\beta_1 = \deg(\alpha)$).

Arithmetic operations – surreal numbers

Hessenberg natural arithmetic – alternative characterisation

Let $\leq$ be the natural partial order on $\mathbf{On}^n$. Given $f, g : \mathbf{On}^n \to \mathbf{On}$, say that $f \leq g$ if for every $\overrightarrow{\alpha}$ minimal such that $f(\overrightarrow{\alpha}) \neq g(\overrightarrow{\alpha})$ we have $f(\overrightarrow{\alpha}) < g(\overrightarrow{\alpha})$.

- $+$ is the $\leq$-least function that is strictly increasing in both arguments
- $\cdot$ is the $\leq$-least function such that $\alpha\beta + \alpha'\beta' > \alpha'\beta + \alpha\beta'$ for $\alpha' < \alpha, \beta' < \beta$
- $\alpha \mapsto \omega^\alpha$ is the $\leq$-least function such that $\omega^\alpha > \omega^{\alpha'}k, 0$ for $\alpha' < \alpha, k \in \mathbb{N}$

Remark. The partial order $\leq$ on $\mathbf{On}^n$ is well founded, but $\leq$ on the functions $\mathbf{On}^n \to \mathbf{On}$ is not.

Surreal arithmetic

Let $\leq_s$ be the natural partial order on $\mathbf{No}^n$. Given $f, g : \mathbf{No}^n \to \mathbf{No}$, say that $f \leq_s g$ if for every $\overrightarrow{x}$ minimal such that $f(\overrightarrow{x}) \neq g(\overrightarrow{x})$ we have $f(\overrightarrow{x}) <_s g(\overrightarrow{x})$.

- $+$ is the $\leq_s$-least function that is strictly increasing in both arguments
- $\cdot$ is the $\leq_s$-least function such that $xy + x'y' > x'y + xy'$ for $x' < x, y' < y$
- $x \mapsto \omega^x$ is the $\leq_s$-least function such that $\omega^x > \omega^{x'}k, 0$ for $x' < x, k \in \mathbb{N}$

Remark. The partial order $\leq_s$ on $\mathbf{No}^n$ is well founded, but $\leq_s$ on the functions $\mathbf{No}^n \to \mathbf{No}$ is not.

Theorem (Conway). $(\mathbf{No}, <, +, \cdot)$ is a (saturated) real closed field.
No contains a canonical (i.e. "$\leq_s$-least") copy of $\mathbb{R}$ and of **On**.

Other surreal functions

f	...is $\leq_s$-least such that:
$+$	$x+z<y+z, z+x<z+y$ for $x<y$ (more restrictively: $(\mathbf{No},<,+)$ is an ordered group) [Conway '76]
$\cdot$	$xy+x'y'>x'y+xy'$ for $x'<x, y'<y$ (more restrictively: $(\mathbf{No},<,+,\cdot)$ is an ordered ring) [Conway '76]
exp	$\exp:(\mathbf{No},+,<)\xrightarrow{\sim}(\mathbf{No}^{>0},\cdot,<)$, $\exp(r)=e^r$ for $r\in\mathbb{R}$, $\exp(x)>x^n$ for $x>\mathbb{N}, n\in\mathbb{N}$ $\exp(\varepsilon)=1+\varepsilon+\ldots+\frac{\varepsilon^n}{n!}+o(\varepsilon^n)$ for $\varepsilon\prec 1$ and $n\in\mathbb{N}$ [Gonshor '80]
$f_{\restriction[-1,1]^k}$ restricted analytic	for $k=1$, define $x\mapsto(f(x),f'(x),f''(x),\ldots)\in\mathbf{No}^\omega$; [ok, for $k>1$ I can't fit it on this slide.] $f(x+\varepsilon)=f(x)+f'(x)\varepsilon+\ldots+\frac{f^{(n)}(x)}{n!}\varepsilon^n+o(\varepsilon^n)$ for $x\preceq 1, \varepsilon\prec x, n\in\mathbb{N}$ [Neumann '49+Alling '87]
∂	$\partial(x+y)=\partial x+\partial y$, $\partial(xy)=\partial x\cdot y+x\cdot\partial y$, $\partial\mathbb{R}=0$, $\partial\exp(x)=\exp(x)\cdot\partial x$ and if $x>\mathbb{N}$, then $\partial x>0$ ($\equiv\partial$ is an exp-compatible H-field derivation with $\ker(\partial)=\mathbb{R}$) [Berarducci-M '18]
L_ω	$L_\omega(\exp(x))=L_\omega(x)+1$; $L_\omega(x)<\log(\ldots(\log(x)))$; $L_\omega(x+\varepsilon)=L_\omega(x)+\frac{1}{x\log(x)\log(\log(x))\cdots}\varepsilon+\ldots$ [Bagayoko-van der Hoeven-M]
ω^x	$\omega^x>\omega^{x'}k$ for $x'<x$ and $k\in\mathbb{N}$ [Conway '76]
$\lfloor\cdot\rfloor$	$\lfloor x\rfloor\leq x<\lfloor x\rfloor+1$ for all x [Conway '76]

(See also Costin-Ehrlich-Friedman, Berarducci-M '19 for EB-summable functions.)

First order theory of surreal functions

Theorem (Conway '76). $(\mathbf{No}, <, +, \cdot)$ is a real closed field.

Theorem (van den Dries-Ehrlich '01). $(\mathbf{No}, <, +, \cdot, \exp, f)_{f \text{ restricted analytic}}$ is a model of $\mathrm{Th}(\mathbb{R}, <, +, \cdot, \exp, f)_{f \text{ restricted analytic}}$.

Theorem (Aschenbrenner-van den Dries-van der Hoeven '19). $(\mathbf{No}, <, +, \cdot, \partial)$ is a model of the theory of *LE*-series (which is model complete in the language of ordered valued differential fields).

Theorem ($\sim$Shepherdson '64). $(\mathbf{Oz}^{\geq 0}, <, +, \cdot)$ is a model of Open Induction (where $\mathbf{Oz} = \lfloor \mathbf{No} \rfloor$).
But it's not a model of Peano Arithmetic: the fraction field is real closed (it's **No**!).

We do not know much about *EB*-summable functions, and definitely nothing about L_ω.

What's so special about the simplest functions?

Definition. Call $A \subseteq \mathbf{No}$ initial if for all $x \in A$, $y \in \mathbf{No}$, $y \leq_s x$ implies $y \in A$ (so A is $\leq_s$-downward closed).

Theorem (Ehrlich 2001, Fornasiero 2006). If $A \subseteq \mathbf{No}$ is initial, then the $\mathcal{L}$-structure generated by A, denoted by $\langle A \rangle_{\mathcal{L}}$, is initial for the following languages:

- $\mathcal{L} = \{+, -\}$ (hence $\langle A \rangle_{\mathcal{L}} =$ the group generated by A);
- $\mathcal{L} = \{+, -, \frac{1}{n} \cdot\}$ (hence $\langle A \rangle_{\mathcal{L}} =$ the divisible group generated by A);
- $\mathcal{L} = \{+, -, \cdot\}$ (hence $\langle A \rangle_{\mathcal{L}} =$ the ring generated by A);
- $\mathcal{L} = \{+, -, \cdot, (\cdot)^{-1}\}$ (hence $\langle A \rangle_{\mathcal{L}} =$ the field generated by A);
- $\mathcal{L} = \{+, -, \cdot, (\cdot)^{-1}, f\}_{f \text{ restricted algebraic}}$ (hence $\langle A \rangle_{\mathcal{L}} =$ the real closed field generated by A);
- $\mathcal{L} = \{+, -, \cdot, (\cdot)^{-1}, f\}_{f \text{ restricted analytic}}$; $\mathcal{L} = \{+, -, \cdot, (\cdot)^{-1}, f, \exp\}_{f \text{ restricted analytic}}$;
- (added by Ehrlich-Kaplan '20) $\mathcal{L} = \{+, -, \cdot, (\cdot)^{-1}, \exp, f\}_{f \text{ in a convergent Weierstrass system } W}$.

Remark. This is *not* true if we skip some functions. For instance, $A = \{0, 1, 2\}$ is initial, but $\langle 0, 1, 2 \rangle_{\{\cdot\}} = \{0, 1, 2, 4, \ldots\}$ is not.

Proof sketch

Proposition. If $A \subseteq \mathbf{No}$ is initial, then the closure of A under $+$ is initial.
Proof. Pick $(x, y) \in A^2$ $\leq_s$-minimal such that $x + y \notin A$.

- Let $z \in \mathbf{No}$ such that $z \notin A$ and $z \leq_s x + y$.
- By definition of $<$, $a < z \Leftrightarrow a < x + y$ for all $a \in A$.
- Let $A' = \{x', y', x' + y' : (x', y') <_s (x, y)\}$. Note that $A' \subseteq A$, $x, y \in A'$.
- Now take an automorphism σ of $(\mathbf{No}, <)$ fixing A' and such that $\sigma(x + y) = z$.
- Define $z +^\sigma w := \sigma(\sigma^{-1}(z) + \sigma^{-1}(w))$. Then (x, y) is $\leq_s$-minimal such that $x + y \neq x +^\sigma y$.
- Since $+ \leq_s +^\sigma$, we have $x + y \leq_s x +^\sigma y = z \leq_s x + y$, hence $x + y = z$.
- Therefore, $A \cup \{x + y\}$ is initial. Now continue by induction.

General case. For the language $\mathcal{L} = \{+, -\}$, first close A under $+$, and change the definition of A' so that it is closed under $+$ as well. Likewise for $\mathcal{L} = \{+, -, \cdot\}$. In general, work by induction on the size of the language (as long as you remember to enumerate $+, \cdot$ first!).

Initial substructures

Theorem (Fornasiero 2006, Ehrlich-Kaplan 2020). If $A \subseteq \mathbf{No}$ is initial, then $\langle A\rangle_{\mathcal{L}}$ is initial for:

- $\mathcal{L} = \{+, -, \frac{1}{n}\cdot\}$ (hence $\langle A\rangle_{\mathcal{L}}$ = the divisible group generated by A);
- $\mathcal{L} = \{+, -, \cdot, (\cdot)^{-1}, f\}_{f \text{ restricted algebraic}}$ (hence $\langle A\rangle_{\mathcal{L}}$ = the real closed field generated by A);
- $\mathcal{L} = \{+, -, \cdot, (\cdot)^{-1}, \exp?, f\}_{f \text{ in a convergent Weierstrass system } W}$.

Corollary. If $M \models T$, then there is an embedding $\iota : M \to \mathbf{No}$ with $\iota(M)$ initial, for T the theory of divisible ordered groups, real closed fields, or T_{an}, $T_{\mathrm{an,exp}}$, $T_{W,\mathrm{exp}}$.

Proof. Write $M = \bigcup_\alpha M_\alpha$ where $M_{\alpha+1} = M_\alpha\langle x_\alpha\rangle_{\mathcal{L}}$. We may assume that each M_α is a model (add $\frac{1}{n}\cdot$ and/or the algebraic functions to $\mathcal{L}$ if necessary: we just need $M_\alpha\langle x_\alpha\rangle_{\mathcal{L}} = \mathrm{dcl}_{\mathcal{L}}(M_\alpha x_\alpha)$).

- Given $\iota_\alpha : M_\alpha \to \mathbf{No}$ with $\iota_\alpha(M_\alpha)$ initial, and the cut satisfied by x_α over M_α, map x_α to the simplest y_α in the corresponding cut over $\iota_\alpha(M_\alpha)$.
- Then $\iota_\alpha(M_\alpha) \cup \{y_\alpha\}$ is initial and y_α has the same type as x_α by o-minimality.
- Therefore, $M_{\alpha+1} \cong \iota_\alpha(M_\alpha)\langle y_\alpha\rangle_{\mathcal{L}}$, and the latter is initial.

Remark. We used (1) $\mathrm{Th}_{\mathcal{L}}(\mathbf{No})$ o-minimal, (2) $\langle A\rangle_{\mathcal{L}}$ model of T for all A, (3) $\langle A\rangle_{\mathcal{L}}$ initial for all A initial.

Question. Let $\mathcal{L}$ be such that $\langle A\rangle_{\mathcal{L}}$ is initial for all A initial. Let $\mathcal{C}_{\mathcal{L}}$ be the class of the $\mathcal{L}$-structures with initial embeddings into $\mathbf{No}$. Is $\mathcal{C}_{\mathcal{L}}$ elementary? Does $\mathcal{C}_{\mathcal{L}}$ have the amalgamation property?

Remark. Ehrlich, Ehrlich-Kaplan give some algebraic characterisation for $\mathcal{C}_{\mathcal{L}}$ for some $\mathcal{L}$.

Existence of simplest functions

The existence of $\leq_s$-minimum functions is ad hoc for each case. One builds a very plausible candidate, then verifies that it is correct.

For the traditional $+, \cdot, \exp$, Conway and Gonshor use "recursive" or "genetic" definition. These are roughly equivalent to taking the simplest function satisfying a collection of strict inequalities.

f	...is $\leq_s$-least such that:
$+$	$x+z<y+z, z+x<z+y$ for $x<y$ (drop "$+$" is a group operation)
$\cdot$	$xy+x'y'>x'y+xy'$ for $x'<x, y'<y$
exp	... does not fit here

Conway proceeds by transfinite recursion. Pick $x, y \in \mathbf{No}$. Let $x+y$ be the simplest value that fits the conditions just w.r.t. $x'+y'$ for $(x', y') <_s (x, y)$. One verifies that such value always exist (here strict inequalities really help), and that the resulting function satisfies the requirements.

One then verifies that $x+y=y+x$ (easy by symmetry), $x+(y+z)=(x+y)+z$, the existence of inverses, divisibility…

Simplest derivation

The genetic approach does not scale well. As of now, there is no genetic definition of ∂.

f	...is $\leq_s$-least such that:
∂	$\partial(x+y) = \partial x + \partial y, \partial(xy) = \partial x \cdot y + x \cdot \partial y, \partial\mathbb{R} = 0, \partial \exp(x) = \exp(x) \cdot \partial x$ and if $x > \mathbb{N}$, then $\partial x > 0$ ($\equiv \partial$ is an exp-compatible H-field derivation with $\ker(\partial) = \mathbb{R}$)

The construction of ∂ goes as follows:

- Write each $x \in$ **No** in normal form $x = \sum_{i<\alpha} r_i \omega^{y_i} = \sum_{i<\alpha} r_i e^{\gamma_i}$.
- Require $\partial x = \sum_{i<\alpha} r_i e^{\gamma_i} \partial \gamma_i$.
- Iterate and impose $\partial x = \sum_{i_1, i_2, \ldots, i_n} r_{i_1} s_{i_1 i_2} \ldots e^{\gamma_{i_1}} e^{\mu_{i_1 i_2}} \ldots \partial \lambda_{i_1 i_2 \ldots i_n}$, where for each $i_1 \ldots i_n$ we take n minimum such that $\lambda_{i_1 i_2 \ldots i_n}$ is log-atomic: its normal form is $e^{\lambda'}$, and λ' is log-atomic too.
- Then (1) define ∂ on the log-atomics as the simplest satisfying some inequalities.
- And (2) verify that the summands range in an anti-well ordered set (or more precisely, form a 'Noetherian' family), and thus can be summed.

Step (2) uses the simplicity of exp in a crucial way: one proves that certain manipulations of the normal form $\sum_{i<\alpha} r_i e^{\gamma_i}$ (such as truncating the tail) move surreal numbers downward for $\leq_s$, and thus they can only be applied finitely many times.

Further questions

Question. Suppose T is some o-minimal theory in a language $\mathcal{L}$ with no relation symbols beyond $<$, with quantifier elimination and universal axiomatisation (or similar). Enumerate $\mathcal{L} = \{<, f_0, f_1, \ldots\}$. Is there an $\mathcal{L}$-structure on **No** such that each f_α is the $\leq_s$-minimum function that makes **No** a model of $T \restriction_{\{<, f_\beta\}_{\beta<\alpha}}$?

And can this be done for other theories such as $\mathrm{Th}(\mathbf{No}, <, +, \cdot, \partial)$ (which is only "o-minimal at infinity")?

Question. Does every model of $\mathrm{Th}(\mathbf{No}, <, +, \cdot, \partial)$ with $\ker(\partial) \cong \mathbb{R}$ embed initially into **No**?