FLUTTERS and CHAMELEONS

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Leeds virtual logic seminar

Thursday June 25, 2020

The infinitary Ramsey principle $\omega \longrightarrow (\omega)^\omega_k$, where $\omega =_{\text{df}} \{0, 1, 2, \ldots\}$ and $2 \leqslant k < \omega$, says that if $\pi : [\omega]^\omega \longrightarrow k$ then for some infinite $X \subseteq \omega$, π is constant on the set $[X]^\omega$ of infinite subsets of X.

Solovay, in famous work soon after Cohen's invention of forcing, used a strongly inaccessible cardinal to construct a model of $\mathsf{ZF} + \mathsf{DC}$ in which various principles hold which contradict AC:

LM: every set of real numbers is Lebesgue measurable;
PB: every set of real numbers has the property of Baire;
UP: every uncountable set of real numbers has a perfect subset.

Mathias showed in 1968 that in Solovay's model, this principle holds:

RAM: all colourings are Ramsey; in symbols, $\omega \longrightarrow (\omega)^\omega$;

and in 1969 that using a Mahlo cardinal, the Solovay model satisfies

NoMAD: no maximal infinite AD family of infinite subsets of ω.

DIGRESSION

PROPOSITION *There is an arithmetical subset N of the square $[\omega]^\omega \times [\omega]^\omega$ such that for each A, $\{B \mid_B (A,B) \in N\}$ is CR$^+$ and for each B $\{A \mid_A (A,B) \in N\}$ is CR$^-$.*

COROLLARY *There is no analogue for the Ellentuck topology to the theorems of Fubini and Kuratowski–Ulam.*

Proof : Let us say that A is *much denser than* B if

$$\lim \overline{\overline{A \cap [\tilde{B}(n), \tilde{B}(n+1))}} = \infty,$$

where $\tilde{B} : \omega \longrightarrow \omega$ is the monotonic enumeration of B. Let $N =_{\rm df} \{\langle A,\ B\rangle \mid_{A,B} A$ is much denser than $B\}$

End of digression

It is natural to ask whether these large cardinals are necessary; in some cases the answer is known:

Specker had shown in the 1950s that UP implies that the true ω_1 is strongly inaccessible in L and in each $L[\alpha]$ for α a real; Shelah showed that LM implies the same thing, but, surprisingly, PB does not.

More recently Törnquist has shown that NoMAD holds in Solovay's original model; Shelah and Horowitz have extended his work to show that even that inaccessible is unnecessary to get a model of NoMAD; Törnquist and Schrittesser have shown that if all sets are Ramsey then NoMAD.

But it is still open, even after fifty years whether RAM implies that ω_1 is inaccessible to reals.

Let Ω be a countably infinite set: in this paper, it will usually be either ω, $\omega \times \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$.

We define equivalence relations on $\mathcal{P}(\Omega)$:

$$B \sim_* C \iff_{\mathrm{df}} B \triangle C \text{ is finite;}$$

$$B \sim_k C \iff_{\mathrm{df}} \begin{cases} B \sim_* C \;\&\; \overline{\overline{B \setminus C}} = \overline{\overline{C \setminus B}} & \text{if } k = 0, \\ B \sim_* C \;\&\; \overline{\overline{B \setminus C}} \equiv \overline{\overline{C \setminus B}} \mod k & \text{if } k > 1; \end{cases}$$

Remark All those equivalence relations have the property that if $B \sim C$ then $\Omega \setminus B \sim \Omega \setminus C$.

Warning Many authors write E_0 for the equivalence relation $\sim_*$, but our notation fits better with our other equivalence relations.

DEFINITION If R and S are equivalence relations on Ω with every R-class being a union of S-classes—when we say S *is nested in* R— then an (R, S, Ω)-*selector* is a function that chooses from each R-class an S-class.

REMARK Many of the above equivalence relations are nested; in this notation an E_0-selector is an $(\sim_*, =, \Omega)$ selector. The equality relation on $[\omega]^\omega$ is nested in the relation $\sim_k$ which is nested in $\sim_*$ for each $k = 0, 2, 3, \ldots$.

EXAMPLE A $(\sim_*, \sim_k, \Omega)$-*selector* is a function choosing one $\sim_k$-class from each $\sim_*$-class on Ω.

PROPOSITION $A \sim_{\mathbb{Z}} B \iff \forall^\infty k\ A \sim_k B \iff \exists^\infty k\ A \sim_k B.$

EXPLANATION Here $\exists^\infty k$ means "there are infinitely many k" and $\forall^\infty k$ "for all but finitely many k"; so $\forall^\infty k \Phi(k)$ is equivalent to $\neg\exists^\infty k \neg\Phi(k)$.

REMARK As Ω is countably infinite, each $\sim_*$-class is a disjoint union of $\aleph_0$ $\sim_0$-classes, and for each $k \geqslant 2$, of exactly k $\sim_k$-classes.

Let us write $sel(T, S)$ to mean that S is nested in T and there is a function that selects from each T class an S-class.

PROPOSITION *Suppose that R is nested in S and S in T. The following are easily checked:*

i) $sel(T, S)$ & $sel(S, R) \implies sel(T, R)$;

ii) $sel(T, R) \implies sel(T, S)$;

iii) $sel(T, R)$ *need not imply* $sel(S, R)$.

PROPOSITION $sel(\sim_*, =) \implies sel(\sim_k, =)$ *for each* $k = 0, 2, 3, \ldots$.

Proof : Let F be an E_0-selector. Let $X \in [\omega]^\omega$, with $\omega \setminus X$ infinite. Let $Y = F(X)$. From Y we may define, for positive n, Y_n to be Y together with the first n elements of $\omega \setminus Y$, and Y_{-n} to be Y minus its first n elements. Now let $m = \overline{\overline{X \setminus Y}} - \overline{\overline{Y \setminus X}}$; then $Y_m \sim_0 X$, and so for every $k > 1$, $Y_m \sim_k X$. $\dashv$

DEFINITION For $k = 0, 2, 3, \ldots$ we write $\mathbb{Z}_k$ for the ring $\mathbb{Z}/k\mathbb{Z}$; we identify $\mathbb{Z}_0$ and $\mathbb{Z}$. A *k-chameleon* is a map $\chi : \mathcal{P}(\omega) \longrightarrow \mathbb{Z}_k$ such that

$$n \notin A \subseteq \omega \implies \chi(A \cup \{n\}) = \chi(A) + 1$$

Notice the cyclicity: the additive group of $\mathbb{Z}_k$ is cyclic, generated by 1, and is of order k when $k > 1$, infinite when $k = 0$.

PROPOSITION (ZF) *For $k = 0, 2, 3, \ldots$, the existence of a k-chameleon is equivalent to the existence of a $(\sim_*, \sim_k, \omega)$-selector.*

Proof : Given such a selector, let $(B)_{\sim_k}$ be the chosen member of $(A)_{\sim_*}$ and define

$$\chi(A) = \begin{cases} \overline{\overline{A \smallsetminus B}} - \overline{\overline{B \smallsetminus A}} & \text{if } k = 0, \\ \\ \overline{\overline{A \smallsetminus B}} - \overline{\overline{B \smallsetminus A}} \mod k & \text{if } k > 1. \end{cases}$$

Given a k-chameleon, define a $(\sim_*, \sim_k, \omega)$-selector by choosing from each $\sim_*$ class the set of its members assigned value 0 by the chameleon. $\dashv$

A *$\mathbb{Z}$-chameleon* is an alternative name for a 0-chameleon.

COROLLARY *If there is a* $(\sim_*, =, \omega)$ *selector, there is a* $\mathbb{Z}$*-chameleon.*

REMARK Plainly the existence of selectors, and therefore of chameleons, is guaranteed by the Axiom of Choice. For $k > 1$, the Axiom of Choice for families of sets of size k (commonly notated $\mathbf{C_k}$) will be enough, for using $\mathbf{C_k}$, one may choose one $\sim_k$-class from the k of them into which any $\sim_*$-class splits. Further, $\mathbf{C_k}$ for finite k follows from the principle that each set has a linear ordering.

Constructing a $(\sim_*, \sim_{\mathbb{Z}}, \omega)$ selector seems to require a stronger form of AC: for implications between the existence of $\mathbb{Z}$-chameleons (in a more general setting) and weak forms of the axiom of choice, see [Mo].

The Belgian economist Luc Lauwers and the set theorists Giorgio Laguzzi and his collaborators have established serious links between $\mathbb{Z}$-chameleons and the social welfare relations of mathematical economics.

THEOREM *Let $\sim$ be an analytic equivalence relation on $\mathcal{P}(\omega)$ with all classes countable. Then for some infinite $a \subseteq \omega$, $\sim\restriction [a]^\omega$ is hyperfinite.*

Suppose that $\sim$ is $\Sigma^1_1(d)$ where d is a real parameter.

LEMMA *If $X \sim Y$ then $\langle X, d\rangle =_{\mathrm{HYP}} \langle Y, d\rangle$.*

Proof : if $X \sim Y$ then $X \in \{Z \mid Z \sim Y\}$, a countable set that is $\Sigma^1_1(d, Y)$; by Harrison all members are HYP in d and Y. Use symmetry $\dashv$

Now let M be a countable transitive model of enough set theory, with d in M. Let a be Mathias generic over M. (To be more precise, suppose that F is in M a selective ultrafilter and that a is $(M, \mathbb{P}_F)$ generic.)

PROPOSITION *If b and c are in $[a]^\omega$ and $b \sim c$ then $b \triangle c$ is finite.*

Proof : by the Lemma, if $b \sim c$ then b is HYP in c and d, so $b \cup c$ is HYP in c and d; But $b \cup c$ is also Mathias generic over M; by arguments as in the proof of Theorem 8.2 of *Happy Families*, it cannot be HYP in d and a subset of itself from which its difference is infinite. So $b \setminus c$ is finite. By symmetry, $c \setminus b$ is finite. Hence the result. $\dashv$

Proof of the Theorem: by the Proposition we know that on $[a]^\omega$, $b \sim c \implies b \triangle c$ is finite. So define $b \sim_n c$ to hold if $b \sim c$ and $b \triangle c \subseteq n$. Each $\sim_n$ is an equivalence relation with finite classes and the union of these relations, which increase with n, is $\sim$. $\dashv$

REMARK Since $b \triangle c$ finite implies that $b =_{\text{Turing}} c$, the result shows that if $\sim$ is $=_{\text{Turing}}$ then restricted to $[a]^\omega$ the equivalence relation is exactly $\sim_*$; the same conclusion holds for $=_{\text{HYP}}$, even though that is a co-analytic but not analytic relation, as the Lemma (but not its proof) still applies.

EXAMPLE The relation $\approx$ where $X \approx Y$ if the symmetric difference is not only finite but of even cardinality, is hyperfinite: for each n say that $X \approx_n Y$ if X results from Y by making an even number of changes, all less than n. But on no $[a]^\omega$ is $\approx$ exactly $\sim_*$.

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The Baumgartner method

Another method for constructing chameleons emerged in correspondence in 1973 between Mathias and Baumgartner. Fix an integer $k \geqslant 2$. For an infinite subset A of ω, and $0 \leqslant i < k$ define

$$\mathfrak{b}_i^k(A) = \{n \in \omega \mid \overline{\overline{n \cap A}} \equiv i \pmod{k}\},$$

then $\bigcup_i \mathfrak{b}_i^k(A) = \omega$ and for $i < j < k$, $\mathfrak{b}_i^k(A) \cap \mathfrak{b}_j^k(A) = \varnothing$.

Remark $A \sim_k B \iff \forall i < k\ \mathfrak{b}_i^k(A) \sim_* \mathfrak{b}_i^k(B)$.

A filter on ω is *feeble* if there is a partition of ω into disjoint finite intervals s_i such that every $X \in F$ meets all but finitely many s_i's.

Proposition ([M3]) *Suppose that there is a non-feeble filter on ω. Then there is a non-Ramsey set.*

Jalali-Naini [J-N] and Talagrand [T] later showed that a filter on ω is feeble if and only if it is meagre; so if PB, every filter on ω is feeble.

PROPOSITION *Let U be a free ultrafilter and let $2 \leqslant k < \omega$. Then there is a k-chameleon.*

Proof : Let $X \in [\omega]^\omega$. Then ω being the disjoint union of the k infinite sets $\mathfrak{b}_0^k(X), \mathfrak{b}_1^k(X), \ldots \mathfrak{b}_{k-1}^k(X)$, exactly one of them, say $\mathfrak{b}_j^k(X)$, is in U. We define $\pi_k^U(X)$ to be that j. If $X = Y'$ then for $i \geqslant 1$, $\mathfrak{b}_i^k(Y) = \mathfrak{b}_{i+1}^k(X)$, and for $i = 0$, $\mathfrak{b}_0^k(Y) = \mathfrak{b}_1^k(X)$ plus a finite set. Hence $\pi_k^U(X) = \pi_k^U(Y)+1$. $\dashv$

Call $\rho : [\omega]^\omega \longrightarrow 2$ invariant if $A \sim_* B \implies \rho(A) = \rho(B)$. Write $\omega \xrightarrow{\triangle} (\omega)^\omega$ to mean that all invariant colourings are Ramsey. Then from his 2012 Singapore thesis:

THEOREM (Dongxu Shao) ($\mathsf{AD}+\ \mathsf{DC}$) *If $\omega \xrightarrow{\triangle} (\omega)^\omega$ then $\omega \longrightarrow (\omega)^\omega$.*

The proof in (M3) extends to prove:

THEOREM (ZF) *If $\omega \xrightarrow{\triangle} (\omega)^\omega$ and there is a non-feeble filter, then there is a 2-chameleon.*

Power series, partitions and more equivalences

An algebraic k-chameleon is a map $\bar{\chi} : \mathbb{Z}_k[[X]] \longrightarrow \mathbb{Z}_k$ such that

$$\bar{\chi}(P + X^n) = \bar{\chi}(P) + 1$$

We write $c_n(P)$ for the coefficient of X^n in the member P of the formal power series ring $\mathbb{Z}_k[[X]]$, so that P may be written $\sum_{n\in\omega} c_n(P)X^n$ or $\sum_n c_n(P)X^n$; and then for $r \in \mathbb{Z}_k$ we define $A_r(P) = \{n \mid c_n(P) = r\}$.

We write S for the formal power series $\sum_n X^n$.

Remember that in $\mathbb{Z}_k[[X]]$, $(1 - X)S = 1$.

We consider polynomials to be series with almost all coefficients 0.

For series P and Q in $\mathbb{Z}[[X]]$ or $\mathbb{Z}_k[[X]]$ define

$$\begin{aligned} P \sim^* Q &\iff_{\text{df}} P - Q \text{ is a polynomial} \\ P \sim^0 Q &\iff_{\text{df}} P \sim^* Q \;\&\; (P-Q)(1) = 0 \\ P \sim^k Q &\iff_{\text{df}} P \sim^* Q \;\&\; (P-Q)(1) \equiv 0 \mod k \end{aligned}$$

A k-partition is a sequence $\langle A_g \mid g \in \mathbb{Z}_k \rangle$ of possibly empty subsets of ω, with $\bigcup_{g \in \mathbb{Z}_k} A_g = \omega$ and $A_g \cap A_h = \varnothing$ whenever $g \neq h$.

Write $\mathbf{P}_k$ for the set of all k-partitions.

A bijective correspondence between partitions and series

Given $P \in \mathbb{Z}_k[[X]]$ let $\mathcal{A}_P = \langle A_r(P) \mid_r r \in \mathbb{Z}_k \rangle$.

Given $\mathcal{B} \in \mathbf{P}_k$, let $P_{\mathcal{B}} = \sum_{g \in \mathbb{Z}_k} \sum_{n \in B_g} gX^n$

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The shift of a partition

If $\mathcal{A} = \langle A_g \mid_g g \in \mathbb{Z}_k \rangle$, $\mathfrak{s}(\mathcal{A}) =_{\mathrm{df}} \langle A_{g-1} \mid_g g \in \mathbb{Z}_k \rangle$.

PROPOSITION $\qquad \mathfrak{s}(\mathcal{A}_P) = \mathcal{A}_{P+S}$.

An equivalence relation on partitions

$$\mathcal{A} \,{}^*\!\!\sim \mathcal{B} \iff_{\mathrm{df}} P_{\mathcal{A}} \sim^* P_{\mathcal{B}}$$

Thus if $\mathcal{A} \,{}^*\!\!\sim \mathcal{B}$ and $k > 1$, $A_g \sim_* B_g$ for each of the finitely many $g \in \mathbb{Z}_k$; if $k = 0$, then $A_g \sim_* B_g$ for finitely many g and $A_g = B_g$ for the rest.

A *k-flutter* is a map $\phi : \mathbf{P}_k \longrightarrow \mathbb{Z}_k$ such that

$$\mathcal{A} \,{}^*\!\!\sim \mathcal{B} \implies \phi(\mathcal{A}) = \phi(\mathcal{B})$$
$$\phi(\mathfrak{s}(\mathcal{A})) = \phi(\mathcal{A}) + 1$$

The first condition on $\bar{\phi}$ may be weakened to $\bar{\phi}(P + X^n) = \bar{\phi}(P)$.

An algebraic k-flutter is a map $\bar{\phi} : \mathbb{Z}_k[[X]] \longrightarrow \mathbb{Z}_k$ such that

$$P \sim^* Q \implies \bar{\phi}(P) = \bar{\phi}(Q)$$
$$\bar{\phi}(P + S) = \bar{\phi}(P) + 1$$

THEOREM (ZF) *For each* $k = 0, 2, 3, \ldots$ *the following are equivalent:*

i) there exists a k-chameleon

ii) there exists an algebraic k-chameleon

iii) there exists an algebraic k-flutter

iv) there exists a k-flutter

The Henle method

DEFINITION For each $x \in [\omega]^\omega$, let $\tilde{x}$ be the monotonic enumeration of x; let $x' = \{\tilde{x}(n) \mid 0 < n \in \omega\}$, (often called the *shift* of x) so that for χ a k-chameleon, $\chi(x') = \chi(x) - 1 \bmod k$; and for $2 < k < \omega$ and $\ell < k$ let $\mathfrak{h}^k_\ell(x) = \{\tilde{x}(kn + \ell) \mid n \in \omega\}$.

REMARK Note that if $b = a'$, then $\mathfrak{h}^3_0(b) = \mathfrak{h}^3_1(a)$, $\mathfrak{h}^3_1(b) = \mathfrak{h}^3_2(a)$ and $\mathfrak{h}^3_2(b) = (\mathfrak{h}^3_0(a))'$.

More generally, $\mathfrak{h}^k_i(b) = \mathfrak{h}^k_{i+1}(a)$ for $i < k-1$, and $\mathfrak{h}^k_{k-1}(b) = (\mathfrak{h}^k_0(a))'$.

LEMMA *Let $K \subseteq [\omega]^\omega$ be such that for all $A \in [\omega]^\omega$, $A \in K \iff A' \in K$. Then $D \sim_* E \in K \implies D \in K$.*

Proof : If $D \sim_* E$ there are m, n and B such that $D^{(m)} = B = E^{(n)}$. Then $E \in K \implies B \in K \implies D \in K$. $\dashv$

A *barren* extension [HMW] is one which adds no new maps from an ordinal into the ground model.

THEOREM *If* $\omega \xrightarrow{\triangle} (\omega)^\omega$, *then the Hausdorff extension is barren.*

Proof : a modest refinement of the argument in [HMW]. We write conditions as (p), where $p \in [\omega]^\omega$, and (p) is its $\sim_*$-class. Suppose that κ is an ordinal and that $(p_0) \Vdash \dot{f} : \hat{\kappa} \longrightarrow \hat{V}$. For $p \in [p_0]^\omega$, define $\psi(p)$ to be the least ordinal ζ such that for no $x \in V$ does (p) force $\dot{f}(\hat{\zeta}) = \hat{x}$.

Define

$$\pi(p) = \begin{cases} 0 & \text{if } \psi(\mathfrak{h}_0^2(p)) = \psi(\mathfrak{h}_1^2(p)) \\ 1 & \text{if } \psi(\mathfrak{h}_0^2(p)) \neq \psi(\mathfrak{h}_1^2(p)) \end{cases}$$

Note that π is invariant. So let $\bar{p} \in [p_0]^\omega$ be homogeneous for it, and let $\eta = \psi(\bar{p})$.

Let u, v be two disjoint members of $[\bar{p}]^\omega$ such that for two distinct members a, b of the ground model $(u) \Vdash \dot{f}(\hat{\eta}) = \hat{a}$ and $(v) \Vdash \dot{f}(\hat{\eta}) = b$.

Note that any infinite subset x of $\bar{p}$ which has infinite intersection with both u and v will have $\psi(x) = \eta$, but that any infinite subset y of either u or v will have $\psi(y) > \eta$.

Now let q be an infinite subset of $\bar{p}$ whose members come in turn from $u, u, v, v, u, u, v, v, \ldots$. Then $\psi(\mathfrak{h}^2_0(q)) = \eta = \psi(\mathfrak{h}^2_1(q))$, so that $\pi(q) = 0$.

But let r be an infinite subset of $\bar{p}$ whose members come in turn from $u, u, u, v, u, u, u, v, \ldots$. Then $\mathfrak{h}^2_0(r) \subseteq u$ so $\psi(\mathfrak{h}^2_0(r)) > \eta = \psi(\mathfrak{h}^2_1(r))$, and $\pi(r) = 1$.

Thus $\bar{p}$ is not homogeneous for π, which contradiction shows that no such $\dot{f}$ exists and that the extension is therefore barren. $\dashv$

There has been much work on inner models of the form $L(\mathbb{R})[\mathfrak{U}]$ when they are barren extensions of $L(\mathbb{R})$ by a generic for the Hausdorff extension, by di Prisco, Todorcevic, Dobrinen, Hathaway, Larson, Zapletal, Raghavan and their collaborators.

New chameleons for old

PROPOSITION (ZF) *If there is a k-chameleon and either $k = 0$ or ℓ divides $k > 0$, then there is an ℓ-chameleon.*

PROPOSITION (ZF) *If χ is a k-chameleon and ψ is an ℓ-chameleon and $k > 1$ and $\ell > 1$ are co-prime, then $A \mapsto (\chi(A), \psi(A))$ is a $k\ell$-chameleon.*

PROPOSITION (ZF). *Let p be prime, $n > 0$. Suppose that there is a p^n-chameleon, χ. Then there is a p^{n+1}-chameleon.*

We could prove more if we assumed that all invariant colourings are Ramsey: indeed by early June 2013 we had the following:

THEOREM *Let k and ℓ be integers > 1. Then assuming $\omega \xrightarrow{\triangle} (\omega)^\omega$, there is a k-chameleon iff there is an ℓ-chameleon.*

That result inspired the following result of Nathan Bowler, proved in mid-June 2013:

COROLLARY *If* $\omega \xrightarrow{\triangle} (\omega)^\omega$, *there is no 2-chameleon.*

To sum up:

THEOREM *If* $\omega \xrightarrow{\triangle} (\omega)^\omega$ *all filters on* ω *are feeble and there is no chameleon of any kind.*

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