Constructing The Constructible Universe Constructively

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History

- The constructible universe was developed by Gödel in papers published in 1939 and 1940 to show the consistency of the Axiom of Choice and the Generalised Continuum Hypothesis with ZF.
- There are 2 main approaches to building L both of which are formalisable in KP[1] :
  - Syntactically as the set of definable subsets of M (See Devlin - *Constructibility*)
  - Using Gödel functions (See Barwise - *Admissible Sets*) or
  - Using Rudimentary Functions (See Gandy, Jensen, Mathias)
- The syntactic approach was then modified for IZF by Lubarsky (*Intuitionistic L* - 1993)
- And then for IKP by Crosilla (*Realizability models for constructive set theories with restricted induction* - 2000)

[1] In fact significantly weaker systems - see Mathias: *Weak Systems of Gandy, Jensen and Devlin*, 2006

Non-constructive Principles

- $\varphi \vee \neg\varphi$
- $\neg\neg\varphi \to \varphi$
- $(\varphi \to \psi) \to (\neg\varphi \vee \psi)$

- Foundation: $\forall a(\exists x(x \in a) \to \exists x \in a\, \forall y \in a(y \notin x))$
- Axiom of Choice / Well - Ordering Principle
- Definition by cases which differentiate between successor and limit ordinals

Remark

$\neg\varphi$ is interpreted as $\varphi \to (0 = 1)$.

Ordinals

Definition

An *ordinal* is a transitive set of transitive sets.

Remarks

- If α is an ordinal then so is $\alpha + 1 := \alpha \cup \{\alpha\}$.
- If X is a set of ordinals then $\bigcup X$ is an ordinal.
- $\beta \in \alpha \not\Rightarrow \beta + 1 \in \alpha + 1$.
- $\forall \alpha \; (0 \in \alpha + 1)$ implies excluded middle!

Trichotomy

- α is *trichotomous* $\forall \beta \in \alpha \; \forall \gamma \in \alpha \; (\beta \in \gamma \vee \beta = \gamma \vee \gamma \in \beta)$.
- It is consistent with IZF that the collection of trichotomous ordinals is a set!

Ordinals

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- $\beta \in \alpha \not\Rightarrow \beta + 1 \in \alpha + 1$.
- $\forall \alpha \ (0 \in \alpha + 1)$ implies excluded middle!

Definition

An ordinal α is a *weak additive limit* if $\forall \beta \in \alpha \ \exists \gamma \in \alpha \ (\beta \in \gamma)$.

An ordinal α is a *strong additive limit* if $\forall \beta \in \alpha \ (\beta + 1 \in \alpha)$.

Formula Complexity

Definition

The collection of Σ_0 formulae is the smallest collection of formulae closed under conjunction, disjunction, implication, negation and bounded quantification.

A formula is Σ_1 (Π_1) if it is of the form $\exists x \varphi(x)$ ($\forall x \varphi(x)$) for some Σ_0 formula $\varphi(v)$.

The collection of Σ formulae is the smallest collection containing the Σ_0 formulae which is closed under conjunction, disjunction, bounded quantification and unbounded existential quantification.

Example

$\forall x \in a \, \exists b \, (Trans(b) \; \wedge \; x \in b)$ is Σ but not Σ_1.

IKP

Definition (IKP*)

- Extensionality
- Empty Set
- Pairing
- Unions
- Set Induction (For any formula $\varphi(u)$, $\forall a(\forall x \in a\ \varphi(x) \to \varphi(a)) \to \forall a\ \varphi(a))$
- Bounded Collection (For any Σ_0 formula $\varphi(u, v)$ and set a, $\forall x \in a\ \exists y\ \varphi(x, y) \to \exists b\ \forall x \in a\ \exists y \in b\ \varphi(x, y))$
- Bounded Separation (For any Σ_0 formula $\varphi(u)$ and set a, $\{x \in a : \varphi(x)\}$ is a set)

Definition (IKP)

IKP is IKP* plus strong infinity
$(\exists a\ (Ind(a)\ \wedge\ \forall b\ (Ind(b) \to \forall x \in a(x \in b))))$[2].

[2] $Ind(a) \equiv \emptyset \in a\ \wedge\ \forall x \in a\ (x \cup \{x\} \in a)$

Basic Properties of IKP^*

- $\forall a, b\, \exists c, d\, (c = \langle a, b\rangle \;\wedge\; d = a \times b)$
- (Σ-Reflection) For any Σ formula φ,

$$\mathrm{IKP}^* \vdash \varphi \leftrightarrow \exists a\, \varphi^{(a)}$$ [3]

- (Strong Σ-Collection) For any Σ formula $\varphi(u, v)$ and set a,

$$\mathrm{IKP}^* \vdash \forall x \in a\, \exists y\, \varphi(x, y) \rightarrow \exists b\, \forall x \in a\, \exists y \in b\, \varphi(x, y) \;\wedge\; \forall y \in b\, \exists x \in a\, \varphi(x, y)$$

Remark

Δ separation; the assertion that whenever $\forall x \in a\, (\varphi(x) \leftrightarrow \psi(x))$ holds for φ a Σ formula and ψ a Π formula, $\{x \in a : \varphi(x)\}$ is a set, is not provable in IKP.

[3] $\varphi^{(a)}$ is the result of replacing each unbound quantifier with bounded by a.

Notation

- $x \times y := \{\langle u, v\rangle : u \in x \ \wedge \ v \in y\}$,
- For x an ordered pair
  - $1^{st}(x) := \{u : \exists v \ \langle u, v\rangle \in x\}$,
  - $2^{nd}(x) := \{v : \exists u \ \langle u, v\rangle \in x\}$,
- $\langle x, y, z\rangle := \langle x, \langle y, z\rangle\rangle$.
- $x''\{u\} := \{v : v \in 2^{nd}(x) \ \wedge \ \langle u, v\rangle \in x\}$

Gödel Functions

Definition

- $\mathcal{F}_p(x, y) := \{x, y\}$,
- $\mathcal{F}_\cap(x, y) := x \cap \bigcap y$
- $\mathcal{F}_\cup(x, y) := \bigcup x$,
- $\mathcal{F}_\setminus(x, y) := x \setminus y$,
- $\mathcal{F}_\times(x, y) := x \times y$,
- $\mathcal{F}_\rightarrow(x, y) := x \cap \{z \in 2^{nd}(y) : y \textit{ is an ordered pair } \wedge z \in 1^{st}(y)\}$,
- $\mathcal{F}_\forall(x, y) := \{x''\{z\} : z \in y\}$,

Gödel Functions

Definition

- $\mathcal{F}_{dom}(x, y) \coloneqq dom(x) = \{1^{st}(z) : z \in x \wedge z \text{ is an ordered pair}\}$,
- $\mathcal{F}_{ran}(x, y) \coloneqq ran(x) = \{2^{nd}(z) : z \in x \wedge z \text{ is an ordered pair}\}$,
- $\mathcal{F}_{123}(x, y) \coloneqq \{\langle u, v, w\rangle : \langle u, v\rangle \in x \ \wedge \ w \in y\}$,
- $\mathcal{F}_{132}(x, y) \coloneqq \{\langle u, w, v\rangle : \langle u, v\rangle \in x \ \wedge \ w \in y\}$,
- $\mathcal{F}_{=}(x, y) \coloneqq \{\langle v, u\rangle \in y \times x : u = v\}$,
- $\mathcal{F}_{\in}(x, y) \coloneqq \{\langle v, u\rangle \in y \times x : u \in v\}$.

Remark

Let $\mathcal{I}$ be the finite set indexing the above operations.

Generating Constructible Sets

Lemma (Barwise: Admissible Sets, Lemma II.6.1)

For every Σ_0 formula $\varphi(v_1, \ldots, v_n)$ with free variables among $v_1, \ldots, v_n$, there is a term $\mathcal{F}_\varphi$ built up from the Gödel functions such that

$$\mathrm{IKP} \vdash \mathcal{F}_\varphi(a_1, \ldots, a_n) = \{\langle x_n, \ldots, x_1 \rangle \in a_n \times \ldots \times a_1 : \varphi(x_1, \ldots, x_n)\}.$$

Proof.

- Call a formula $\varphi(x_1, \ldots, x_n)$ a termed-formula or *t-formula* if there is a term $\mathcal{F}_\varphi$ such that the conclusion of the lemma holds.
- Proceed by induction on Σ_0 formulae to show that every such formula is a t-formula.

□

$\wedge$, $\vee$, $\rightarrow$ & $\neg$

Suppose that $\varphi(v_1, \ldots, v_n)$ and $\psi(v_1, \ldots, v_n)$ are t-formulae.

$$\mathcal{F}_\psi(a_1, \ldots, a_n) = \{\langle x_n, \ldots, x_1\rangle \in a_n \times \ldots \times a_1 : \psi(x_1, \ldots, x_n)\}$$

$\varphi \wedge \psi$

$$\begin{aligned}\mathcal{F}_{\varphi\wedge\psi}(a_1, \ldots, a_n) &= \mathcal{F}_\varphi(a_1, \ldots, a_n) \cap \mathcal{F}_\psi(a_1, \ldots, a_n)\\ &= \mathcal{F}_\cap(\mathcal{F}_\varphi, \mathcal{F}_p(\mathcal{F}_\psi, \mathcal{F}_\psi))\end{aligned}$$

$\varphi \vee \psi$

$$\begin{aligned}\mathcal{F}_{\varphi\vee\psi}(a_1, \ldots, a_n) &= \mathcal{F}_\varphi(a_1, \ldots, a_n) \cup \mathcal{F}_\psi(a_1, \ldots, a_n)\\ &= \mathcal{F}_\cup(\mathcal{F}_p(\mathcal{F}_\varphi, \mathcal{F}_\psi), \mathcal{F}_\varphi)\end{aligned}$$

$\wedge$, $\vee$, $\rightarrow$ & $\neg$

$\varphi \rightarrow \psi$

$$\{\langle x_n \ldots, x_1\rangle \in a_n \times \ldots \times a_1 : \varphi(x_1, \ldots, x_n) \rightarrow \psi(x_1, \ldots, x_n)\}$$
$$=$$
$$(a_1 \times \ldots \times a_n) \cap \{z \in \mathcal{F}_\psi(a_1, \ldots, a_n) : z \in \mathcal{F}_\varphi(a_1, \ldots, a_n)\}$$
$$=$$
$$\mathcal{F}_\rightarrow\Big(a_n \times \ldots \times a_1,\ \langle \mathcal{F}_\varphi(a_1, \ldots, a_n), \mathcal{F}_\psi(a_1, \ldots, a_n)\rangle\Big)$$

$\neg\varphi$

$$\neg\varphi(v_1, \ldots, v_n) \equiv (\varphi(v_1, \ldots, v_n) \rightarrow 0 = 1)$$

Existentials

Suppose that $\psi(v_1, \ldots, v_{n+1})$ is a t-formula.

$$\mathcal{F}_\psi(a_1, \ldots, a_n) = \{\langle x_n, \ldots, x_1\rangle \in a_n \times \ldots \times a_1 : \psi(x_1, \ldots, x_n)\}$$

$\varphi(v_1, \ldots, v_n) \equiv \exists v_{n+1} \in v_j \ \psi(v_1, \ldots, v_{n+1})$

- Let $\theta(v_1, \ldots, v_n) \equiv v_{n+1} \in v_j$.
- Then $\psi \wedge \theta$ is a t-formula.
- $\mathcal{F}_{\psi\wedge\theta}(a_1, \ldots, a_n, \bigcup a_j) =$
$$\left\{ \langle x_{n+1}, x_n \ldots x_1\rangle : \begin{array}{l} \forall i \in [1, n]\ x_i \in a_i \ \wedge \ x_{n+1} \in x_j \\ \wedge \ \psi(x_1, \ldots, x_{n+1}) \end{array} \right\}$$
- $\mathcal{F}_\varphi(a_1, \ldots, a_n) = \mathcal{F}_{ran}(\mathcal{F}_{\psi\wedge\theta}(a_1, \ldots, a_n, \bigcup a_j), \mathcal{F}_\setminus(a_1, a_1))$.

Universals

Suppose that $\psi(v_1, \ldots, v_{n+1})$ is a t-formula.

$$\mathcal{F}_\psi(a_1, \ldots, a_n) = \{\langle x_n, \ldots, x_1 \rangle \in a_n \times \ldots \times a_1 : \psi(x_1, \ldots, x_n)\}$$

$\varphi(v_1, \ldots, v_n, b) \equiv \forall v_{n+1} \in b \; \psi(v_1, \ldots, v_{n+1}),\; b \notin \{v_1, \ldots, v_n\}$

First note that $\mathcal{F}_\forall(\mathcal{F}_\psi(a_1, \ldots, a_n, b), b) = \{ran(\mathcal{F}_\psi(a_1, \ldots, a_n, \{z\})) : z \in b\}$.
Therefore $\mathcal{F}_\varphi(a_1, \ldots, a_n, b)$ can be expressed as

$$\begin{aligned}
& \{\langle x_n, \ldots, x_1 \rangle \in a_n \times \ldots \times a_1 : \forall x_{n+1} \in b \; \psi(x_1, \ldots x_n)\} \\
= \; & (a_n \times \ldots \times a_1) \cap \\
& \{w : \forall x_{n+1} \in b \; \langle x_{n+1}, w \rangle \in \mathcal{F}_\psi(a_1, \ldots, a_n, \{x_{n+1}\})\} \\
= \; & (a_n \times \ldots \times a_1) \cap \\
& \bigcap \{ran(\mathcal{F}_\psi(a_1, \ldots, a_n, \{x_{n+1}\})) : x_{n+1} \in b\} \\
= \; & \mathcal{F}_\cap\Big(a_n \times \ldots \times a_1,\; \mathcal{F}_\forall(\mathcal{F}_\psi(a_1, \ldots, a_n, b), b)\Big).
\end{aligned}$$

$\varphi(v_1, \ldots, v_n) \equiv \forall v_{n+1} \in v_j \; \psi(v_1, \ldots, v_{n+1})$

Let $\theta(v_1, \ldots, v_n, b) \equiv \forall v_{n+1} \in b \; (v_{n+1} \in v_j \to \psi(v_1, \ldots, v_{n+1}))$ which is a t-formula. Then

$$\{\langle x_n, \ldots, x_1\rangle \in a_n \times \ldots \times a_1 : \forall x_{n+1} \in x_j \; \psi(x_1, \ldots, x_{n+1})\}$$
$$= \{\langle x_n, \ldots, x_1\rangle \in a_n \times \ldots \times a_1 : \theta(x_1, \ldots, x_n, \bigcup a_j)\}.$$

Bounded Separation

Theorem (Barwise: Corollary 6.2)

For any Σ_0 formula $\varphi(v_1, \ldots, v_n)$ with free variables among $v_1, \ldots v_n$ there is a term $\mathcal{F}_\varphi$ of n arguments built from the Gödel functions such that:

$$\mathrm{IKP}^* \vdash \mathcal{F}_\varphi(a, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \{x_i \in a : \varphi(x_1, \ldots, x_n)\}.$$

Proof.

- Let $\mathcal{F}_\varphi$ be such that IKP^* deduces that $\mathcal{F}_\varphi(a_1, \ldots, a_n) = \{\langle x_n, \ldots, x_1\rangle \in a_n \times \ldots \times a_1 : \varphi(x_1, \ldots, x_n)\}$
- Then our required set can be built from
$$\mathcal{F}_\varphi(\{x_1\}, \ldots, \{x_{i-1}\}, a_i, \{x_{i+1}\}, \ldots \{x_n\})$$
by using $\mathcal{F}_{ran}$ $n - i$ times and then $\mathcal{F}_{dom}$.

∎

$\mathcal{L}$

Definition

For a set b, $\mathcal{D}(b) := b \cup \{\mathcal{F}_i(x, y) : x, y \in b \ \wedge \ i \in \mathcal{I}\}$.

Definition

For α an ordinal, $\mathcal{L}_\alpha := \bigcup_{\beta\in\alpha} \mathcal{D}(\mathcal{L}_\beta \cup \ \{\mathcal{L}_\beta\})$.

$$\mathcal{L} := \bigcup_\alpha \mathcal{L}_\alpha.$$

Remarks

- $\mathcal{L}_{\alpha+1} = \mathcal{D}(\mathcal{L}_\alpha \cup \{\mathcal{L}_\alpha\})$.
- If α is a strong additive limit then $\mathcal{L}_\alpha = \bigcup_{\beta\in\alpha} \mathcal{L}_\beta$.

Basic Properties

Lemma

For any ordinals α, β;

1. If $\beta \subseteq \alpha$ then $\mathcal{L}_\beta \subseteq \mathcal{L}_\alpha$,
2. $\mathcal{L}_\alpha \in \mathcal{L}_{\alpha+1}$,
3. If $x, y \in \mathcal{L}_\alpha$ then for any $i \in \mathcal{I}$, $\mathcal{F}_i(x, y) \in \mathcal{L}_{\alpha+1}$,
4. If for all $\beta \in \alpha$, $\beta + 1 \in \alpha$ then $\mathcal{L}_\alpha$ is transitive,
5. $\mathcal{L}$ is transitive.

IKP in $\mathcal{L}$

Theorem

For every axiom, φ, of IKP^*, $\mathrm{IKP}^* \vdash \varphi^{\mathcal{L}}$. Moreover, $\mathrm{IKP}^* + \text{``}\textit{strong infinity}\text{''} \vdash (\textit{strong infinity})^{\mathcal{L}}$.

Proof of Σ_0-Collection.

- Suppose that $\varphi(x, y, z)$ is a Σ_0 formula.
- Assume that $a, z \in \mathcal{L}$ and $\forall x \in a\ \exists y \in \mathcal{L}\ (\varphi(x, y, z))^{\mathcal{L}}$.
- Then $\forall x \in a\ \exists \alpha\ (\exists y \in \mathcal{L}_\alpha\ \varphi(x, y, z))$.
- By Σ-collection in V, there is a β such that $\forall x \in a\ \exists \alpha \in \beta\ (\exists y \in \mathcal{L}_\alpha\ \varphi(x, y, z))$.
- So $\forall x \in a\ \exists y \in \mathcal{L}_\beta\ \varphi(x, y, z)$.

□

IKP in $\mathcal{L}$

Theorem

For every axiom, φ*, of* IKP^*, $\mathrm{IKP}^* \vdash \varphi^{\mathcal{L}}$*. Moreover,* $\mathrm{IKP}^* + \text{“}\textit{strong infinity}\text{”} \vdash (\textit{strong infinity})^{\mathcal{L}}$.

Proof of Strong Infinity.

- For all $n \in \omega$, $n + 1 = \mathcal{F}_{\cup}(n, \mathcal{F}_p(n, n)) \in \mathcal{L}_{2n+3}$.
- So

$$\omega = \{n \in \mathcal{L}_\omega : n = \emptyset \ \vee \ \exists m \in n \ (n = m \cup \{m\})\}$$

is in $\mathcal{L}$ by bounded separation.

□

Axiom of Constructibility

We want to prove that $(\mathrm{V} = \mathcal{L})^{\mathcal{L}}$. But, $\mathcal{L} = \bigcup_{\alpha \in \mathrm{ORD} \cap \mathrm{V}} \mathcal{L}_\alpha$ and we don't know if $\mathrm{ORD} \cap \mathcal{L} = \mathrm{ORD} \cap \mathrm{V}$. However, $(\mathrm{V} = \mathcal{L})^{\mathcal{L}}$ will be immediate from the following:

Lemma (Lubarsky)

For every ordinal α there is an ordinal $\alpha^* \in \mathcal{L}$ such that $\mathcal{L}_\alpha = \mathcal{L}_{\alpha^*}$

Definition (Hereditary Addition)

For ordinals α and γ, *hereditary addition* is defined inductively on α as

$$\alpha +_H \gamma := \left(\bigcup \{\beta +_H \gamma : \beta \in \alpha\} \cup \{\alpha\} \right) + \gamma$$

where "$+$" is the usual ordinal addition. Also

$$(\alpha +_H \gamma)^- := \left(\bigcup \{\beta +_H \gamma : \beta \in \alpha\} \cup \{\alpha\} \right).$$

α^*

Lemma (Lubarsky)

For every ordinal α there is an ordinal $\alpha^* \in \mathcal{L}$ such that $\mathcal{L}_\alpha = \mathcal{L}_{\alpha^*}$

Proof.

- Proof by induction on α.
- Fix $k \in \omega$ such that for all ordinals α and τ,

$$\{\gamma \in \mathcal{L}_\tau : \mathcal{D}(\mathcal{L}_\gamma \cup \{\mathcal{L}_\gamma\}) \subseteq \mathcal{L}_\alpha\} \in \mathcal{L}_{\tau+k}.$$

- $\alpha^* := \{\gamma \in \mathcal{L}_{(\alpha+_H k)^-} : \mathcal{D}(\mathcal{L}_\gamma \cup \{\mathcal{L}_\gamma\}) \subseteq \mathcal{L}_\alpha\} \in \mathcal{L}_{\alpha+_H k}$.
- **Claim:** If $\beta \in \alpha$ then $\beta^* \in \alpha^*$.
- Therefore $\mathcal{L}_\alpha = \bigcup_{\beta \in \alpha} \mathcal{D}(\mathcal{L}_\beta \cup \{\mathcal{L}_\beta\}) = \bigcup_{\beta \in \alpha} \mathcal{D}(\mathcal{L}_{\beta^*} \cup \{\mathcal{L}_{\beta^*}\})$

$$\subseteq \bigcup_{\gamma \in \alpha^*} \mathcal{D}(\mathcal{L}_\gamma \cup \{\mathcal{L}_\gamma\}) = \mathcal{L}_{\alpha^*}.$$

□

Alternative Definition of Definability I

Definition (IKP)

For a set b, $\mathsf{Def}(b) := \bigcup_{n\in\omega} \mathcal{D}^n(b \cup \{b\})$. For α an ordinal, $\mathrm{L}_\alpha := \bigcup_{\beta\in\alpha} \mathsf{Def}(\mathrm{L}_\beta)$

$$\mathrm{L} := \bigcup_{\alpha} \mathrm{L}_\alpha.$$

Proposition (IKP)

For all ordinals α, β:

1. If $\beta \in \alpha$ then $\mathrm{L}_\beta \subseteq \mathrm{L}_\alpha$,
2. $\mathrm{L}_\alpha \in \mathrm{L}_{\alpha+1}$,
3. L_α is a transitive model of Σ_0 separation,
4. $\mathrm{L}_\alpha = \mathcal{L}_{\omega\cdot\alpha}$.

Alternative Definition of Definability II

Definition (IKP)

Say that a set x is definable over $\langle M, \in \rangle$ if there exists a formula φ and $a_1, \ldots, a_n \in M$ such that

$$x = \{y \in M : \langle M, \in \rangle \models \varphi[y, a_1, \ldots, a_n]\}.$$

We can then define the collection of definable subsets of M as

$$\mathrm{def}(M) := \{x \subseteq M : x \textit{ is definable over } \langle M, \in \rangle\}.$$

Theorem (IKP)

For every transitive set M:

$$\begin{aligned} \mathrm{def}(M) &= \mathrm{Def}(M) \cap \mathcal{P}(M) \\ &= \bigcup_{n \in \omega} \mathcal{D}^n(M) \cap \mathcal{P}(M). \end{aligned}$$

Collection

Idea

IZF is the theory ZF with intuitionistic logic instead of classical logic.

Definition

Let IZF_{rep} denote the theory IKP plus full separation plus the full replacement scheme.

$$(\forall x \in a \; \exists! y \; \varphi(x, y, z) \to \exists b \; \forall x \in a \; \exists y \in b \; \varphi(x, y, z))$$

Let IZF denote the theory IKP plus full separation plus the full collection scheme.

$$(\forall x \in a \; \exists y \; \varphi(x, y, z) \to \exists b \; \forall x \in a \; \exists y \in b \; \varphi(x, y, z))$$

Let IZF_{ref} denote the theory IKP plus full separation plus the reflection scheme.

(*For any formula* φ *and set* x *there is a transitive set* M such that $x \subseteq M$ *and* φ *is absolute between* M *and* V.)

External Cumulative Hierarchy

Definition

Let $M \subseteq N$. We say that M has an external cumulative hierarchy (e.c.h.) in N if there exists a sequence $\langle M_\alpha : \alpha \in \mathrm{Ord} \cap N \rangle$ (which is definable in N) such that;

- $\forall \alpha \in \mathrm{Ord} \cap N \; M_\alpha \in M$,
- $M = \bigcup_{\alpha \in \mathrm{Ord} \cap N} M_\alpha$,
- If $\beta \in \alpha$ then $M_\beta \subseteq M_\alpha$.

Remarks

- When $N = \mathrm{V}$ we will just say that M has an e.c.h.
- If M is a model of IZF containing all of the ordinals then its rank hierarchy is an e.c.h.
- By construction $\langle \mathrm{L}_\alpha : \alpha \in \mathrm{Ord} \rangle$ is an e.c.h. for L.

e.c.h.'s classically

Proposition

Suppose that $M \subseteq N$ are transitive models of ZF [5]. If M has an e.c.h. in N then $\textsc{Ord} \cap M = \textsc{Ord} \cap N$.

Proof.

- Let $\langle M_\alpha : \alpha \in \textsc{Ord} \cap N \rangle$ be an e.c.h.
- Prove inductively that $\forall \gamma \in N\ \exists \beta \in N\ (\gamma \subseteq M_\beta)$.
  - Working in N, $\forall \alpha \in \gamma\ \exists \tau_\alpha \in N\ (\alpha \in M_{\tau_\alpha})$.
  - Using collection and the cumulative nature of the hierarchy, $\exists \beta \in N\ \forall \alpha \in \gamma\ (\alpha \in M_\beta)$.
- Since $M_\beta \in M$ and M is transitive, either $\gamma = M_\beta \cap \textsc{Ord}$ or $\gamma \in M_\beta \cap \textsc{Ord}$.
- Either of which yields that $\gamma \in \textsc{Ord} \cap M$.

□

[5] Much less than this is needed

Submodels

Definition

Let $M \subseteq N$. We say that M is almost universal in N if for any $x \in N$, if $x \subseteq M$ then there exists some $y \in M$ such that $x \subseteq y$.

Theorem

Suppose that N is a model of IZF and $M \subseteq N$ is a transitive (proper) class with an external cumulative hierarchy in N. Then M is a model of IZF iff M is closed under Gödel functions and is almost universal in N.

Remarks

- The e.c.h. is not necessary for the right to left implication.
- It is needed to show that M is almost universal in N:
  - If $a \in N$, $a \subseteq M$ then $\exists \beta \in \mathrm{Ord} \cap N \ (a \subseteq \bigcup_{\alpha \in \beta} \mathrm{V}_\alpha^M)$.
  - If $\beta \notin M$, why should this union be in M?

Open Questions

1. Does $\mathrm{IZF}_{\mathit{ref}}$ prove $(\mathrm{IZF}_{\mathit{ref}})^{\mathrm{L}}$?
2. Does $\mathrm{IZF}_{\mathit{rep}}$ prove $(\mathrm{IZF}_{\mathit{rep}})^{\mathrm{L}}$?
3. Does L have any nice additional properties? For example,
   - Is $\mathcal{P}(\omega) \subseteq \mathrm{L}_{\omega_1}$?
   - Does L satisfy some form of condensation?
4. Which large set axioms (intuitionistic versions of large cardinals axioms) are downwards absolute to L?
5. Is $\textsc{Ord} \cap \mathrm{V} = \textsc{Ord} \cap \mathrm{L}$?

Strange Ordinals - An approach to adding ordinals

Claim

There is a model, M, of $(\mathrm{IZF} + \mathrm{V} = \mathrm{L})$[6] containing a sequence $\langle \alpha_n : n \in \omega \rangle$ such that:

- Each α_n is a distinct ordinal,
- If $n \neq m$ then $\alpha_n \notin \alpha_m$.

- Let $\langle \alpha_n : n \in \omega \rangle$ be such a sequence.
- For $f : \omega \to 2$, let $\delta_f := \bigcup_n (\alpha_n \cup f(n))$.
- Then $\{\delta_f : f \in {}^\omega 2\}$ is an encoding of ${}^\omega 2$ by ordinals.
- So, if $M \subseteq N$ are models of IZF and $\langle \alpha_n : n \in \omega \rangle \in M$, then
$$\mathrm{Ord} \cap M = \mathrm{Ord} \cap N \Rightarrow {}^\omega 2 \cap M = {}^\omega 2 \cap N.$$
- So, if we add a Cohen real to M we add a new function from ω to 2 and therefore new ordinals.

[6] Possibly Kleene's first realizability model?

Strange Ordinals - An approach to adding ordinals

Claim

There is a model, M, of $(\mathrm{IZF} + \mathrm{V} = \mathrm{L})^7$ containing a sequence $\langle \alpha_n : n \in \omega \rangle$ such that:

- Each α_n is a distinct ordinal,
- If $n \neq m$ then $\alpha_n \notin \alpha_m$.

Conclusions

- $\textsc{Ord} \cap \mathrm{V}$ need not equal $\textsc{Ord} \cap \mathrm{L}$!
- Forcing can add ordinals!

[7] Possibly Kleene's first realizability model?