Ordinal Oddities

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What is an Ordinal?

Definition

$\langle A, \prec \rangle$ is a well-ordering if it is a strict total order such that any non-empty subset X of A has an $\prec$-least element.

Definition

An Ordinal α is a transitive set which is well-ordered by $\in$.
Let ORD denote the class of Ordinals.

Proposition

α is an ordinal iff it is a transitive set of transitive sets.

Remark

Because $\in$ is an order, we will often switch between $\in$ and $<$.

Basic Properties

- If α is an ordinal then so is $\alpha + 1 := \alpha \cup \{\alpha\}$,
- If X is a set of ordinals then $\bigcup X$ is an ordinal,
- $\beta < \alpha \implies \beta + 1 \leq \alpha$,
- For any ordinal α, $0 \in \alpha + 1$,
- **Trichotomy:** For any α, β, $\alpha = \beta$ or $\alpha \in \beta$ or $\beta \in \alpha$,
- Every non-empty set of ordinals has an $\in$-least element,
- Every ordinal is one of
  - 0,
  - A successor, $\alpha = \beta + 1$
  - An additive limit. $\forall \beta \in \alpha \; \beta + 1 \in \alpha$

Non-constructive Principles

- (Law of Excluded Middle) $\varphi \vee \neg\varphi$
- (Double Negation Elimination) $\neg\neg\varphi \to \varphi$
- (Some Classical Logical Equivalences) $(\varphi \to \psi) \to (\neg\varphi \vee \psi)$

- Foundation: $\forall a(\exists x(x \in a) \to \exists x \in a\, \forall y \in a(y \notin x))$
- "Least elements" of sets
- Axiom of Choice / Well-Ordering Principle
- Definition by cases which differentiate between successor and limit ordinals

Remark

$\neg\varphi$ is interpreted as $\varphi \to (0 = 1)$.

IZF

Idea

IZF is the theory ZF with intuitionistic logic instead of classical logic.

Definition (IZF)

- Extensionality
- Empty Set
- Power set
- Pairing
- Unions

IZF

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- Extensionality
- Pairing
- Empty Set
- Unions
- Power set
- Set Induction (For any formula $\varphi(u)$, $\forall a(\forall x \in a\ \varphi(x) \to \varphi(a)) \to \forall a\ \varphi(a)$)
- Collection (For any formula $\varphi(u,v)$ and set a, $\forall x \in a\ \exists y\ \varphi(x,y) \to \exists b\ \forall x \in a\ \exists y \in b\ \varphi(x,y)$)
- Separation (For any formula $\varphi(u)$ and set a, $\{x \in a : \varphi(x)\}$ is a set)
- Strong Infinity ($\exists a\ (Ind(a)\ \wedge\ \forall b\ (Ind(b) \to \forall x \in a(x \in b)))$)[1].

[1] $Ind(a) \equiv \emptyset \in a\ \wedge\ \forall x \in a\ (x \cup \{x\} \in a)$

IKP

Definition (IKP^{-Inf})

- Extensionality
- Empty Set
- Pairing
- Unions
- Set Induction (For any formula $\varphi(u)$, $\forall a(\forall x \in a\ \varphi(x) \to \varphi(a)) \to \forall a\ \varphi(a)$)
- Bounded Collection (For any Σ_0 formula $\varphi(u, v)$ and set a, $\forall x \in a\ \exists y\ \varphi(x, y) \to \exists b\ \forall x \in a\ \exists y \in b\ \varphi(x, y)$)
- Bounded Separation (For any Σ_0 formula $\varphi(u)$ and set a, $\{x \in a : \varphi(x)\}$ is a set)

Definition (IKP)

IKP is IKP^{-Inf} plus strong infinity.

Ordinary Ordinal Oddities

Definition

An *ordinal* is a transitive set of transitive sets.

Remarks

- If α is an ordinal then so is $\alpha + 1 := \alpha \cup \{\alpha\}$.
- If X is a set of ordinals then $\bigcup X$ is an ordinal.
- $\beta \in \alpha \not\Rightarrow \beta + 1 \in \alpha + 1$.
- $\forall \alpha \ (0 \in \alpha + 1)$ implies excluded middle!

Trichotomy

- α is *trichotomous* $\forall \beta \in \alpha \ \forall \gamma \in \alpha \ (\beta \in \gamma \vee \beta = \gamma \vee \gamma \in \beta)$.
- It is consistent with IZF that the collection of trichotomous ordinals is a set!

Ordinary Ordinal Oddities

Definition

An *ordinal* is a transitive set of transitive sets.

Remarks

- If α is an ordinal then so is $\alpha + 1 := \alpha \cup \{\alpha\}$.
- If X is a set of ordinals then $\bigcup X$ is an ordinal.
- $\beta \in \alpha \not\Rightarrow \beta + 1 \in \alpha + 1$.
- $\forall\alpha\ (0 \in \alpha + 1)$ implies excluded middle!

Definition

An ordinal α is a *weak additive limit* if $\forall\beta \in \alpha\ \exists\gamma \in \alpha\ (\beta \in \gamma)$.

An ordinal α is a *strong additive limit* if $\forall\beta \in \alpha\ (\beta + 1 \in \alpha)$.

Truth Values

Given a formula φ, an important ordinal is

$$\alpha_\varphi := \{0 \in 1 : \varphi\}.$$

Naively, if we don't assume $\varphi \vee \neg\varphi$ then α_φ is neither 0 not 1.

In general we let

$$\Omega := \mathcal{P}(1) = \{x : x \subseteq 1\}$$

be the class of *truth values*.

If $\Omega = 2$ then the Law of Excluded Middle holds.

Note that

$$0 \in \alpha_\varphi + 1 \implies 0 \in \alpha_\varphi \vee 0 = \alpha_\varphi \implies \varphi \vee \neg\varphi.$$

History

- The constructible universe was developed by Gödel in papers published in 1939 and 1940 to show the consistency of the Axiom of Choice and the Generalised Continuum Hypothesis with ZF.
- There are $2/3$ main approaches to building L both of which are formalisable in KP:[2]
  - Syntactically as the set of definable subsets of M (See Devlin - *Constructibility*)
  - Using Gödel functions (See Barwise - *Admissible Sets*) or
  - Using Rudimentary Functions (See Gandy, Jensen, Mathias)
- The syntactic approach was then modified for IZF by Lubarsky (*Intuitionistic L* - 1993)
- And then for IKP by Crosilla (*Realizability models for constructive set theories with restricted induction* - 2000)

[2] In fact significantly weaker systems - see Mathias: *Weak Systems of Gandy, Jensen and Devlin*, 2006

Gödel Functions

Definition

- $\mathcal{F}_p(x, y) := \{x, y\}$,
- $\mathcal{F}_\cap(x, y) := x \cap \bigcap y$ $(\cap y = \{u : \forall v \in y\ (u \in v)\})$
- $\mathcal{F}_\cup(x, y) := \bigcup x$,
- $\mathcal{F}_\setminus(x, y) := x \setminus y$,
- $\mathcal{F}_\times(x, y) := x \times y$,
- $\mathcal{F}_\to(x, y) := x \cap \{z \in 2^{nd}(y) : y \textit{ is an ordered pair } \wedge z \in 1^{st}(y)\}$,
- $\mathcal{F}_\forall(x, y) := \{x''\{z\} : z \in y\}$, $(x''u = \{v : v \in 2^{nd}(x) \wedge \langle u, v\rangle \in x\})$

Gödel Functions

Definition

- $\mathcal{F}_{dom}(x, y) := dom(x) = \{1^{st}(z) : z \in x \wedge$ z *is an ordered pair*$\}$,
- $\mathcal{F}_{ran}(x, y) := ran(x) = \{2^{nd}(z) : z \in x \wedge$ z *is an ordered pair*$\}$,
- $\mathcal{F}_{123}(x, y) := \{\langle u, v, w\rangle : \langle u, v\rangle \in x \;\wedge\; w \in y\}$,
- $\mathcal{F}_{132}(x, y) := \{\langle u, w, v\rangle : \langle u, v\rangle \in x \;\wedge\; w \in y\}$,
- $\mathcal{F}_{=}(x, y) := \{\langle v, u\rangle \in y \times x : u = v\}$,
- $\mathcal{F}_{\in}(x, y) := \{\langle v, u\rangle \in y \times x : u \in v\}$.

Notation

Let $\mathcal{I}$ *be the finite set indexing the above operations.*

Generating Constructible Sets

Lemma (Barwise: Admissible Sets, Lemma II.6.1, (M.))

For every Σ_0 formula $\varphi(v_1, \ldots, v_n)$ with free variables among $v_1, \ldots, v_n$, there is a term $\mathcal{F}_\varphi$ built up from the Gödel functions such that

$$\mathrm{IKP} \vdash \mathcal{F}_\varphi(a_1, \ldots, a_n) = \{\langle x_n, \ldots, x_1 \rangle \in a_n \times \ldots \times a_1 : \varphi(x_1, \ldots, x_n)\}.$$

Proof.

- Call a formula $\varphi(x_1, \ldots, x_n)$ a termed-formula or *t-formula* if there is a term $\mathcal{F}_\varphi$ such that the conclusion of the lemma holds.
- Proceed by induction on Σ_0 formulae to show that every such formula is a t-formula.

□

Universals

Suppose that $\psi(v_1, \dots, v_{n+1})$ is a t-formula.

$$\mathcal{F}_\psi(a_1, \dots, a_n, a_{n+1}) = \{\langle x_{n+1}, x_n, \dots, x_1\rangle \in a_{n+1} \times a_n \times \dots \times a_1 : \psi(x_1, \dots, x_n, x_{n+1})\}$$

$\varphi(v_1, \dots, v_n, b) \equiv \forall v_{n+1} \in b\ \psi(v_1, \dots, v_{n+1}),\ b \notin \{v_1, \dots, v_n\}$

First note that $\mathcal{F}_\forall(\mathcal{F}_\psi(a_1, \dots, a_n, b), b) = \{ran(\mathcal{F}_\psi(a_1, \dots, a_n, \{z\})) : z \in b\}$.

Therefore $\mathcal{F}_\varphi(a_1, \dots, a_n, b)$ can be expressed as

$$\begin{aligned}
& \{\langle x_n, \dots, x_1\rangle \in a_n \times \dots \times a_1 : \forall x_{n+1} \in b\ \psi(x_1, \dots x_n)\} \\
= \ & (a_n \times \dots \times a_1) \cap \\
& \{w : \forall x_{n+1} \in b\ \langle x_{n+1}, w\rangle \in \mathcal{F}_\psi(a_1, \dots, a_n, \{x_{n+1}\})\} \\
= \ & (a_n \times \dots \times a_1) \cap \\
& \bigcap \{ran(\mathcal{F}_\psi(a_1, \dots, a_n, \{x_{n+1}\})) : x_{n+1} \in b\} \\
= \ & \mathcal{F}_\cap\Big(a_n \times \dots \times a_1,\ \mathcal{F}_\forall(\mathcal{F}_\psi(a_1, \dots, a_n, b), b)\Big).
\end{aligned}$$

Universals

Suppose that $\psi(v_1, \ldots, v_{n+1})$ is a t-formula.

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Therefore $\mathcal{F}_\varphi(a_1, \ldots, a_n, b)$ can be expressed as

$$\mathcal{F}_\cap\Big(a_n \times \ldots \times a_1,\ \mathcal{F}_\forall(\mathcal{F}_\psi(a_1, \ldots, a_n, b), b)\Big).$$

$\varphi(v_1, \ldots, v_n) \equiv \forall v_{n+1} \in v_j\ \psi(v_1, \ldots, v_{n+1})$

Let $\theta(v_1, \ldots, v_n, b) \equiv \forall v_{n+1} \in b\ (v_{n+1} \in v_j \to \psi(v_1, \ldots, v_{n+1}))$ which is a t-formula. Then

$$\{\langle x_n, \ldots, x_1\rangle \in a_n \times \ldots \times a_1 : \forall x_{n+1} \in x_j\ \psi(x_1, \ldots, x_{n+1})\}$$
$$= \{\langle x_n, \ldots, x_1\rangle \in a_n \times \ldots \times a_1 : \theta(x_1, \ldots, x_n, \bigcup a_j)\}.$$

Bounded Separation

Theorem (Barwise: Corollary 6.2)

For any Σ_0 formula $\varphi(v_1, \ldots, v_n)$ with free variables among $v_1, \ldots v_n$ there is a term $\mathcal{F}_\varphi$ of n arguments built from the Gödel functions such that:

$$\mathrm{IKP}^{-Inf} \vdash \mathcal{F}_\varphi(a, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\ = \{x_i \in a : \varphi(x_1, \ldots, x_n)\}.$$

Proof.

- Let $\mathcal{F}_\varphi$ be such that IKP^{-Inf} deduces that $\mathcal{F}_\varphi(a_1, \ldots, a_n) = \{\langle x_n, \ldots, x_1 \rangle \in a_n \times \ldots \times a_1 : \varphi(x_1, \ldots, x_n)\}$
- Then our required set can be built from

$$\mathcal{F}_\varphi(\{x_1\}, \ldots, \{x_{i-1}\}, a_i, \{x_{i+1}\}, \ldots \{x_n\})$$

by using $\mathcal{F}_{ran}$ $n - i$ times and then $\mathcal{F}_{dom}$.

□

L

Definition

For a set b, $\mathcal{D}(b) := b \cup \{\mathcal{F}_i(x, y) : x, y \in b \ \wedge \ i \in \mathcal{I}\}$.

Definition

For α an ordinal, $\mathrm{L}_\alpha := \bigcup_{\beta \in \alpha} \mathcal{D}(\mathrm{L}_\beta \cup \ \{\mathrm{L}_\beta\})$.

$$\mathrm{L} := \bigcup_\alpha \mathrm{L}_\alpha.$$

Definition (Assuming Strong Infinity)

For a set b, $\mathrm{Def}(b) := \bigcup_{n \in \omega} \mathcal{D}^n(b \cup \{b\})$. For α an ordinal, $\mathcal{L}_\alpha := \bigcup_{\beta \in \alpha} \mathrm{Def}(\mathcal{L}_\beta)$

$$\mathrm{L} := \bigcup_\alpha \mathcal{L}_\alpha.$$

The Axioms of L

Proposition (IKP)

For all ordinals α, β:

1. *If $\beta \in \alpha$ then $\mathrm{L}_\beta \subseteq \mathrm{L}_\alpha$ and $\mathcal{L}_\beta \subseteq \mathcal{L}_\alpha$,*
2. $\mathrm{L}_\alpha \in \mathrm{L}_{\alpha+1}$ *and* $\mathcal{L}_\alpha \in \mathcal{L}_{\alpha+1}$,
3. $\mathcal{L}_\alpha$ *is a transitive model of* Σ_0 *separation,*
4. $\mathcal{L}_\alpha = \mathrm{L}_{\omega\cdot\alpha}$.

Theorem

For every axiom, φ, of $\mathrm{IKP}^{-\mathit{Inf}}$, $\mathrm{IKP}^{-\mathit{Inf}} \vdash \varphi^{\mathrm{L}}$. Moreover, $\mathrm{IKP}^{-\mathit{Inf}} + \text{“}\mathit{strong\ infinity}\text{”} \vdash (\mathit{strong\ infinity})^{\mathrm{L}}$.

Theorem

For every axiom, φ, of IZF, $\mathrm{IZF} \vdash \varphi^{\mathrm{L}}$.

Axiom of Constructibility

We want to prove that $(\mathrm{V} = \mathrm{L})^{\mathrm{L}}$. But, $\mathrm{L} = \bigcup_{\alpha \in \mathrm{ORD} \cap \mathrm{V}} \mathrm{L}_\alpha$ and we don't know if $\mathrm{ORD} \cap \mathrm{L} = \mathrm{ORD} \cap \mathrm{V}$. However, $(\mathrm{V} = \mathrm{L})^{\mathrm{L}}$ will be immediate from the following:

Lemma (Lubarsky)

For every ordinal α there is an ordinal $\alpha^* \in \mathrm{L}$ such that $\mathrm{L}_\alpha = \mathrm{L}_{\alpha^*}$

Definition (Hereditary Addition)

For ordinals α and γ, *hereditary addition* is defined inductively on α as

$$\alpha +_H \gamma := \left(\bigcup \{\beta +_H \gamma : \beta \in \alpha\} \cup \{\alpha\} \right) + \gamma$$

where "$+$" is the usual ordinal addition. Also

$$(\alpha +_H \gamma)^- := \left(\bigcup \{\beta +_H \gamma : \beta \in \alpha\} \cup \{\alpha\} \right).$$

α^*

Lemma (Lubarsky)

For every ordinal α there is an ordinal $\alpha^* \in \mathrm{L}$ such that $\mathrm{L}_\alpha = \mathrm{L}_{\alpha^*}$

Proof.

- Proof by induction on α.
- Fix $k \in \omega$ such that for all ordinals α and τ,

$$\{\gamma \in \mathrm{L}_\tau : \mathcal{D}(\mathrm{L}_\gamma \cup \{\mathrm{L}_\gamma\}) \subseteq \mathrm{L}_\alpha\} \in \mathrm{L}_{\tau+k}.$$

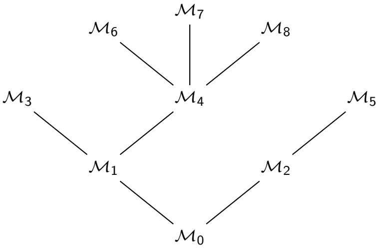
- $\alpha^* := \{\gamma \in \mathrm{L}_{(\alpha+_H k)^-} : \mathcal{D}(\mathrm{L}_\gamma \cup \{\mathrm{L}_\gamma\}) \subseteq \mathrm{L}_\alpha\} \in \mathrm{L}_{\alpha+_H k}$.
- **Claim:** If $\beta \in \alpha$ then $\beta^* \in \alpha^*$.
- Therefore $\mathrm{L}_\alpha = \bigcup_{\beta\in\alpha} \mathcal{D}(\mathrm{L}_\beta \cup \{\mathrm{L}_\beta\}) = \bigcup_{\beta\in\alpha} \mathcal{D}(\mathrm{L}_{\beta^*} \cup \{\mathrm{L}_{\beta^*}\})$

$$\subseteq \bigcup_{\gamma\in\alpha^*} \mathcal{D}(\mathrm{L}_\gamma \cup \{\mathrm{L}_\gamma\}) = \mathrm{L}_{\alpha^*}. \qquad \square$$

Kripke Models

A Kripke model is a collection of "*possible worlds*" along with a binary relation which gives us some information as to how the worlds are related to one another.

Alternatively, a Kripke model is a collection of "*states of knowledge*" and p is related to q indicates that if we know p then it is possible that we shall know q at a later stage.



Kripke Models

Definition

A *Kripke model* is an ordered quadruple $\mathscr{K} = \langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \iota \rangle$ where

- $\mathcal{K}$ is a non-empty set of "*nodes*",
- $\mathcal{D}$ is a function on $\mathcal{K}$,
- $\mathcal{R}$ is a binary, reflexive relation between elements of $\mathcal{K}$,
- ι is a set of functions $\iota_{p,q}$ for each pair $p, q \in \mathcal{K}$ with $p\mathcal{R}q$

such that the following hold.

- For each $p \in \mathcal{K}$, $\mathcal{D}(p)$ is an inhabited class structure.
- If $p\mathcal{R}q$ then $\iota_{p,q} \colon \mathcal{D}(p) \to \mathcal{D}(q)$ is a homomorphism.
- If $p\mathcal{R}q$ and $q\mathcal{R}r$ then $\iota_{p,r} = \iota_{q,r} \circ \iota_{p,q}$.

Forcing(ish)

Now, for atomic formulae φ, let $p \Vdash \varphi$ denote that $\mathcal{D}(p) \models \varphi$. Then $\Vdash$ can be extended to arbitrary formulae by the following prescription:

- For no p do we have $p \Vdash \perp$,
- $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$,
- $p \Vdash \varphi \vee \psi$ iff $p \Vdash \varphi$ or $p \Vdash \psi$,
- $p \Vdash \varphi \to \psi$ iff for any $r \in \mathcal{K}$ with $p\mathcal{R}r$, if $r \Vdash \varphi$ then $r \Vdash \psi$,
- $p \Vdash \forall x\ \varphi(x)$ iff whenever $p\mathcal{R}q$ and $d \in \mathcal{D}(q)$, $q \Vdash \varphi(d)$,
- $p \Vdash \exists x\ \varphi(x)$ iff there is some $d \in \mathcal{D}(p)$ such that $p \Vdash \varphi(d)$.

Validity

Definition

Let $\mathscr{K} = \langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \iota \rangle$ be a Kripke model and $p \in \mathcal{K}$.

- A formula φ is said to be *valid at p* iff $p \Vdash \varphi$.
- A formula φ is *valid in the full Kripke model*, written $\mathscr{K} \Vdash \varphi$, if for every $p \in \mathcal{K}$, $p \Vdash \varphi$.

Fact (Hendtlass, Lubarsky)

It is possible to add a model structure to $\mathscr{K}$, $\mathrm{V}(\mathscr{K})$ such that

$$\mathrm{V}(\mathscr{K}) \models \varphi \iff \forall p \in \mathcal{K}\ p \Vdash \varphi.$$

Theorem (Hendtlass, Lubarsky)

If for each $p, q \in \mathcal{K}$, $\mathcal{D}(p) \models \mathrm{ZF}$ *and* $\mathrm{ORD} \cap \mathcal{D}(p) = \mathrm{ORD} \cap \mathcal{D}(q)$, *then* $\mathrm{V}(\mathscr{K}) \models \mathrm{IZF}$.

Interpreting the initial node

Let $\mathscr{K} = \langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \iota \rangle$ be a Kripke model.

Definition

Define $\mathscr{K}^p$ to be the *truncation* of the Kripke model to $\mathcal{K}^p := \{q \in \mathcal{K} : p\mathcal{R}q\}$. So $\mathcal{K}^p$ is the cone of nodes which are related to p.

Fact

Given $p \in \mathcal{K}$ and $x \in \mathcal{D}(p)$ we can define an interpretation x^p such that if $p\mathcal{R}q$ then $q \Vdash x^p = x^q$.

This gives us a way to talk about the past worlds in the current one.

Same Ordinals, Same Reals

Theorem

Suppose that $\mathrm{N} \subseteq \mathrm{M}$ *are models of* IZF *such that* N *satisfies the following weak incidence of excluded middle:*

for any set $\{a_n : n \in \omega\}$ *of distinct sets, if we have* x *such that*

$$x \in \bigcup_n a_n \text{ and for some } k,\ x \notin \bigcup_{n \neq k} a_n \text{ then } x \in a_k.$$

Further suppose that in N *there is an ordinal* α *such that* $\alpha \notin \omega$ *and* $\omega \nsubseteq \alpha$. *Then*

$$\textsc{Ord} \cap \mathrm{M} = \textsc{Ord} \cap \mathrm{N} \Longrightarrow ({}^{\omega}2)^{\mathrm{M}} = ({}^{\omega}2)^{\mathrm{N}}.$$

The Proof

Theorem

$$\mathrm{ORD} \cap \mathrm{M} = \mathrm{ORD} \cap \mathrm{N} \Longrightarrow ({}^{\omega}2)^{\mathrm{M}} = ({}^{\omega}2)^{\mathrm{N}}.$$

- Fix $\alpha \in \mathrm{N}$ such that $\alpha \notin \omega$ and $\omega \not\subseteq \alpha$,
- Note that this is also true in M.
- Also, $(\alpha + 1) \not\subseteq \omega$
- So, $\{n \cup (\alpha + 1) : n \in \omega\}$ is a set of ω many pairwise incomparable ordinals.
- i.e. If $m \neq n$ then $m \cup (\alpha + 1) \notin n \cup (\alpha + 1)$.
- For $f \in ({}^{\omega}2)^{\mathrm{M}}$ define

$$\delta_f := \bigcup_{n\in\omega} [(n \cup (\alpha + 1)) + f(n)].$$

The Proof

Theorem

$$\mathrm{ORD} \cap \mathrm{M} = \mathrm{ORD} \cap \mathrm{N} \implies ({}^{\omega}2)^{\mathrm{M}} = ({}^{\omega}2)^{\mathrm{N}}.$$

- $\delta_f := \bigcup_{n\in\omega}[(n \cup (\alpha+1)) + f(n)] \in \mathrm{ORD} \cap \mathrm{M} = \mathrm{ORD} \cap \mathrm{N}$.
- Now define a function $g\colon \omega \to 2$ in N,

$$\begin{aligned} g(k) = 1 &\iff (k \cup (\alpha+1)) \in \delta_f \\ &\iff f(k) = 1. \end{aligned}$$

  And so $f \in \mathrm{N}$.
- Note that, in M, if $(k \cup (\alpha+1)) \in \delta_f$ then $(k \cup (\alpha+1)) \in (n \cup (\alpha+1)) + f(n)$ for some n,
- But for $n \neq k$, $(k \cup (\alpha+1)) \notin (n \cup (\alpha+1)) + f(n)$,
- So $(k \cup (\alpha+1)) \in (k \cup (\alpha+1)) + f(k)$ and $f(k) = 1$.

Could it all go wrong!?

Suppose that V is a model of IZF, $\mathbb{P} \in \mathrm{L}$ a partial order and that there exists some set $\{\alpha_p : p \in \mathbb{P}\} \subseteq \mathcal{P}(1)$ such that for all $p, q \in \mathbb{P}$:[3]

1. $\alpha_p \neq 0$ (that is $\neg(\forall x \in \alpha_p\ (x \neq x))$),
2. If $p \neq q$ then $\alpha_p \neq \alpha_q$,
3. $\mathrm{L}_{\alpha_p} = \alpha_p$.

- Let $G \subseteq \mathbb{P}$ be generic.
- Classically, $G \notin \mathrm{L}$ because forcing doesn't add ordinals and *definability* is absolute.
- Intuitionistically, $\mathrm{L}_{\alpha_p \cup \{\alpha_p\}} = 1 \cup \alpha_p \cup \{\alpha_p\}$.
- Define $\delta_G := 1 \cup \{\alpha_p : p \in G\}$

[3] It is unclear how to make all three of these points simultaneously hold!

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2. If $p \neq q$ then $\alpha_p \neq \alpha_q$,
3. $\mathrm{L}_{\alpha_p} = \alpha_p$.

- $\mathrm{L}_{\delta_G} = \bigcup_{\gamma \in \delta_G} \mathcal{D}(\mathrm{L}_\gamma) = \mathrm{L}_1 \cup \bigcup_{p \in G} \mathcal{D}(\mathrm{L}_{\alpha_p})$
  $= \bigcup_{p \in G} 1 \cup \alpha_p \cup \{\alpha_p\}$.
- But $\alpha_p \in \mathrm{L}_{\delta_G} \iff p \in G$
- Therefore, since $\mathrm{L}_{\delta_G}, \mathbb{P} \in \mathrm{L}$,

$$G = \{p \in \mathbb{P} : \alpha_p \in \mathrm{L}_{\delta_G}\} \in \mathrm{L}!$$

[3] It is unclear how to make all three of these points simultaneously hold!

Theorem

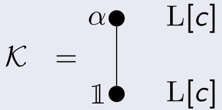
It is consistent to have a model of IZF *such that*

$$\textsc{Ord} \cap \mathrm{V} \neq \textsc{Ord} \cap \mathrm{L}.$$

Sketch.

The desired model will be $\mathrm{V}(\mathscr{K})$ where

- $\mathcal{K}$ is the two node Kripke structure $\{\mathbb{1}, \alpha\}$,
- $\mathcal{D}(\mathbb{1}) = \mathcal{D}(\alpha) = \mathrm{L}[c]$,
- c is a Cohen real over L,
- ι is the identity.



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Sketch.

- Let c^p be the interpretation of c at node p
- Then $p \Vdash c^p \notin \mathrm{L}$.
- So, $\mathrm{V}(\mathscr{K}) \models c \notin \mathrm{L}$.
- Let 1_α be the ordinal in $\mathrm{V}(\mathscr{K})$ which looks like 0 at $\mathbb{1}$ and 1 at α.

$$1_\alpha : \mathcal{K} \to 2 \qquad 1_\alpha(p) = \begin{cases} 0, & \text{if } p = \mathbb{1} \\ 1, & \text{if } p = \alpha. \end{cases}$$

- Then, in $\mathrm{V}(\mathscr{K})$, $1_\alpha \subseteq 1$ and $\mathrm{L}_{1_\alpha} = 1_\alpha$.

Theorem

It is consistent to have a model of IZF *such that*

$$\textsc{Ord} \cap \mathrm{V} \neq \textsc{Ord} \cap \mathrm{L}.$$

Sketch.

- Define δ_c to be an ordinal encoding c, for example,

$$\begin{aligned} \delta_c &= \bigcup_{n\in\omega} (\alpha \cup n) + c(n) \\ &= \{\alpha \cup n : c(n) = 0\} \cup \{\alpha \cup n \cup \{\alpha \cup n\} : c(n) = 1\} \\ &= \{\alpha \cup n : n \in \omega\} \cup \{\{\alpha \cup n\} : c(n) = 1\}. \end{aligned}$$

- Then $c(n) = 1$ if and only if $(\alpha \cup n) \in \delta_c$,
- So, since $c \in \mathrm{L} \iff \delta_c \in \mathrm{L}$,
- $\delta_c \notin \mathrm{L}$.

□

Other Odd Ordinals

Theorem

It is consistent with ZFC *to have a model of* $\mathrm{IZF} + \mathrm{V} = \mathrm{L}$ *plus a non-trivial automorphism of the universe.*

Idea

Find a model of IZF with two non-zero ordinals $\alpha_p, \alpha_q \in \mathcal{P}(1)$ with $\alpha_p \neq \alpha_q$ which are *indistinguishable*.

Theorem

It is consistent with ZFC *plus a measurable cardinal to have a model of* IZF *plus a non-trivial elementary embedding* $j : \mathrm{V} \to \mathrm{M}$ *and an ordinal* κ *such that*

- $\omega \in \kappa$,
- $\forall \alpha \in \kappa \; j(\alpha) = \alpha$,
- $\kappa \in j(\kappa)$,
- $\mathrm{L}_\kappa \models \mathrm{IZF}$,
- κ *is a weak additive limit,*
- $\omega + 1 \notin \kappa$.

The Model Back

Suppose that $\mathscr{K}$ is a Kripke model and that for each node p, $\mathcal{D}(p)$ is a model of ZF. We shall simultaneously define the set of objects at p, $\mathrm{M}^p := \bigcup_\alpha \mathrm{M}^p_\alpha$, inductively through the ordinals.
So suppose that $\{\mathrm{M}^p_\beta : p \in \mathcal{K}\}$ has been defined for each $\beta \in \alpha$ along with transition functions $k_{p,q} : \mathrm{M}^p_\beta \to \mathrm{M}^q_\beta$ for each pair $p\mathcal{R}q$.
The objects of M^p_α are then the collection of functions g such that

- $\mathrm{dom}(g) = \mathcal{K}^p$,
- $g \restriction \mathcal{K}^q \in \mathcal{D}(q)$,
- $g(q) \subseteq \bigcup_{\beta\in\alpha} \mathrm{M}^q_\beta$,
- If $h \in g(q)$ and $q\mathcal{R}r$ then $k_{q,r}(h) \in g(r)$.

Finally, extend $k_{p,q}$ to M^p_α by setting $k_{p,q}(g) := g \restriction \mathcal{K}^q$. Then the objects at node p are $\bigcup_\alpha \mathrm{M}^p_\alpha$.
We now define truth at node p for formulae by the following:

- $p \Vdash g \in h \iff g \restriction \mathcal{K}^p \in h(p)$,
- $p \Vdash g = h \iff g \restriction \mathcal{K}^p = h \restriction \mathcal{K}^p$,
- For logical connectives and quantifiers we use the rules for $\Vdash$.