Big Classes and the Respected Model

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ZFC without Power Set

Under ZF without Replacement the following three principles are equivalent:

- The Reflection Principle *("any formula reflects to a transitive set.")*
- The Collection Scheme $(\forall x \in a\ \exists y\ \varphi(x, y) \rightarrow \exists b \forall x \in a \exists y \in b \varphi(x, y))$
- The Replacement Scheme. $(\forall x \in a\ \exists! y\ \varphi(x, y) \rightarrow \exists b \forall x \in a \exists y \in b \varphi(x, y))$

However, without Power Set the reverse implications break down.

Definition 1

Let ZF− denote the theory consisting of the following axioms:

- Empty set, Extensionality, Pairing, Unions, Infinity,
- the Foundation Scheme, the Separation Scheme,
- the Replacement Scheme.

ZFC without Power Set[1]

Definition 2

- ZF^- denotes the theory ZF− plus the Collection Scheme.
- ZFC^- denotes the theory ZF^- plus the Well-Ordering Principle.
- ZFC^-_{Ref} denotes the theory ZFC^- plus the Reflection Principle.

Remarks

- For μ regular, $\mathrm{H}_\mu \models \mathrm{ZFC}^-_{Ref}$.
- Models of ZFC− can behave very counter-intuitively.

[1] See *What is the Theory* ZFC *without Power Set?* by Gitman, Hamkins and Johnstone.

Going wrong without Collection[1]

Any of the following can occur in ZFC− models:

- ω_1 exists and is singular,
- ω_1 exists and every set of reals is countable,
- For every $n \in \omega$ there is a set of reals of size $\aleph_n$ but none of size $\aleph_\omega$,
- The Łoś ultrapower theorem fails,
- Gaifman's Theorem fails (there is a cofinal, Σ_1-elementary map $j\colon M \to N$ which is not fully elementary),
- The class of Σ_1 formulas is not closed under bounded quantification (i.e. φ is Σ_1 but $\forall x \in a\ \varphi$ is not).

Conclusion

All of these problems go away if we also assume Collection. So ZFC^- is the "*correct*" way to state ZFC without Power Set.

[1]See *What is the Theory* ZFC *without Power Set?* by Gitman, Hamkins and Johnstone.

Big Classes

Definition

A proper class A is called *Big* if for every non-zero ordinal α there is a surjection of A onto α.

Proposition 3

Over ZFC−, *if* $j : \mathrm{V} \to \mathrm{M}$ *is elementary and if* $j \restriction (\mathcal{C} \cup \{\mathcal{C}\})$ *is the identity then* $\mathcal{C}$ *does not surject onto* $\mathrm{crit}(j)$.

Corollary 4

Over ZFC−, *if* $j : \mathrm{V} \to \mathrm{M}$ *is elementary,* $\mathcal{P}(\omega)$ *is not big.*

Proposition 5

In ZF *every proper class is big.*

Easy Examples 1

Proposition 6

In ZF *every proper class is big.*

Proof.

- Given a proper class $\mathcal{C}$, define

$$S := \{\gamma \in \textsc{Ord} \mid \exists x \in \mathcal{C} \ \operatorname{rank}(x) = \gamma\}.$$

- S must be unbounded in the ordinals.
- So, given an ordinal α, we can take the first α many elements of S, $\{\gamma_\beta \mid \beta \in \alpha\}$.
- Then

$$f(x) = \begin{cases} \beta, & \textit{if } \operatorname{rank}(x) = \gamma_\beta \\ 0, & \textit{otherwise} \end{cases}$$

defines a surjection of $\mathcal{C}$ onto α. □

Easy Example 2

Proposition 7

$L_{\aleph_\omega^L}$ is a model of KP containing a proper class which is not big.

- Every cardinal is admissible, and therefore $L_{\aleph_\omega^L} \models \mathrm{KP}$,
- Externally, $\textsc{Card} = \{\aleph_n^L \mid n \in \omega\}$ has cardinality ω,
- So $\textsc{Card}$ does not surject onto $\aleph_1^L$ in L,
- So there can be no surjection in $L_{\aleph_\omega^L}$.

Easy Examples 3

Theorem 8 (Gitman, Hamkins, Johnstone)

Suppose that $\mathrm{V} \models \mathrm{ZFC}$, κ *is a regular cardinal with* $2^\omega < \aleph_\kappa$ *and that* $G \subseteq \mathrm{Add}(\omega, \aleph_\kappa)$ *is* V*-generic. If* $\mathrm{W} = \bigcup_{\gamma<\kappa} \mathrm{V}[G_\gamma]$ *where* $G_\gamma = G \cap \mathrm{Add}(\omega, \aleph_\gamma)$, *(that is* G_γ *is the first* $\aleph_\gamma$ *many of the Cohen reals added by* G*) then* $\mathrm{W} \models \mathrm{ZFC}-$ *has the same cardinals as* V *and the* DC_α*-Scheme holds in* W *for all* $\alpha < \kappa$, *but the* DC_κ*-Scheme and the Collection Scheme fail.*

- $\mathrm{V} \subseteq \mathrm{W} \subseteq \mathrm{V}[G]$ all have the same cardinals.
- In $\mathrm{V}[G]$, $2^\omega = \aleph_\kappa$.
- Therefore there is no surjection of $\mathcal{P}(\omega)$ onto $\aleph_{\kappa+1}$.
- Hence there is no such surjection in W.
- $\mathcal{P}(\omega) \cap \mathrm{W}$ is a proper class in W, so W is a model of ZFC− with a proper class that is not big.

Dependent Choice

Definition 9

For μ an infinite cardinal, the DC_μ-*Scheme* is the assertion that any (definable class) tree of height μ, which has no maximal element and is closed under sequences of length μ has a branch of order type μ.

Notation

For $\mu = \aleph_0$, the above is called the DC-*Scheme*.

Consequences of DC in ZF:

- Baire Category Theorem (equivalent to DC),
- Downward Löwenheim-Skolem Theorem (equivalent to DC),
- Axiom of Choice for countable families,
- Every infinite set has a countably infinite subset,
- $\aleph_1$ is regular.

Reflection

Theorem 10 (Gitman, Hamkins and Johnstone)

Over ZFC^-*, the* $\mathrm{DC}_{\aleph_0}$*-Scheme is equivalent to the Reflection Principle.*

Theorem 11 (Friedman, Gitman and Kanovei)

The Reflection Principle is not provable in ZFC^-*.*

Proposition 12 (M.)

Suppose that $\mathrm{V} \models \mathrm{ZF}^- + \mathrm{DC}_\mu$ *for* μ *an infinite cardinal. Then for any proper class* $\mathcal{C}$*, which is definable over* V*, there is a subset* b *of* $\mathcal{C}$ *of cardinality* μ*.*

Proof

Proper Classes are Big with Dependent Choice

Proposition 13 (M.)

Suppose that $\mathrm{V} \models \mathrm{ZF}^- + \mathrm{DC}_\mu$ *for* μ *an infinite cardinal. Then for any proper class* $\mathcal{C}$*, which is definable over* V*, there is a subset* b *of* $\mathcal{C}$ *of cardinality* μ*.*

Proof

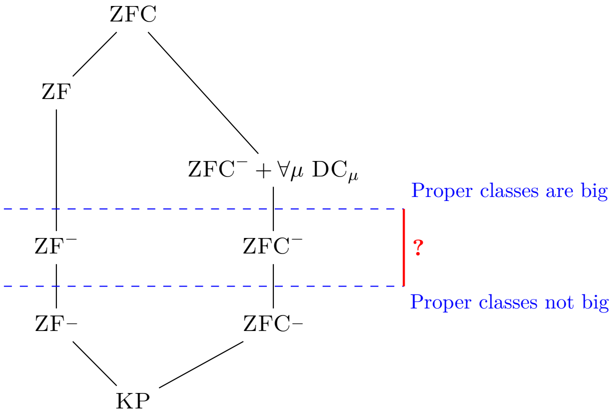
Corollary 14

If $\mathrm{V} \models \mathrm{ZF}^- + \mathrm{DC}_\mu$*, for every cardinal* μ*, then every proper class is big.*

Corollary 15

Over $\mathrm{V} \models \mathrm{ZF}^- + \mathrm{DC}_\mu$ *for every cardinal* μ*,* $j\colon \mathrm{V} \to \mathrm{M}$ *is an elementary embedding then both* $\mathcal{P}(\omega)$ *and* $\mathrm{V}_{\mathrm{crit}(j)}$ *are sets.*

Big Classes Summary



Question

Is every proper class big in $ZF(C)^-$?

Spoiler . . .

Question

Is every proper class big in ZFC^-?

Answer (Joint with V. Gitman) Answer

No!

. . . But we are not going to talk about that . . .

. . . Instead we will see a failed attempt . . .

An attempt

- Start with a model M of ZFC.
- Consider the forcing $\mathbb{P} = \mathrm{Add}(\omega, \textsc{Ord} \times \omega)$ to add $\textsc{Ord}$ many ω blocks of Cohen reals.
- Let $G \subseteq \mathbb{P}$ be generic. Then $\mathrm{M}[G] \models \mathrm{ZFC}^-$.
- Take the symmetric model N such that the blocks form an amorphous proper class.[1]

Assertion 16

N *is a model of* ZF^- *with an infinite class which doesn't surject onto* ω.

[1] That is an infinite class A such that for any subclass B either B or $A \setminus B$ is finite.

A Contradiction

Theorem 17

Suppose that $\langle \mathrm{N}, A\rangle$ satisfies;

1. N *models* $\mathrm{ZF}-$ *in the language with a predicate for* A,
2. $A \subseteq \mathrm{N}$ *and* $\langle \mathrm{N}, A\rangle \models$ *"A is a proper class"*,
3. $\langle \mathrm{N}, A\rangle \models$ *"if $B \subseteq A$ is infinite then B is a proper class".*

Then the Collection Scheme fails in $\langle \mathrm{N}, A\rangle$. In fact, $\langle \mathrm{N}, A\rangle$ does not have a cumulative hierarchy and therefore the Power Set also fails.

To prove that the Collection Scheme fails consider the sentence

$$\forall n \in \omega \, \exists y \, (|y| = n \ \wedge \ y \subseteq A).$$

What Does this mean?

- Suppose that $\mathrm{M} \models \mathrm{ZF}^-$.
- A class forcing $\mathbb{P} \in \mathrm{M}$ is *pretame* if for any $\mathbb{P}$-generic G, $\mathrm{M}[G] \models \mathrm{ZF}^-$.
- Adding Ord many Cohen reals is pretame.
- But the Collection Scheme failed in N.
- Therefore the symmetric submodel of a pretame class forcing need not preserve the Collection Scheme.

Remarks

- In fact, it is unclear what the symmetric submodel actually satisfies!
- In Gitik's model where every cardinal in singular, the forcing is pretame (and Power Set fails) but the symmetric submodel satisfies ZF!

Second Order Set Theory

There are two sorts: *sets* (e.g. $x, y, z, \ldots$) and *classes* (e.g. $A, B, C, \ldots$).
Use separate variables and quantifiers for sets and classes.

Convention

A model of a second-order theory will be a triple $\mathscr{V} = \langle \mathrm{V}, \in, \mathcal{C} \rangle$ where

- V consists of the sets,
- $\mathcal{C}$ consists of the classes,
- Every set is a class (i.e. $\mathrm{V} \subseteq \mathcal{C}$),
- Classes are made up of sets (i.e. for every $C \in \mathcal{C}$, $C \subseteq \mathrm{V}$).

Gödel-Bernays

Definition 18 (GB)

The axioms of GB are:

1. Set axioms: $\mathrm{V} \models \mathrm{ZF}$,
2. Class Replacement: If F is a function and a is a set then $F \restriction a$ is a set,
3. First-Order Comprehension: If $\varphi(x, A)$ is a first-order formula then $\{x \mid \varphi(x, A)\}$ is a class.

Example

If $\langle M, \in \rangle$ is a model of ZF and $\mathrm{def}(M)$ is the collection of definable subsets of M then $\langle M, \in, \mathrm{def}(M) \rangle \models \mathrm{GB}$.

More Theories

Definition 19

GBc is GB plus *set choice* (every set can be well-ordered), GBC is GB plus *Global Choice* (there is a bijection $F\colon \mathrm{V} \to \mathrm{Ord}$).

Definition 20

KM is GBC plus *Second-Order Comprehension*: If $\varphi(x, A)$ is a second-order formula then $\{x \mid \varphi(x, A)\}$ is a class.

Theorem 21

KM *implies the consistency of* ZFC.
If κ *is inaccessible then* $\langle V_\kappa, \in, \mathcal{P}(V_\kappa)\rangle \models \mathrm{KM}$.

Class Forcing

- Let $\mathscr{V} = \langle V, \in, \mathcal{C} \rangle$ be a model of GB.
- A *class forcing* is a partial order $\mathbb{P}$ such that $\mathbb{P} \in \mathcal{C}$.
- G is $\mathscr{V}$*-generic* for $\mathbb{P}$ if it meets every dense subclass $D \in \mathcal{C}$ of $\mathbb{P}$.
- In which case, the forcing extension is $\mathscr{V}[G] = \langle V[G], \in, \mathcal{C}[G] \rangle$.
- For the extension to be defined $\mathbb{P}$ needs to satisfy the *forcing theorem* which consists of 2 lemmas:
  1. *truth:* Anything true in the generic extension is forced to be true by an element in the generic.
  2. *definability:* The forcing relation $\Vdash$ is definable in the ground model.
- **Warning:** In class forcing, the forcing theorem may fail, even for atomic formulas!

The Forcing Theorem

Theorem 22 (Holy, Krapf, Lücke, Njegomir, Schlicht)

Let M *be a countable transitive model of* GB*. Then there is a partial order* $\mathbb{P}$*, which is definable over* M *and does not satisfy the forcing theorem for atomic formulae over* M*.*

Theorem 23 (Holy, Krapf, Lücke, Njegomir, Schlicht)

If $\mathbb{P}$ *satisfies the definability lemma for either "*$u \in v$*" or "*$u = v$*" then* $\mathbb{P}$ *satisfies the forcing theorem for all (*$\mathcal{L}_{\in}$*-)formulas over* M*.*

Definition 24 (Stanley / Friedman)

Let $\mathscr{V} \models \mathrm{GB}^{-}$. A forcing $\mathbb{P}$ is said to be *pretame* if for every generic G, $\mathscr{V}[G] \models \mathrm{GB}^{-}$.
Let $\mathscr{V} \models \mathrm{GB}$. A forcing $\mathbb{P}$ is said to be *tame* if for every generic G, $\mathscr{V}[G] \models \mathrm{GB}$.

Pretameness

Remarks

- If $\mathbb{P}$ is pretame then it satisfies the forcing theorem.
- $\mathbb{P}$ is pretame if and only if $\mathscr{V}[G] \models \mathrm{GB}-$.
- Both pretamesss and tameness can be formally defined by those forcings which satisfy a combinatorial property.

Formalisation

- Tame Forcing
  - Forcing GCH to fail / hold at every regular cardinal,
  - Making all supercompact cardinals Laver indestructible,
  - Adding a global well-order.
- Pretame Forcing
  - Adding ORD many Cohen reals $(\mathrm{Add}(\omega, \mathrm{ORD}))$,
- Satisfies Forcing Theorem
  - Adding a surjection $F\colon \omega \to \mathrm{ORD}$ $(\mathrm{Col}(\omega, \mathrm{ORD}))$.

(Set) Symmetric submodels

- Suppose that M is a model of ZFC and $\mathbb{P} \in$ M is a forcing. $(\mathrm{Add}(\omega, \omega))$
- Let $\mathcal{G} \in$ M be a group of order preserving automorphisms of $\mathbb{P}$. *(The automorphisms generated by bijections of ω)*
- Let $\mathcal{K} = \mathcal{P}(\mathcal{G})$.
- $\mathcal{F} \in$ M is a *normal filter of subgroups of* $\mathcal{G}$ if
  - $\mathcal{F} \subseteq \mathcal{K}$ and $\mathcal{G} \in \mathcal{F}$
  - If $H \in \mathcal{F}$ and $K \in \mathcal{F}$ then $H \cap K \in \mathcal{F}$,
  - If $H \in \mathcal{F}$ and $H \subseteq K$ where $K \in \mathcal{K}$ then $K \in \mathcal{F}$,
  - (Normality) If $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$ then $\pi H \pi^{-1} \in \mathcal{F}$.

  (The filter generated by fixing finite subsets of ω)
- We shall then call the triple $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ a *symmetric system*.

The Symmetric Model

Definition 25

Say that a name $\dot{x}$ is *symmetric* if

$$sym(\dot{x}) := \{\pi \in \mathcal{G} \mid \pi\dot{x} = \dot{x}\} \in \mathcal{F}.$$

Let $\mathrm{HS}_{\mathcal{F}}$ denote the class of *hereditarily symmetric names*.

The *symmetric model* given by $\mathcal{F}$ is

$$\mathrm{N}_G := \{\dot{x}^G \mid \dot{x} \in \mathrm{M}^{\mathbb{P}} \ \wedge \ \dot{x} \in \mathrm{HS}_{\mathcal{F}}\}$$

The Symmetry Lemma 26

Let φ be a formula, $p \in \mathbb{P}$, $\pi \in \mathcal{G}$ and $\dot{x} \in \mathrm{M}^{\mathbb{P}}$. Then

$$p \Vdash \varphi(\dot{x}) \iff \pi p \Vdash \varphi(\pi\dot{x}).$$

Theorem 27

N_G is a model of ZF with $\mathrm{M} \subseteq \mathrm{N}_G \subseteq \mathrm{M}[G]$.

Class Symmetric submodels

- Suppose that $\langle \mathrm{M}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \rangle$[2] is a model of fourth order ZFC and $\mathbb{P}$ is a pretame class forcing. $(\mathrm{Add}(\omega, \textsc{Ord}))$
- Let $\mathcal{G} \subseteq \mathcal{C}_2$ be a group of order preserving automorphisms of $\mathbb{P}$. (*The automorphisms generated by bijections of* Ord)
- Let $\mathcal{K} \in \mathcal{C}_3$ denote the collection of subclasses of $\mathcal{G}$.
- $\mathcal{F} \in \mathcal{C}_3$ is a *normal filter of subgroups of* $\mathcal{G}$ if
    - $\mathcal{F} \subseteq \mathcal{K}$ and $\mathcal{G} \in \mathcal{F}$
    - If $H \in \mathcal{F}$ and $K \in \mathcal{F}$ then $H \cap K \in \mathcal{F}$,
    - If $H \in \mathcal{F}$ and $H \subseteq K$ where $K \in \mathcal{K}$ then $K \in \mathcal{F}$,
    - (Normality) If $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$ then $\pi H \pi^{-1} \in \mathcal{F}$.

  (*The filter generated by fixing finite subsets of* Ord)
- We shall then call the triple $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ a *symmetric system*.

[2] $\mathcal{C}_1$ are the classes, $\mathcal{C}_2$ the hyper-classes and $\mathcal{C}_3$ the hyper-hyper-classes. For simplicity, take $\langle \mathrm{M}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \rangle = \langle \mathrm{V}_\kappa, \mathrm{V}_{\kappa+1}, \mathrm{V}_{\kappa+2}, \mathrm{V}_{\kappa+3} \rangle$ where κ is inaccessible.

The Symmetric Model

Definition 28

Say that a (class) name $\dot{X}$ is *symmetric* if

$$sym(\dot{X}) \coloneqq \{\pi \in \mathcal{G} \mid \pi\dot{X} = \dot{X}\} \in \mathcal{F}.$$

Let $\mathrm{HS}_{\mathcal{F}}$ denote the collection of *hereditarily symmetric names*.

Definition 29

The *symmetric model* given by $\mathcal{F}$ is $\langle \mathrm{N}_G, \mathcal{C}_G \rangle$ where

$$\mathrm{N}_G \coloneqq \{\dot{x}^G \mid \dot{x} \in \mathrm{M}^{\mathbb{P}} \ \wedge \ \dot{x} \in \mathrm{HS}_{\mathcal{F}}\}$$

$$\mathcal{C}_G \coloneqq \{\dot{\Gamma}^G \mid \dot{\Gamma} \in \mathcal{C}_1^{\mathbb{P}} \ \wedge \ \dot{\Gamma} \in \mathrm{HS}_{\mathcal{F}}\}.$$

The Symmetry Lemma 30

Let φ be a formula, $p \in \mathbb{P}$, $\pi \in \mathcal{G}$, $\dot{x} \in \mathrm{M}^{\mathbb{P}}$ and $\dot{\Gamma} \in \mathcal{C}_1$. Then

$$p \Vdash \varphi(\dot{x}, \dot{\Gamma}) \iff \pi p \Vdash \varphi(\pi\dot{x}, \pi\dot{\Gamma}).$$

Examples of Class Symmetric Models

- Adding a Dedekind finite class (a class X such that there is no injection from ω into X),
- Adding an amorphous class (a class which cannot be partitioned into two infinite subclasses),
- Gitik's Model (a model of ZF where every cardinal has cofinality ω),
- The Morris Model (ZF plus for every α there exists a set A_α which is the countable union of countable sets, and $\mathcal{P}(A_\alpha)$ can be partitioned into $\aleph_\alpha$ many non-empty sets),
- Monro's Model (ZF with a Dedekind finite proper class which surjects onto V). Formal Statement

Formalising the Contradiction

- Let $\langle \mathrm{M}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \rangle$ be a model of fourth-order set theory.
- The forcing $\mathbb{P} = \mathrm{Add}(\omega, \textsc{Ord} \times \omega)$ is pretame.
- So if $G \subseteq \mathbb{P}$ is generic then $\mathrm{M}[G] \models \mathrm{ZFC}^-$.
- Let $\mathcal{G}$ be the class of automorphisms of $\mathbb{P}$ generated by the wreath product of the automorphisms of $\textsc{Ord}$ with the automorphisms of ω.
- Let $\mathcal{F}$ be the filter generated by fixing finite subsets of $\textsc{Ord} \times \omega$.
- Then $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a symmetric system.
- Let N_G be the symmetric model given by $\mathcal{F}$.
- Then N_G contains an amorphous proper class. So, Collection fails in N_G.
- So the symmetric submodel of a pretame class forcing need not satisfy ZFC^-.
- In fact, I have no idea what theory N_G satisfies ...

Replacement

- Let N_G be the symmetric submodel.
- Suppose that $p \Vdash \dot{f}$ is a total function on $\dot{a}$ where $\dot{f}$ and $\dot{a}$ are hereditarily symmetric names.
- We want a name for the range of $\dot{f}$.
- Using Collection, we can find some set of hereditarily symmetric names c containing witnesses to elements being in the range of $\dot{f}$.
- Let

$$\dot{b} = \{\langle \dot{y}, s\rangle \mid \dot{y} \in c \wedge \exists \langle \dot{x}, r\rangle \in \dot{a}\,(s \in d_{\dot{x},r}{}^{3} \wedge s \Vdash \dot{f}(\dot{x}) = \dot{y})\}.$$

- Want to conclude that for any $\pi \in \mathrm{sym}(\dot{a}) \cap \mathrm{sym}(\dot{f})$, $\pi\dot{b} = \dot{b}$.
- However, in general, $\{\pi(\langle \dot{y}, s\rangle) \mid \pi \in \mathrm{sym}(\dot{a}) \cap \mathrm{sym}(\dot{f})\}$ will not be a set!

[3] These sets are determined using pretameness

Hereditary Respect

Definition 31 (Karagila)

Say that a name $\dot{x}$ is *respected* if $\{\pi \in \mathcal{G} \mid \mathbb{1} \Vdash \pi\dot{x} = \dot{x}\} \in \mathcal{F}$. Let $\mathrm{HR}_{\mathcal{F}}$ denote the class of *hereditarily respected names*.

Definition 32

The *respected model* given by $\mathcal{F}$ is $\langle \mathrm{N}_G, \mathcal{C}_G \rangle$ where

$$\mathrm{N}_G \coloneqq \{\dot{x}^G \mid \dot{x} \in \mathrm{M}^{\mathbb{P}} \ \wedge \ \dot{x} \in \mathrm{HR}_{\mathcal{F}}\}$$

$$\mathcal{C}_G \coloneqq \{\dot{x}^G \mid \dot{x} \in \mathcal{C}_1^{\mathbb{P}} \ \wedge \ \dot{x} \in \mathrm{HR}_{\mathcal{F}}\}.$$

Remark

If $\dot{a}, \dot{f} \in \mathrm{HR}_{\mathcal{F}}$, $p \Vdash$ '$\dot{f}$ is a total function on $\dot{a}$' and $\{\pi \mid \pi p = p\} \in \mathcal{F}$ then $\dot{b} \in \mathrm{HR}_{\mathcal{F}}$,

$$\dot{b} = \{\langle \dot{y}, s\rangle \mid \dot{y} \in c \ \wedge \ \exists \langle \dot{x}, r\rangle \in \dot{a} \ (s \in d_{\dot{x},r} \ \wedge \ s \Vdash \dot{f}(\dot{x}) = \dot{y})\}.$$

Tenacity

Definition 33 (Karagila)

Let $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ be a symmetric system. A condition $p \in \mathbb{P}$ is said to be *$\mathcal{F}$-tenacious* if there exists some $H \in \mathcal{F}$ such that for every $\pi \in H$, $\pi p = p$.
$\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is said to be *$\mathcal{F}$-tenacious* if there is a dense subset of $\mathcal{F}$-tenacious conditions.

Example

The forcing to add an amorphous proper class is tenacious. As is Cohen's model adding a Dedekind finite set (/ class)

Theorem 34 (Karagila, Hayut)

Over ZFC, *every (set) symmetric system is equivalent to a tenacious one.*

Tenacity

Definition 33 (Karagila)

Let $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ be a symmetric system. A condition $p \in \mathbb{P}$ is said to be *$\mathcal{F}$-tenacious* if there exists some $H \in \mathcal{F}$ such that for every $\pi \in H$, $\pi p = p$.
$\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is said to be *$\mathcal{F}$-tenacious* if there is a dense subset of $\mathcal{F}$-tenacious conditions.

The proof relies on the completion of the corresponding Boolean algebra, which does not always exist for class forcings ...

Question 1

If every class symmetric system ($\mathbb{P}$ pretame) equivalent to a tenacious one?

The Respected Model 1

Fix $\mathscr{M}$ a model of fourth-order set theory and $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ a symmetric system such that $\mathbb{P}$ satisfies the forcing theorem. Given a $\mathbb{P}$-generic G, let $\mathscr{N}_G$ be the respective Respected Model.

Theorem 34 (M.)

If for any $\mathbb{P}$-generic $G \subseteq \mathbb{P}$ $\mathscr{M}[G]$ is closed under Gödel operations, then so is $\mathscr{N}_G$.

Theorem 35 (M.)

If $\mathbb{P}$ is pretame and $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is tenacious then $\mathscr{N}_G \models \mathrm{GB}-$ (with second-order Replacement but not Collection).

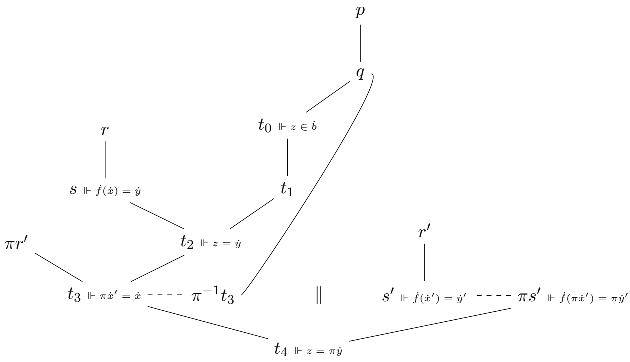
Observation

If $\mathscr{N}_G$ is the Respected submodel formed by adding an amorphous proper class as before, then Collection fails in $\mathscr{N}_G$.[4]

[4] but, unlike in the symmetric case, we can prove Replacement!

Replacement in Respected Model

Suppose $p \Vdash \dot{f}$ is a total function on $\dot{a}$.
$\dot{b} = \{\langle \dot{y}, s\rangle \mid \dot{y} \in c \ \wedge \ \exists \langle \dot{x}, r\rangle \in \dot{a} \ (s \in d_{\dot{x},r} \ \wedge \ s \Vdash \dot{f}(\dot{x}) = \dot{y})\}$.
$q \leq p$ (satisfies some condition) and
$\pi \in \mathrm{resp}(\dot{a}) \cap \mathrm{resp}(\dot{f}) \cap \mathrm{sym}(q)$.
Claim: $q \Vdash \pi\dot{b} = \dot{b}$ (and therefore so does $\mathbb{1}$).



The Respected Model 2

Theorem 36 (M.)

If $\mathbb{P}$ is tame then $\mathcal{N}_G \models \mathrm{GB}$.

Remark

This does not require any assumption about tenacity.
Essentially because, using tameness,

$$\mathbb{1} \Vdash \exists u \, \forall x \, (\mathrm{rank}(x) < \alpha \rightarrow x \in u)$$

and $\pi\mathbb{1} = \mathbb{1}$ for any automorphism π.

Symmetric vs Respected

Proposition 37

Suppose $\mathbb{P}$ is a pretame class forcing and $\langle\mathbb{P},\mathcal{G},\mathcal{F}\rangle$ is a tenacious symmetric system. Suppose further that for any $\dot{x}\in\mathrm{HR}_{\mathcal{F}}$, $\{\pi\dot{x}\mid\pi\in\mathcal{G}\}\in\mathrm{V}$.

Then $\dot{x}\in\mathrm{HR}_{\mathcal{F}}$ iff there is some $\dot{y}\in\mathrm{HS}_{\mathcal{F}}$ such that $\mathbb{1}\Vdash\dot{x}=\dot{y}$.

Idea

If $\dot{x}\in\mathrm{HR}_{\mathcal{F}}$ then $\dot{y}$ will be

$$\dot{y}=\bigcup\{\pi\dot{x}\mid\pi\in\mathrm{resp}(\dot{x})\}.$$

Symmetric vs Respected

Proposition 37

Suppose $\mathbb{P}$ is a pretame class forcing and $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a tenacious symmetric system. Suppose further that for any $\dot{x} \in \mathrm{HR}_{\mathcal{F}}$, $\{\pi\dot{x} \mid \pi \in \mathcal{G}\} \in \mathrm{V}$.

Then $\dot{x} \in \mathrm{HR}_{\mathcal{F}}$ iff there is some $\dot{y} \in \mathrm{HS}_{\mathcal{F}}$ such that $\mathbb{1} \Vdash \dot{x} = \dot{y}$.

Theorem 38 (M.)

Assuming the above hypothesis, $\mathcal{N}_G \models \mathrm{GB}^-$. In particular, $\mathcal{N}_G$ satisfies Collection.

Corollary 39

If $\mathbb{P}$ is a set then the Symmetric Model and the Respected Model are the same.

Questions

Question 1

If every class symmetric system ($\mathbb{P}$ pretame) equivalent to a tenacious one?

Question 2

What conditions do we have to put on a (pretame) tenacious symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ to ensure $\mathscr{N}_G$ satisfies Collection?

Question 3

What about Power Set?

Question 4

Is the Respected Model actually different to the Symmetric Model?

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Union of ZF^- models

Construction 40 (Zarach)

Suppose that $\mathrm{M} \models \mathrm{ZFC}$, $\mathbb{P} \in \mathrm{M}$, $\omega(\mathbb{P})$ *is the finite support product of* ω *many copies of* N *and* $h\colon \mathbb{P} \cong \omega(\mathbb{P})$ *be an order isomorphism. Let* G *be* $\mathbb{P}$*-generic over* $\mathcal{M}$ *and* $H = h``G$ *be* $\omega(\mathbb{P})$*-generic. Let* $G_n = H \restriction \{n\}$ *be the* n^{th} *generic and let* $\mathrm{M}_n = \mathrm{M}[G_0 \times \cdots \times G_{n-1}]$. *Consider*

$$\mathrm{N} = \bigcup_n \mathrm{M}_n.$$

Theorem 41 (M., Gitman)

N *is a model of* $\mathrm{ZFC}^-_{Ref} + \neg\mathrm{DC}_{|\mathcal{P}^{\mathrm{V}[G]}(\mathbb{P})|^+}$. *In particular,* $\mathcal{P}(\mathbb{P})$ *is a proper class that does not surject onto every ordinal!*

A solution

Theorem 42 (M., Gitman)

$\langle \mathrm{N}, \in, \mathrm{M}\rangle$ is a model of $\mathrm{ZFC}^-_{Ref} + \neg\mathrm{DC}_{|\mathcal{P}^{\mathrm{V}[G]}(\mathbb{P})|^+}$. In particular, $\mathcal{P}(\mathbb{P})$ is a proper class that does not surject onto every ordinal!

Corollary 43 (M., Gitman)

One can have models V of ZFC^-_{Ref} with an elementary embedding $j\colon \mathrm{V} \to \mathrm{M}$ for which $\mathcal{P}(\omega)$ is a proper class.

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A note on Injections

Theorem 44 (Monro)

Let $\mathrm{ZF}(\mathrm{K})$ *be the theory with the language of* ZF *plus a one-place predicate* K *and let* M *be a countable transitive model of* ZF. *Then there is a model* N *such that* N *is a transitive model of* $\mathrm{ZF}(\mathrm{K})$ *and*

$$\begin{aligned} \mathrm{N} \quad \models \quad & \mathrm{K} \textit{ is a proper class which is Dedekind-finite} \\ & \textit{and can be mapped onto the universe.} \end{aligned}$$

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Proper Classes Are Big with Reflection

Proposition 45 (M.)

Suppose that $V \models ZF^- + DC_\mu$ *for* μ *an infinite cardinal. Then for any proper class* $\mathcal{C}$*, which is definable over* V*, there is a subset* b *of* $\mathcal{C}$ *of cardinality* μ*.*

Proof

- We shall prove that for any $\nu \leq \mu$ there is a subset b of $\mathcal{C}$ and a bijection between b and ν.
- Suppose not and let δ be the least cardinal for which this fails.
- Let $\varphi(\alpha, s, y) \equiv (s \cup \{y\} \subseteq \mathcal{C} \ \wedge \ y \notin s \ \wedge \ \mathrm{len}(s) = \alpha)$.
- This satisfies the hypothesis of DC_δ.
- So there is a function f with domain δ and whose range gives a subset of $\mathcal{C}$ of cardinality δ. Contradiction. □

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Pretameness

$D \in \mathcal{C} \cap \mathcal{P}(\mathbb{P})$ is said to be *dense below $p \in \mathbb{P}$* if for every $r \leq p$ there is some $s \leq r$ such that $s \in D$.

A set d is said to be *predense below $p \in \mathbb{P}$* if for every $r \leq p$ there is some $s \in d$ which is compatible with r.

A class forcing $\mathbb{P}$ is said to be *pretame* is for every $p \in \mathbb{P}$ and every set length sequence of dense subclasses $\langle D_i \mid i \in I \rangle \in \mathcal{C}$ of $\mathbb{P}$, there is some $q \leq p$ and $\langle d_i \mid i \in I \rangle \in \mathrm{V}$ such that for every $i \in I$, $d_i \subseteq D_i$ and d_i is predense below q.

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