Determinacy and inner models

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- Games of length ω

- Games of countable length $\alpha > \omega$

- Games on reals

Games in Set Theory

Fix set $A \subseteq {}^\omega\omega$. $G(A)$

I	$\sigma(\emptyset)$ $\cancel{n_0}$	$\sigma(\emptyset, n_1)$ $\cancel{n_2}$	$\dots$
II	n_1	n_3	$\dots$

Strategy σ for I

I wins iff $(n_0, n_1, \dots) \in A \subseteq {}^\omega\omega$

σ is winning strategy for I iff $\sigma * x \in A$ for all $x \in {}^\omega\omega$

$G(A)$ (or A) is <u>determined</u> iff one of the players has a winning strategy

Which games are determined?

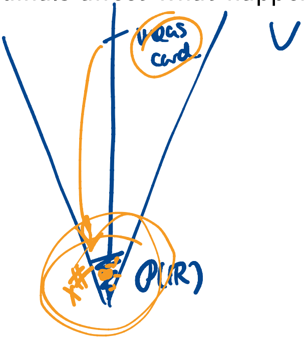
- Gale-Stewart, 1953: Assume ZFC. Then every open and every closed set is determined.
- Martin, 1975: Assume ZFC. Then every Borel set of reals is determined.
- Martin, 1970: Assume ZFC and that there is a measurable cardinal. Then every analytic set is determined.
- Martin-Steel, 1985: Assume ZFC and there are n Woodin cardinals with a measurable cardinal above them all. Then every $\boldsymbol{\Sigma}^1_{n+1}$ set is determined.
- Gale-Stewart, 1953: Assuming AC there is a set of reals which is not determined.

Determinacy and large cardinals

Are large cardinals necessary for the determinacy of these sets of reals?

Yes, in some sense

How can these large cardinals affect what happens with the sets of reals?



They don't, their countable sisters have

The main goal of inner model theory is to construct L-like models, which we call mice, for stronger and stronger large cardinals.

Gödel's constructible universe L

Definition

Let E be a set or a proper class. Let

$$\begin{aligned} J_0[E] &= \emptyset \\ J_{\alpha+1}[E] &= \mathrm{rud}_E(J_\alpha[E] \cup \{J_\alpha[E]\}) \\ J_\lambda[E] &= \bigcup_{\alpha<\lambda} J_\alpha[E] \text{ for limit } \lambda \\ L[E] &= \bigcup_{\alpha\in\mathrm{Ord}} J_\alpha[E] \end{aligned}$$

nice closure operator

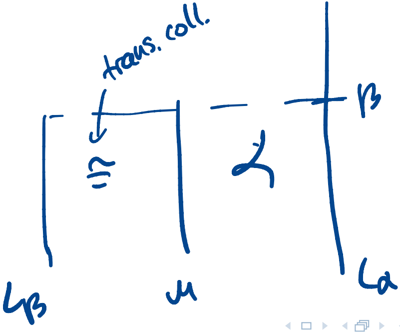
Note that rud_E denotes the closure under functions which are rudimentary in E (i.e. basic set operations like minus, union and pairing or intersection with E).

Basic properties of L

Condensation Let α be an infinite ordinal and let

$$M \prec (L_\alpha, \in).$$

Then the transitive collapse of M is equal to L_β for some ordinal $\beta \leq \alpha$.



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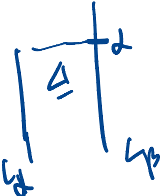
Comparison Let L_α and L_β for ordinals α and β be initial segments of L. Then one is an initial segment of the other, that means



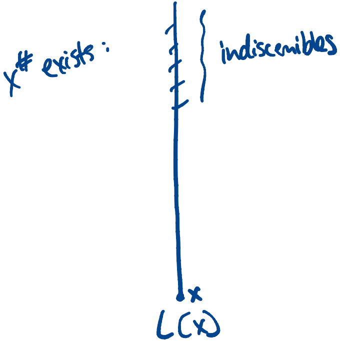
$$L_\alpha \trianglelefteq L_\beta \text{ or } L_\beta \trianglelefteq L_\alpha.$$

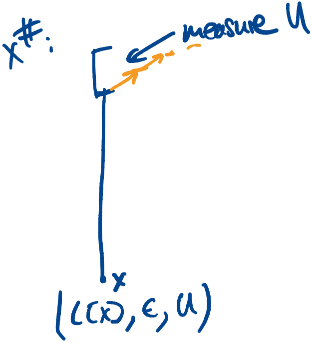
An equivalence for Analytic Determinacy

Theorem (Harrington, Martin)

The following are equivalent.

1. *All analytic games are determined.*
2. $x^\#$ *exists for all reals* x.





An equivalence for Projective Determinacy

Theorem (Neeman, Woodin)

Open

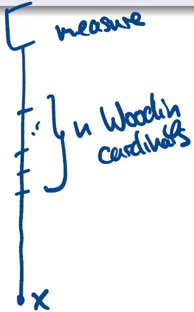
Let $n \geq 1$. Then the following are equivalent.

1. Σ^1_{n+1}*-determinacy.*

   Σ^1_n-det + Σ^1_{n+1}-det

2. *For every $x \in \mathbb{R}$ the ω_1-iterable countable model of set theory with n Woodin cardinals $M_n^\#(x)$ exists.*

   f.a. x $M_{n-1}^\#(x)$ ex. + $M_n^\#$ exists



For (a) $\Rightarrow$ (b) see (M, Schindler, Woodin) "Mice with Finitely many Woodin Cardinals from Optimal Determinacy Hypotheses", JML 2020.

For (b) $\Rightarrow$ (a) see (Neeman) "Optimal proofs of determinacy II", JML 2002.

Why stop playing at ω?

Let $\alpha < \omega_1$. Let $A \subseteq {}^{\alpha}(\omega^\omega)$

I	x_0	x_2	…	x_ω		$x_{\omega+2}$	…
II	x_1	x_3	…		$x_{\omega+1}$		…

1st round   2nd round   … α rounds

I wins iff $(x_\eta)_{\eta < \omega\alpha} \in A$.

winning strategies, determined

What we know

Theorem (Mycielski, 1964)

AD_{ω_1}, *determinacy for arbitrary games of length* ω_1, *is inconsistent.*

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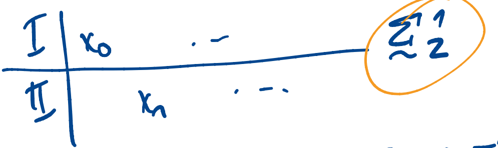
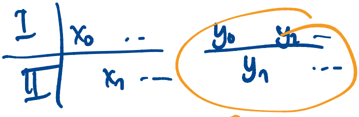
Proposition

$\mathrm{Det}_{\omega\cdot(n+1)}(\boldsymbol{\Pi}^1_1)$ implies $\mathrm{Det}_{\omega}(\boldsymbol{\Pi}^1_{n+1})$.

$\omega\cdot 2$



I wins iff $(x_n)_{n<\omega} \in p[A]$, $A \in \boldsymbol{\Pi}^1_1$



witness for $x \in p[A]$

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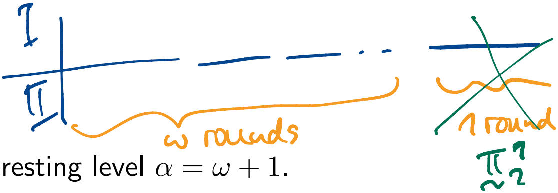
Theorem (Neeman, 2004)

Let $\alpha > 1$ *be a countable ordinal and suppose that there are* $-1+\alpha$ *Woodin cardinals with a measurable cardinal above them all. Then* $\mathrm{Det}_{\omega\cdot\alpha}(\mathbf{\Pi}^1_1)$ *holds.*



Let's focus on the first interesting level $\alpha = \omega + 1$.

Theorem (Aguilera-M, JSL 2020)

Suppose $\mathrm{Det}_{\omega\cdot(\omega+1)}(\mathbf{\Pi}^1_1)$. *Then there is a premouse with* $\omega + 1$ *Woodin cardinals.*

L-like model

In fact, the proof will only use $\mathrm{Det}_{\omega^2}(\mathbf{\Pi}^1_2)$.

A short sketch of the proof

In a first step, we use the determinacy hypothesis to show the following lemma.

Lemma

Suppose $\mathrm{Det}_{\omega^2}(\mathbf{\Pi}^1_2)$*. Then there is a club* $\mathcal{C}$ *in* $\mathcal{P}_{\omega_1}(\mathbb{R})$ *such that for all* $A \in \mathcal{C}$*,*

1. $M_1(A) \cap \mathbb{R} = A$*, and*
2. $M_1(A) \models \mathrm{AD}$*.*

Wdn card.

$A = \mathbb{R}^{M_1(A)}$

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Theorem (M, MALQ 2019)

For $A \in \mathcal{C}$ *as above,* $M_1(A) \vDash \mathrm{DC}$.

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For $A \in \mathcal{C}$ *as above,* $M_1(A) \models \mathrm{DC}$.

This allows us to argue that in fact

$$M_1(A) \models \mathrm{DC} + \mathrm{AD} + \text{“}\Sigma^2_1 \text{ has the scale property”} + \Theta = \theta_0 + \text{Mouse Capturing.}$$

very nice ☺

properties

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Now, we use AD to translate $M_1(A)$ into a premouse with $\omega + 1$ Woodin cardinals.

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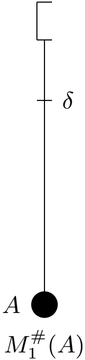

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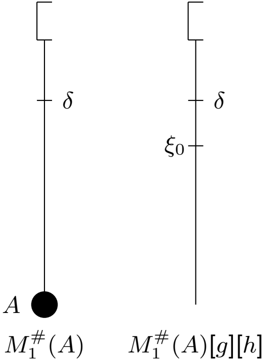

- g generic for a Prikry-type forcing, adds a premouse $\mathcal{M}$ with ω Woodin cardinals
- h is $\mathrm{Col}(\omega, {<}\lambda)$-generic for λ the sup of these Woodins

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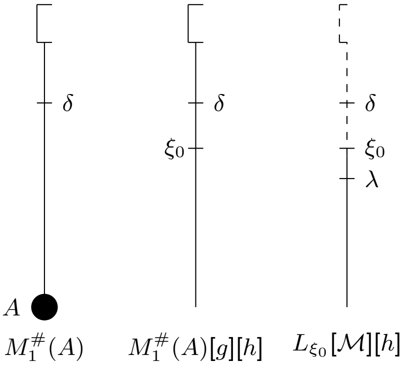

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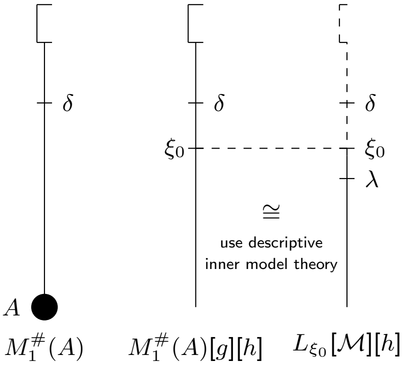

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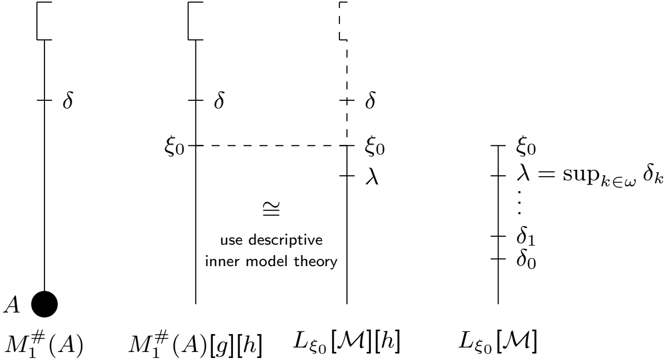

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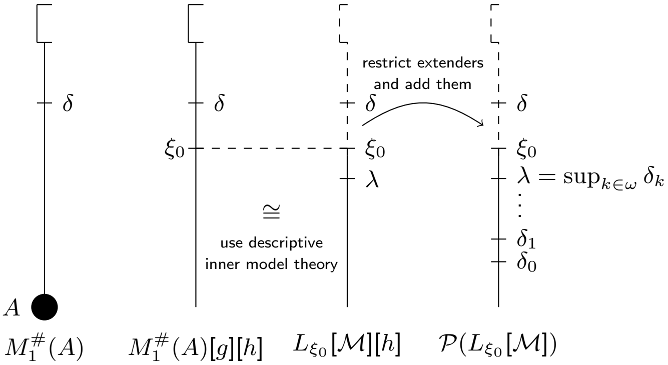

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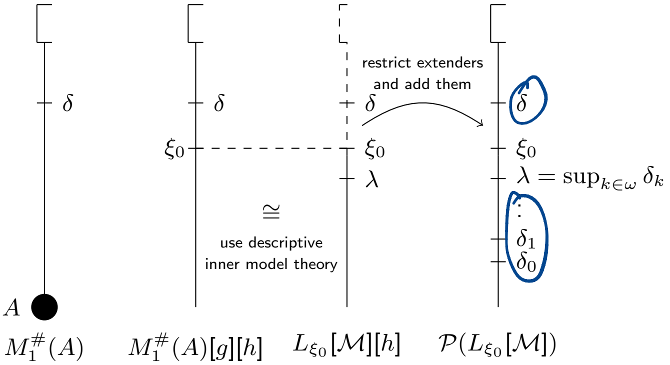

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Games on reals

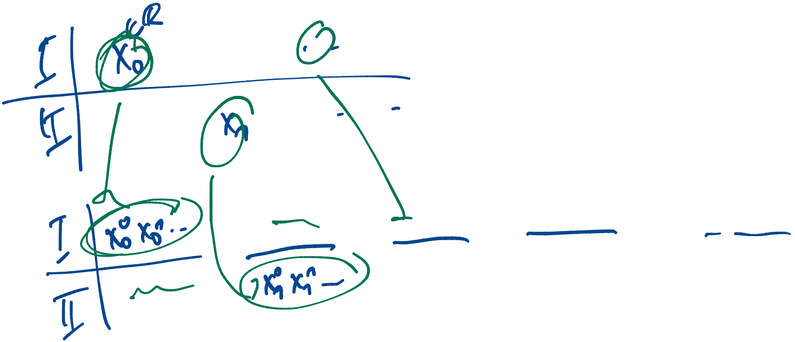
As common, we write $\mathbb{R}$ for ω^ω.

$$\begin{array}{c|cccccc} \mathrm{I} & x_0 \in \mathbb{R} & & x_2 & & \ldots \\ \hline \mathrm{II} & & x_1 & & x_3 & & \ldots \end{array}$$

Games on reals

Proposition

Suppose $\mathrm{Det}_{\omega^2}(\mathbf{\Pi}^1_1)$. Then $\mathrm{Det}^{\mathbb{R}}_{\omega}(\mathbf{\Pi}^1_1)$ holds, i.e. all analytic games on reals of length ω are determined.



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Theorem (Aguilera-M, NDJFL 2020)

The following are equivalent:

1. *Projective determinacy for games on $\mathbb{R}$;*
2. *$M_n^\sharp(\mathbb{R})$ exists for all $n \in \mathbb{N}$.*

measure

n Woodin cards

$\mathbb{R}$

- Games of length ω

$x^{\#}$, $M_n^{\#}(x)$ ctble objects

- Games of countable length $\alpha > \omega$

$M_{\omega+1}^{\#}(x)$, "sisters of large cardinals"

$M_\alpha^{\#}(x)$, $\alpha < \omega_1$

- Games on reals

$M_\omega^{\#}(\mathbb{R})$

"There is an ever changing list of questions in set theory the answers to which would greatly increase our understanding of the universe of sets. The difficulty of course is the ubiquity of independence: almost always the questions are independent."

(W. H. Woodin in Suitable Extender Models I)