The Existence Property among Set Theories

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Gent – Leeds Seminar

Tarmstedt, 13 August 2020

Plan of the Talk

1. Intuitionism

2. The Existence Property and other properties

3. The Existence Property and Collection

Existentialism

All flavors of constructivism seem to demand that:

The correctness of an existential claim $(\exists x \in A)\varphi(x)$ is to be guaranteed by warrants from which both an object $x_0 \in A$ and a further warrant for $\varphi(x_0)$ are constructible.

Bishop: Man and God

When a man proves a positive integer to **exist**, he should show how to **find it**.

If God has mathematics of his own that needs to be done, let him do it himself.

Basic Assumptions

Let T be a theory whose language, $L(T)$, encompasses the language of set theory. Moreover, for simplicity, we shall assume that $L(T)$ has a constant ω denoting the set of von Neumann natural numbers and for each n a constant $\bar{n}$ denoting the n-th element of ω.

The Disjunction Property

1. T has the **disjunction property**, **DP**, if whenever

$$T \vdash \psi \vee \theta$$

holds for sentences ψ and θ of T, then

$$T \vdash \psi \quad \text{or} \quad T \vdash \theta.$$

Existence Properties

1. T has the **numerical existence property**, **NEP**, if whenever

   $$T \vdash (\exists x{\in}\omega)\varphi(x)$$

   holds for a formula $\varphi(x)$ with at most the free variable x, then

   $$T \vdash \varphi(\bar{n})$$

   for some n.

2. T has the **term existence property**, **TEP**, if whenever

   $$T \vdash \exists x \varphi(x)$$

   holds for a formula $\varphi(x)$ with at most the free variable x, then

   $$T \vdash \varphi(t)$$

   for some closed term t.

Existence Properties continued

1. T has the **existence property**, **EP**, if whenever

$$T \vdash \exists x \varphi(x)$$

holds for a formula $\varphi(x)$ having at most the free variable x, then there is a formula $\vartheta(x)$ with exactly x free, so that

$$T \vdash \exists! x\, \vartheta(x) \quad \text{and} \quad T \vdash \exists x\, [\vartheta(x) \wedge \varphi(x)].$$

2. Slight Problem: Even classical set theories can have the existence property.

   For example, set theories with axioms like $V = OD$ or $V = L$.

   And for the wrong reason: Excluded middle.

Some Ancient History

- **Gödel** (1932) observed that intuitionistic propositional logic has the **DP**.
- **Gentzen** (1934): Intuitionistic predicate logic has the **DP** and **EP**.
- **Kleene** (1945): **HA** has the **DP** and **NEP**.

Intuitionistic Zermelo-Fraenkel set theory, **IZF**

- * **Extensionality**
- ▶ **Pairing**, **Union**, **Infinity**
- ▶ **Full Separation**
- ▶ **Powerset**
- # **Collection**

$$(\forall x \in a)\, \exists y\; \varphi(x, y) \quad \rightarrow \quad \exists b\, (\forall x \in a)\, (\exists y \in b)\; \varphi(x, y)$$

- * **Set Induction**

$$(IND_{\in}) \qquad \forall a\, (\forall x \in a\; \varphi(x) \,\rightarrow\, \varphi(a)) \rightarrow \forall a\; \varphi(a),$$

Myhill's $\mathbf{IZF}_R$:

IZF with **Replacement** instead of **Collection**

Constructive Zermelo-Fraenkel set theory, **CZF**

* **Extensionality**
- **Pairing**, **Union**, **Infinity**
- **Bounded Separation**

Subset Collection

For all sets A, B there exists a "sufficiently large" set of multi-valued functions from A to B.

Strong Collection

$$(\forall x \in a)\, \exists y\ \varphi(x, y) \ \rightarrow$$
$$\exists b\, [\, (\forall x \in a)\, (\exists y \in b)\ \varphi(x, y) \ \wedge\ (\forall y \in b)\, (\exists x \in a)\ \varphi(x, y)\,]$$

* **Set Induction scheme**

The variant $\mathbf{CZF}_E$ of Constructive Zermelo-Fraenkel set theory

Instead of **Subset Collection**, $\mathbf{CZF}_E$ has
Exponentiation: A, B sets $\Rightarrow A^B$ set.

Two types of set existence axioms

- **Explicit set existence axioms**: e.g. Separation, Replacement, Exponentiation
- **Non-explicit** set existence axioms: e.g. in classical set theory Axioms of Choice
- **Non-explicit set existence axioms** in intuitionistic set theory: e.g. Axioms of Choice, (Strong) Collection, Subset Collection, Regular Extension Axiom

Some History

Let $\mathbf{IZF}_R$ result from **IZF** by replacing Collection with Replacement, and let **CST** be Myhill's constructive set theory.

Theorem 1. (Myhill)
$\mathbf{IZF}_R$ and **CST** have the **DP**, **NEP**, and the **EP**.

Theorem 2. (Beeson)
IZF has the **DP** and the **NEP**.

Theorem 3. (Friedman, Scedrov)
IZF does not have the **EP**.

Realizability Theorem

Realizability with truth.

Theorem: (R)

For every theorem θ of **CZF***, there exists an application term s such that*

$$\mathbf{CZF} \vdash \ (s \Vdash_t \theta).$$

Moreover, the proof of this soundness theorem is effective in that the application term s can be effectively constructed from the **CZF** *proof of θ.*

The Main Theorem

Theorem: (R)

The **DP** *and the* **NEP** *hold true for* **CZF**, **CZF** + **REA** *and* **CZF** + *Large Set Axioms.*
One can also add Subset Collection and the following choice principles:

$\mathbf{AC}_\omega, \mathbf{DC}, \mathbf{RDC}, \mathbf{PAx}$.

Theorem:

The **DP** *and the* **NEP** *hold true for* **IZF**, **IZF** + **REA** *and* **IZF** + *Large Set Axioms.*
One can also add $\mathbf{AC}_\omega, \mathbf{DC}, \mathbf{RDC}, \mathbf{PAx}$.

Failure of **EP** for **IZF**

Collection is

$$\forall x \in a\, \exists y\, A(x,y) \;\rightarrow\; \exists b\, \forall x \in a\, \exists y \in b\, A(x,y).$$

This is in **IZF** equivalent to

$$\exists b\, [\forall x \in a\, \exists y\, A(x,y) \;\rightarrow\; \forall x \in a\, \exists y \in b\, A(x,y)]$$

- Let $B(z)$ be a formula expressing that z is an uncountable cardinal. Let $B^*(z)$ result from $B(z)$ by replacing every atomic subformula D of $B(z)$ by

$$D \;\vee\; \forall uv(u \in v \;\vee\; \neg\, u \in v).$$

EP fails for **IZF** for the following instance:

$$\exists y\, [\forall x \in 1\, \exists z\, B^*(z) \;\rightarrow\; \forall x \in 1\, \exists z \in y\, B^*(z)]\,.$$

Problems

- (Beeson 1985) Does any reasonable set theory **with Collection** have the existential definability property?

Conjectures

Conjecture 1. **CZF** does not have the **existence property**.

Conjecture 2. $\mathbf{CZF}_E$ has the **existence property**.

Theorem 1. (Andrew Swan) **CZF** does not have the **existence property**.

The Weak Existence Property

T has the **weak existence property**, **wEP**, if whenever

$$T \vdash \exists x \varphi(x)$$

holds for a formula $\varphi(x)$ having at most the free variable x, then there is a formula $\vartheta(x)$ with exactly x free, so that

$$\begin{aligned} T &\vdash \exists! x\, \vartheta(x), \\ T &\vdash \forall x\, [\vartheta(x) \to \exists u\, u \in x], \\ T &\vdash \forall x\, [\vartheta(x) \to \forall u \in x\, \varphi(u)]. \end{aligned}$$

Extended E-recursive functions

- We would like to have unlimited application of sets to sets, i.e. we would like to assign a meaning to the symbol

$$\{a\}(x)$$

where a and x are sets.

- Known as **E-recursion** or **set recursion**
- However, we shall introduce an extended notion of E-computability, christened **$E_\wp$-computability**, rendering the function

$$exp(a,b) = {}^{a}b$$

computable as well.

- Classically, $E_\wp$-computability is related to **power recursion**, where the power set operation is regarded to be an initial function. Notion studied by Yiannis Moschovakis and Larry Moss.

Realizability with sets of witnesses

We use the expression $a \neq \emptyset$ to convey the positive fact that the set a is inhabited, that is $\exists x\, x \in a$.
We define a relation

$$a \Vdash_{\mathfrak{wt}} B$$

between sets and set-theoretic formulae.

$$a \bullet f \Vdash_{\mathfrak{wt}} B$$

will be an abbreviation for

$$\exists x [a \bullet f \simeq x \ \wedge \ x \Vdash_{\mathfrak{wt}} B]$$

$$
\begin{array}{rcl}
a \Vdash_{\mathfrak{wt}} A & \text{iff} & A \text{ holds true, whenever } A \text{ is an atomic formula} \\
a \Vdash_{\mathfrak{wt}} A \wedge B & \text{iff} & \jmath_0 a \Vdash_{\mathfrak{wt}} A \ \wedge\ \jmath_1 a \Vdash_{\mathfrak{wt}} B \\
a \Vdash_{\mathfrak{wt}} A \vee B & \text{iff} & a \neq \emptyset \wedge (\forall d \in a)\big(\big[\jmath_0 d = 0 \wedge \jmath_1 d \Vdash_{\mathfrak{wt}} A\big] \ \vee \\
 & & \qquad \big[\jmath_0 d = 1 \ \wedge\ \jmath_1 d \Vdash_{\mathfrak{wt}} B\big]\big) \\
a \Vdash_{\mathfrak{wt}} \neg A & \text{iff} & \neg A \wedge \forall c\, \neg c \Vdash_{\mathfrak{wt}} A \\
a \Vdash_{\mathfrak{wt}} A \to B & \text{iff} & (A \to B) \wedge \forall c \left[c \Vdash_{\mathfrak{wt}} A \ \to\ a \bullet c \Vdash_{\mathfrak{wt}} B\right] \\
a \Vdash_{\mathfrak{wt}} (\forall x \in b)\, A & \text{iff} & (\forall c \in b)\; a \bullet c \Vdash_{\mathfrak{wt}} A[x/c] \\
a \Vdash_{\mathfrak{wt}} (\exists x \in b)\, A & \text{iff} & a \neq \emptyset \wedge (\forall d \in a)[\jmath_0 d \in b \wedge \jmath_1 d \Vdash_{\mathfrak{wt}} A[x/\jmath_0 d]] \\
a \Vdash_{\mathfrak{wt}} \forall x A & \text{iff} & \forall c \ \ a \bullet c \Vdash_{\mathfrak{wt}} A[x/c] \\
a \Vdash_{\mathfrak{wt}} \exists x A & \text{iff} & a \neq \emptyset \wedge (\forall d \in a)\; \jmath_1 d \Vdash_{\mathfrak{wt}} A[x/\jmath_0 d]
\end{array}
$$

$$
\begin{array}{rll}
a \Vdash_{\mathfrak{wt}} A & \text{iff} & A \text{ holds true, whenever } A \text{ is an atomic formula} \\
a \Vdash_{\mathfrak{wt}} A \wedge B & \text{iff} & \jmath_0 a \Vdash_{\mathfrak{wt}} A \ \wedge\ \jmath_1 a \Vdash_{\mathfrak{wt}} B \\
a \Vdash_{\mathfrak{wt}} A \vee B & \text{iff} & a \neq \emptyset \wedge (\forall d \in a)\big(\big[\jmath_0 d = 0 \ \wedge\ \jmath_1 d \Vdash_{\mathfrak{wt}} A\big] \vee \\
 & & \qquad \big[\jmath_0 d = 1 \ \wedge\ \jmath_1 d \Vdash_{\mathfrak{wt}} B\big]\big) \\
a \Vdash_{\mathfrak{wt}} \neg A & \text{iff} & \neg A \wedge \forall c\, \neg c \Vdash_{\mathfrak{wt}} A \\
a \Vdash_{\mathfrak{wt}} A \rightarrow B & \text{iff} & (A \rightarrow B) \wedge \forall c \left[c \Vdash_{\mathfrak{wt}} A \ \rightarrow\ a \bullet c \Vdash_{\mathfrak{wt}} B\right] \\
a \Vdash_{\mathfrak{wt}} (\forall x \in b)\, A & \text{iff} & (\forall c \in b)\ a \bullet c \Vdash_{\mathfrak{wt}} A[x/c] \\
a \Vdash_{\mathfrak{wt}} (\exists x \in b)\, A & \text{iff} & a \neq \emptyset \wedge (\forall d \in a)(\, \jmath_0 d \in b \wedge \jmath_1 d \Vdash_{\mathfrak{wt}} A[x/\jmath_0 d]\,) \\
a \Vdash_{\mathfrak{wt}} \forall x A & \text{iff} & \forall c \ \ a \bullet c \Vdash_{\mathfrak{wt}} A[x/c] \\
a \Vdash_{\mathfrak{wt}} \exists x A & \text{iff} & a \neq \emptyset \wedge (\forall d \in a)\ \jmath_1 d \Vdash_{\mathfrak{wt}} A[x/\jmath_0 d]
\end{array}
$$

$$\Vdash_{\mathfrak{wt}} B \text{ iff } \exists a\, a \Vdash_{\mathfrak{wt}} B.$$

If we use indices of $E_{\wp}$-recursive functions rather than $E_{\exp}$-recursive functions, we notate the corresponding notion of realizability by $a \Vdash^{\wp}_{\mathfrak{wt}} B$.

Corollary

(i) $\mathbf{CZF}_E \vdash (\Vdash_{\mathfrak{wt}} B) \to B$.

(ii) $\mathbf{CZF} + \mathbf{Pow} \vdash (\Vdash^{\wp}_{\mathfrak{wt}} B) \to B$.

A variant of **wEP** uniform in parameters, **uwEP**, is the following: if

$$T \vdash \forall u \, \exists x A(u, x)$$

holds for a formula $A(u, x)$ having at most the free variables u, x, then there is a formula $C(u, x)$ with exactly u, x free, so that

$$\begin{aligned} T &\vdash \forall u \, \exists ! x \, C(u, x), \\ T &\vdash \forall u \, \forall x \, [C(u, x) \rightarrow \exists z \, z \in x], \\ T &\vdash \forall u \, \forall x \, [C(u, x) \rightarrow \forall z \in x \, A(u, z)]. \end{aligned}$$

Theorem $\mathbf{CZF}_E$ and $\mathbf{CZF} + \mathbf{Pow}$ both have the weak existence property. Indeed, they both satisfy the stronger property **uwEP**.

Even better

- **THEOREM** If
$$\mathbf{CZF}_E \vdash \exists x\, A(x,u)$$
then one can effectively construct a Σ^E formula $C(y,u)$ such that
$$\mathbf{CZF}_E \vdash \exists! y\; C(y,u)$$
$$\mathbf{CZF}_E \vdash \forall y [\, C(y,u) \to \exists x\; x \in y]$$
$$\mathbf{CZF}_E \vdash \forall y\, [C(y,u) \to \forall x \in y\; A(x,u)]$$

Even better

- **THEOREM** If
$$\mathbf{CZF} + \mathbf{Pow} \vdash \exists x\, A(x, u)$$
then one can effectively construct a Σ^P formula $C(y, u)$ such that
$$\mathbf{CZF} + \mathbf{Pow} \vdash \exists! y\, C(y, u)$$
$$\mathbf{CZF} + \mathbf{Pow} \vdash \forall y [\, C(y, u) \rightarrow \exists x\; x \in y]$$
$$\mathbf{CZF} + \mathbf{Pow} \vdash \forall y\, [C(y, u) \rightarrow \forall x \in y\; A(x, u)]$$

Conservativity

THEOREM
$\mathbf{CZF}_E$ is conservative over $\mathbf{IKP}(\mathcal{E})$ for Σ^E formulae.

THEOREM
$\mathbf{CZF} + \mathbf{Pow}$ is conservative over $\mathbf{IKP}(\mathcal{P})$ for Σ^P formulae.

Towards **EP**

As a result of the preceding theorems, to establish **EP** for $\mathbf{CZF}_E$ and $\mathbf{CZF} + \mathbf{Pow}$ it suffices to do this for $\mathbf{IKP}(\mathcal{E})$ and $\mathbf{IKP}(\mathcal{P})$ for the special cases of provable Σ^E and Σ^P formulae, respectively.

This is were **ordinal analysis** enters the stage.

A sequent calculus formulation of $\mathbf{IKP}(\mathcal{E})$

- The formulas of $\mathbf{IKP}(\mathcal{E})$ are the same as those of **IKP** except we also allow *exponentiation bounded quantifiers* of the form

$$(\forall x \in {}^{a}b)A(x) \quad \text{and} \quad (\exists x \in {}^{a}b)A(x).$$

- These are treated as quantifiers in their own right, not abbreviations. Quantifiers $\forall x$, $\exists x$ will be referred to as unbounded, whereas the other quantifiers (including the exponentiation bounded ones) will be referred to as bounded.
- A $\Delta_0^{\mathcal{E}}$-formula of $\mathbf{IKP}(\mathcal{E})$ is one that contains no unbounded quantifiers.
- The system $\mathbf{IKP}(\mathcal{E})$ derives intuitionistic sequents of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of formulae and Δ contains at most one formula.
- The formula "$\mathrm{fun}(x, a, b)$" means "x is a function from a to b".

The axioms of $\mathbf{IKP}(\mathcal{E})$

Logical axioms:	$\Gamma, A, \Rightarrow A$ for every $\Delta_0^{\mathcal{E}}$–formula A.
Extensionality:	$\Gamma \Rightarrow a = b \wedge B(a) \to B(b)$ for every $\Delta_0^{\mathcal{E}}$-formula $B(a)$.
Pair:	$\Gamma \Rightarrow \exists x[a \in x \wedge b \in x]$
Union:	$\Gamma \Rightarrow \exists x(\forall y \in a)(\forall z \in y)(z \in x)$
Infinity:	$\Gamma \Rightarrow \exists x\,[(\exists y \in x)\, y \in x \;\wedge\; (\forall y \in x)(\exists z \in x)\, y \in z]$.
$\Delta_0^{\mathcal{E}}$ *–Separation:*	$\Gamma \Rightarrow \exists x\; x = \{y \in a \mid A(y)\}$ for every $\Delta_0^{\mathcal{E}}$ formula $A(b)$.
$\Delta_0^{\mathcal{E}}$ *–Collection:*	$\Gamma \Rightarrow (\forall x \in a)\exists y B(x, y) \to \exists z(\forall x \in a)(\exists y \in z)B(x, y)$ for every $\Delta_0^{\mathcal{E}}$ formula $B(b, c)$.
Set Induction:	$\Gamma \Rightarrow \forall u\,[(\forall x \in u)\, G(x) \;\to\; G(u)] \to \forall u\, G(u)$ for every formula $G(b)$.
Exponentiation:	$\Gamma \Rightarrow \exists z\,(\forall x \in {}^{a}b)(x \in z)$.

Rules of $\mathbf{IKP}(\mathcal{E})$

$$(\mathcal{E}b\exists L)\ \frac{\Gamma, \mathrm{fun}(c,a,b) \wedge F(c) \Rightarrow \Delta}{\Gamma, (\exists x \in {}^{a}b)F(x) \Rightarrow \Delta} \qquad (\mathcal{E}b\exists R)\ \frac{\Gamma \Rightarrow \mathrm{fun}(c,a,b) \wedge F(c)}{\Gamma \Rightarrow (\exists x \in {}^{a}b)F(x)}$$

$$(\mathcal{E}b\forall L)\ \frac{\Gamma, \mathrm{fun}(c,a,b) \rightarrow F(c) \Rightarrow \Delta}{\Gamma, (\forall x \in {}^{a}b)F(x) \Rightarrow \Delta} \qquad (\mathcal{E}b\forall R)\ \frac{\Gamma \Rightarrow \mathrm{fun}(c,a,b) \rightarrow F(c)}{\Gamma \Rightarrow (\forall x \in {}^{a}b)F(x)}$$

This final section provides a relativised ordinal analysis for intuitionistic exponentiation Kripke-Platek set theory $\mathbf{IKP}(\mathcal{E})$. Given sets *a* and *b*, set-exponentiation allows the formation of the set ${}^{a}b$, of all functions from *a* to *b*. A problem that presents itself in this case is that it is not clear how to formulate a term structure in such a way that we can read off a term's level in the pertinent 'exponentiation hierarchy' from that term's syntactic structure. Instead we work with a term structure similar to that used in $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$, and a term's level becomes a **dynamic property** *inside* the infinitary system. Making this work in a system for which we can prove all the necessary embedding and cut-elimination theorems turned out to be a major technical hurdle. The end result of the section is a characterisation of $\mathbf{IKP}(\mathcal{E})$ in terms of provable height of the exponentiation hierarchy.

An Exponentiation-hierarchy

$$
\begin{aligned}
E_0 &:= \emptyset \\
E_1 &:= \text{some transitive set} \\
E_{\alpha+2} &:= \{X \mid X \text{ is definable over } \langle E_{\alpha+1}, \in\rangle \text{ with parameters}\} \\
&\quad \cup \{f \mid \mathrm{fun}(f, a, b) \text{ for some } a, b \in E_\alpha.\} \\
E_\lambda &:= \bigcup_{\beta<\lambda} E_\beta \quad \text{for } \lambda \text{ a limit ordinal.} \\
E_{\lambda+1} &:= \{X \mid X \text{ is definable over } \langle E_{\alpha+1}, \in\rangle \text{ with parameters}\}
\end{aligned}
$$

for λ a limit ordinal.

Terms

The terms of $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$ are defined as follows

1. $\mathbb{E}_{\alpha}$ is an $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$ term for each $\alpha < \Omega$.
2. a_i^{α} is an $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$ term for each $\alpha < \Omega$ and each $i < \omega$, these terms will be known as free variables.
3. If $F(a, \bar{b})$ is a $\Delta_0^{\mathcal{E}}$ formula of $\mathbf{IKP}(\mathcal{E})$ containing exactly the free variables indicated, and $t, \bar{s} := s_1, ..., s_n$ are $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$ terms then
$$[x \in t \mid F(x, \bar{s})]$$
is also a term of $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$.

Observe that $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$ terms do not come with 'levels' as in the other infinitary systems. This is because it is not clear how to immediately read off the location of a given term within the E hierarchy, just from the syntactic information available within that term.

Operator Controlled Derivability in $\mathbf{IRS}_\Omega^{\mathbb{E}}$

$\mathbf{IRS}_\Omega^{\mathbb{E}}$ derives intuitionistic sequents of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of $\mathbf{IRS}_\Omega^{\mathbb{E}}$ formulae and Δ contains at most one formula. For $\mathcal{H}$ an operator and α, ρ ordinals we define the relation $\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow \Delta$ by recursion on α.

If $\Gamma \Rightarrow \Delta$ is an axiom and $\alpha \in \mathcal{H}$ then $\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow \Delta$.

It is always required that $\alpha \in \mathcal{H}$, this requirement is not repeated for each inference rule below.

$$(\mathbb{E}\text{-Lim})_\infty \quad \frac{\mathcal{H}[\delta] \stackrel{\alpha_\delta}{\underset{\rho}{\vdash}} \Gamma, s \in \mathbb{E}_\delta \Rightarrow \Delta \text{ for all } \delta < \gamma}{\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\vdash}} \Gamma, s \in \mathbb{E}_\gamma \Rightarrow \Delta} \qquad \begin{array}{r} \gamma \text{ a limit} \\ \alpha_\delta < \alpha \\ \gamma \in \mathcal{H} \end{array}$$

$$(b\forall L) \quad \frac{\begin{array}{l} \mathcal{H} \stackrel{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, s \in t \to A(s) \Rightarrow \Delta \\ \mathcal{H} \stackrel{\alpha_1}{\underset{\rho}{\vdash}} \Gamma \Rightarrow t \in \mathbb{E}_\beta \\ \mathcal{H} \stackrel{\alpha_2}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \in \mathbb{E}_\gamma \end{array}}{\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\vdash}} \Gamma, (\forall x \in t)A(x) \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0, \alpha_1, \alpha_2 < \alpha \\ \beta, \gamma \in \mathcal{H} \\ \gamma < \alpha \\ \gamma \leq \beta \end{array}$$

$$(b\forall R)_\infty \quad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \in t \to F(s) \text{ all } s \qquad \mathcal{H} \overset{\alpha_1}{\underset{\rho}{\vdash}} \Gamma \Rightarrow t \in \mathbb{E}_\beta}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow (\forall x{\in}t)F(x)} \quad \begin{array}{r} \alpha_0, \alpha_1 < \alpha \\ \beta \in \mathcal{H} \\ \beta < \alpha \end{array}$$

$$(b\exists L)_\infty \quad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, s \in t \wedge F(s) \Rightarrow \Delta \text{ all } s \qquad \mathcal{H} \overset{\alpha_1}{\underset{\rho}{\vdash}} \Gamma \Rightarrow t \in \mathbb{E}_\beta}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, (\exists x{\in}t)F(x) \Rightarrow \Delta} \quad \begin{array}{r} \alpha_0, \alpha_1 < \alpha \\ \beta \in \mathcal{H} \\ \beta < \alpha \end{array}$$

$$
(b\exists R) \quad \frac{\mathcal{H} \stackrel{\alpha_0}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \in t \wedge A(s) \qquad \mathcal{H} \stackrel{\alpha_1}{\underset{\rho}{\vdash}} \Gamma \Rightarrow t \in \mathbb{E}_\beta \qquad \mathcal{H} \stackrel{\alpha_2}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \in \mathbb{E}_\gamma}{\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow (\exists x \in t)A(x)} \qquad \begin{array}{r} \alpha_0, \alpha_1, \alpha_2 < \alpha \\ \beta, \gamma \in \mathcal{H} \\ \gamma < \alpha \\ \gamma \leq \beta \end{array}
$$

$$(\mathcal{E}b\forall L)\qquad \frac{\begin{array}{l}\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\mid\!\!-}} \Gamma, \mathsf{fun}(p,s,t) \to A(p) \Rightarrow \Delta \\ \mathcal{H} \overset{\alpha_1}{\underset{\rho}{\mid\!\!-}} \Gamma \Rightarrow s \in \mathbb{E}_\beta \\ \mathcal{H} \overset{\alpha_2}{\underset{\rho}{\mid\!\!-}} \Gamma \Rightarrow t \in \mathbb{E}_\gamma \\ \mathcal{H} \overset{\alpha_3}{\underset{\rho}{\mid\!\!-}} \Gamma \Rightarrow p \in \mathbb{E}_\delta\end{array}}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid\!\!-}} \Gamma, (\forall x \in {}^s t) A(x) \Rightarrow \Delta} \qquad \begin{array}{r}\alpha_0, \alpha_1, \alpha_2, \alpha_3 < \alpha \\ \beta, \gamma, \delta \in \mathcal{H} \\ \delta < \alpha \\ \delta \leq \max(\beta, \gamma) + 2\end{array}$$

$$(\mathcal{E}b\forall R)_\infty\qquad \frac{\begin{array}{l}\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\mid\!\!-}} \Gamma \Rightarrow \mathsf{fun}(p,s,t) \to F(p) \text{ all } p \\ \mathcal{H} \overset{\alpha_1}{\underset{\rho}{\mid\!\!-}} \Gamma \Rightarrow s \in \mathbb{E}_\beta \\ \mathcal{H} \overset{\alpha_2}{\underset{\rho}{\mid\!\!-}} \Gamma \Rightarrow t \in \mathbb{E}_\gamma\end{array}}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid\!\!-}} \Gamma \Rightarrow (\forall x \in {}^s t) F(x)} \qquad \begin{array}{r}\alpha_0, \alpha_1, \alpha_2 < \alpha \\ \beta, \gamma \in \mathcal{H} \\ \max(\beta, \gamma) + 2 \leq \alpha\end{array}$$

$$(\mathcal{E}b\exists L)_\infty \quad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, \mathsf{fun}(p,s,t) \wedge F(p) \Rightarrow \Delta \text{ all } p \qquad \mathcal{H} \overset{\alpha_1}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \in \mathbb{E}_\beta \qquad \mathcal{H} \overset{\alpha_2}{\underset{\rho}{\vdash}} \Gamma \Rightarrow t \in \mathbb{E}_\gamma}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, (\exists x \in {}^s t) F(x) \Rightarrow \Delta}$$

$\alpha_0, \alpha_1, \alpha_2 < \alpha$
$\beta, \gamma \in \mathcal{H}$
$\max(\beta, \gamma) + 2 \leq \alpha$

$$(\mathcal{E}b\exists R) \quad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma \Rightarrow \mathsf{fun}(p,s,t) \wedge A(p) \qquad \mathcal{H} \overset{\alpha_1}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \in \mathbb{E}_\beta \qquad \mathcal{H} \overset{\alpha_2}{\underset{\rho}{\vdash}} \Gamma \Rightarrow t \in \mathbb{E}_\gamma \qquad \mathcal{H} \overset{\alpha_3}{\underset{\rho}{\vdash}} \Gamma \Rightarrow p \in \mathbb{E}_\delta}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow (\exists x \in {}^s t) A(x)}$$

$\alpha_0, \alpha_1, \alpha_2, \alpha_3 < \alpha$
$\beta, \gamma, \delta \in \mathcal{H}$
$\delta < \alpha$
$\delta \leq \max(\beta, \gamma) + 2$

$$(\forall L)\quad \frac{\mathcal{H} \mid\!\frac{\alpha_0}{\rho}\, \Gamma, F(s) \Rightarrow \Delta \qquad \mathcal{H} \mid\!\frac{\alpha_1}{\rho}\, \Gamma \Rightarrow s \in \mathbb{E}_\beta}{\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Gamma, \forall x F(x) \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0 + 3, \alpha_1 + 3 < \alpha \\ \beta < \alpha \\ \beta \in \mathcal{H} \end{array}$$

$$(\forall R)_\infty \quad \frac{\mathcal{H}[\beta] \mid\!\frac{\alpha_\beta}{\rho}\, \Gamma, s \in \mathbb{E}_\beta \Rightarrow F(s) \text{ all } s, \beta < \Omega}{\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow \forall x F(x)} \qquad \beta < \alpha_\beta + 3 < \alpha$$

$$(\exists L)_\infty \quad \frac{\mathcal{H}[\beta] \overset{\alpha_\beta}{\underset{\rho}{\vdash}} \Gamma, s \in \mathbb{E}_\beta, F(s) \Rightarrow \Delta \text{ all } s, \beta < \Omega}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow \forall x F(x)} \qquad \beta < \alpha_\beta + 3 < \alpha$$

$$(\exists R) \quad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma \Rightarrow F(s) \qquad \mathcal{H} \overset{\alpha_1}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \in \mathbb{E}_\beta}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow \exists x F(x)} \qquad \begin{array}{r} \alpha_0 + 3, \alpha_1 + 3 < \alpha \\ \beta < \alpha \\ \beta \in \mathcal{H} \end{array}$$

(Reflection)
$$\frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma \Rightarrow A}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow \exists z\, A^z} \qquad \begin{array}{r} \alpha_0 + 1, \Omega < \alpha \\ A \text{ is a } \Sigma^{\mathcal{E}}\text{-formula} \end{array}$$

(Cut)
$$\frac{\begin{array}{l} \mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, A(s_1, ..., s_n) \Rightarrow \Delta \\ \mathcal{H} \overset{\alpha_1}{\underset{\rho}{\vdash}} \Gamma \Rightarrow A(s_1, ..., s_n) \\ \mathcal{H} \overset{\alpha_2}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s_i \in \mathbb{E}_{\beta_i} \quad i = 1, ..., n \end{array}}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0, \alpha_1, \alpha_2 < \alpha \\ \|A(\bar{s})\|_{\bar{\beta}} < \rho \\ \bar{\beta} \in \mathcal{H} \end{array}$$

Cut-Elimination and Collapsing

Theorem 1. *Suppose* $\mathbf{IKP}(\mathcal{E}) \vdash \Rightarrow A$ *for some* $\Sigma^{\mathcal{E}}$ *formula* A*, then there exists an* $n < \omega$*, which we may compute from the derivation, such that*

$$\mathcal{H}_\sigma \;\overset{\psi_\Omega \sigma}{\underset{\psi_\Omega \sigma}{\vdash}} \;\Rightarrow\; A \quad \textit{where } \sigma := \omega_m(\Omega \cdot \omega^m).$$

Theorem 2. *If* A *is a* $\Sigma^{\mathcal{E}}$*-sentence and* $\mathbf{IKP}(\mathcal{E}) \vdash \Rightarrow A$ *then there is an ordinal term* $\alpha < \psi_\Omega \varepsilon_{\Omega+1}$*, which we may compute from the derivation, such that*

$$E_\alpha \models A.$$

Finally

use **Kleene's slash method** to read of a term that witnesses an existential $\Sigma^{\mathcal{E}}$ theorem.

Besten Dank!

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Besten Dank!