Borel sets in effective descriptive set theory

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Ghent-Leeds online seminar
10 September 2020

Acknowledgements

- This is an ongoing project with Philip Welch (University of Bristol) and Merlin Carl (Europa-Universität Flensburg).
- This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 794020 (IMIC).

Problem 1: Borel ranks of definable sets

Effective descriptive set theory studies simply definable subsets of the Baire space ω^ω.

- A Σ^1_1 set is a projection of a closed subset $[T]$ of $\omega^\omega \times \omega^\omega$, where T is a computable tree. Equivalently, it is definable by a Σ^1_1-formula

$$\exists y \; \varphi(x, y),$$

  where φ is Σ_0.
- A Π^1_1 set is a complement of a Σ^1_1 set.
- A Σ^1_2 set is a projection of a Π^1_1 set, etc.

Where in the Borel hierarchy do these sets appear?

Δ^1_1 sets

An ordinal is called computable if it is coded by a computable real. ω_1^{ck} is the supremum of computable ordinals.

Fact

The supremum of Borel ranks of Δ^1_1 sets is ω_1^{ck}.

This uses an effective version of Lusin's separation theorem: Any two disjoint Σ^1_1 sets are separated by a hyperarithmetic set, i.e. a Borel set with a computable code.

$L_{\omega_1^{ck}}$ is the least admissible set. An *admissible set* is a transitive model of KP: Axioms of set theory with only Σ_1-collection and Δ_0-separation. In particular, Σ_1-recursion is allowed.

Theorem (Louveau 1980 "Louveau Separation")

Given a Δ^1_1 set that is also $\mathbf{\Sigma}^0_\alpha$, there is a $\mathbf{\Sigma}^0_\alpha$-code in $L_{\omega_1^{ck}}$.

Thus KP suffices to calculate Borel ranks of Δ^1_1 sets.

Σ^1_1 Borel sets

A Σ^1_1 set is either

- Truly Σ^1_1 (i.e. not Borel), or
- Borel.

Assuming Σ^1_1-determinacy, all truly Σ^1_1 sets are Wadge equivalent. It remains to understand Σ^1_1 Borel sets.

What is the supremum of Borel ranks of Σ^1_1 Borel sets?

This was calculated by Kechris, Marker and Sami (1989). We simplified the result. Let τ denote the supremum of ordinals Π_1-definable over $L_{\omega_1^V}$.

Proposition (Welch, Carl, S.)

The supremum equals τ.

Thus we need witnesses to Σ_2-statements in $L_{\omega_1^V}$ to calculate ranks of Σ^1_1 Borel sets.

Δ^1_2 Borel sets

A Borel code is a subset of ω that codes a tree which describes the way the Borel set is built up from basic open sets.

An ∞-Borel set is defined by allowing wellordered unions and intersections. An ∞-Borel code is a set of ordinals coding a tree which describes the way the ∞-Borel set is built up from basic open sets.

Do all Δ^1_2 Borel sets have ∞-Borel codes in $L_{\omega_1^V}$?

A set is absolutely Δ^1_2 if it has a uniform Δ^1_2-definition in generic extensions.

Proposition (Welch, Carl, S.)

Suppose that either

a. ω_1^V is inaccessible in L, or

b. V is a generic extension of L by proper forcing.

Then any absolutely Δ^1_2 Borel set has an ∞-Borel code of the same rank in L_τ.

There is no such result for Σ^1_2, since Π^1_2 singletons can exist outside of L.

Δ^1_2 Borel sets

Proposition (Welch, Carl, S.)

Under additional assumptions, any absolutely Δ^1_2 Borel set has an ∞-Borel code of the same rank in L_τ.

We ultimately aim to obtain this result in ZFC. This would simultaneously generalise:

- The above result of Kechris, Marker and Sami
- The Mansfield-Solovay theorem: Countable Δ^1_2 sets are contained in L
- Stern's theorem on Δ^1_2 Borel sets that corresponds to the first case.
- Shoenfield absoluteness

Problem 2: The length of ranks

Fix a class of sets such as Π^1_1 or Σ^1_2. A rank in this class is an abstraction of the quasiordering given by the halting times of infinite computations. The essential property is that rank comparison is both Π^1_1 and Σ^1_1 (for Π^1_1-ranks).

For instance, any Π^1_1-set can be written in a canonical way as an increasing union of Borel subsets, inducing a rank.

Example

Let WO denote the Π^1_1 set of wellorders on ω.
Let $\mathrm{WO}_{\leq\alpha}$ denote the Borel subset of wellorders of order type $\leq\alpha$.

Ranks often arise from transfinite iterations of derivation processes such as the Cantor-Bendixson derivative.

Theorem (Welch, Carl, S.)

The supremum of lengths of countable ranks in the following classes equals τ:

a. Π^1_1-*ranks*
b. Σ^1_2-*ranks*

Σ^1_2-ranks

Fact

A Π^1_1 set is Borel if and only if it admits a countable Π^1_1-rank.

This holds by the boundedness theorem for Π^1_1-ranks.

What does it mean for a Σ^1_2-set to admit a countable rank?

Theorem (Welch, Carl, S.)

The following conditions are equivalent for any Π^1_2-singleton x:

a. $x \in L$.

b. *x is covered by a countable Δ^1_2-set.*

c. *The complement of $\{x\}$ admits a countable Σ^1_2-rank.*

Decision times

Hamkins' and Kidder's infinite time Turing machine (ittm) is a Turing machine that may run for ordinal time via a limit rule.

A set A of reals is called ittm-semidecidable by a program p if

$$A = \{x \mid p(x)\downarrow\}.$$

The decision time of an ittm-program is the supremum of (transfinite) halting times over all real inputs.

What is the supremum of countable decision times?

Theorem (Welch, Carl, S.)

1. *The supremum of countable decision times of ittm-decidable sets equals σ.*
2. *The supremum of countable decision times of ittm-semidecidable sets equals τ.*

Any ittm-semidecidable set with countable decision time is Borel:

$$\text{Decision time } \leq \omega \cdot \alpha \Longrightarrow \text{ Borel rank } \lessapprox \alpha + 1$$

Borel ranks

σ and τ

Definition

Let σ (τ) denote the supremum of ordinals Σ_1-definable (Σ_2-definable) in $L_{\omega_1^V}$.

Fact

1. *σ is least with $L_\sigma \prec_{\Sigma_1} L$.*
2. *σ equals δ^1_2, the supremum lengths of Δ^1_2-wellorders on ω.*

Lemma (Welch, Carl, S.)

τ equals the supremum of ordinals Π_1-definable in $L_{\omega_1^V}$.

τ

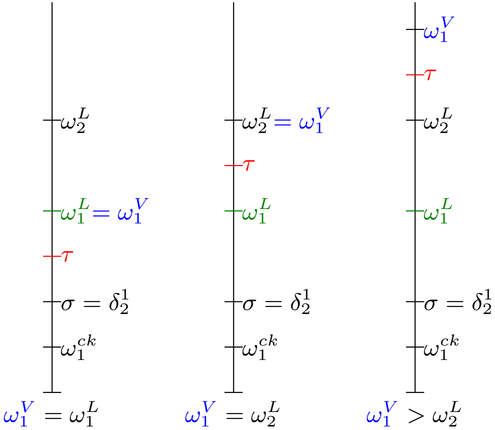
Let τ_* be least such that L_{τ_*} and $L_{\omega_1^V}$ agree on Σ_2-truth. Let τ^* be least with $L_{\tau^*} \prec_{\Sigma_2} L_{\omega_1^V}$.

Then $\tau_* \leq \tau \leq \tau^*$.

Lemma (Welch, Carl, S.)

1. *If* $\omega_1^L = \omega_1^V$*, then* $\tau_* = \tau = \tau^*$.
2. *If* $\omega_1^L < \omega_1^V$*, then* $\tau_* < \omega_1^L < \tau < \tau^*$.

τ



ω_2^L
$\omega_1^L = \omega_1^V$
τ
$\sigma = \delta_2^1$
ω_1^{ck}
$\omega_1^V = \omega_1^L$
$\omega_2^L = \omega_1^V$
τ
ω_1^L
$\sigma = \delta_2^1$
ω_1^{ck}
$\omega_1^V = \omega_2^L$
ω_1^V
τ
ω_2^L
ω_1^L
$\sigma = \delta_2^1$
ω_1^{ck}
$\omega_1^V > \omega_2^L$

The lower bound

Lemma (Kechris, Marker, Sami)

For any $\alpha < \tau$, there is a Π^1_1 Borel set A of Borel rank at least α.

Proof.

Let α_x denote the order type of $x \in \mathrm{WO}$.

Suppose that $\delta > \omega^\alpha$ is a Π_1-singleton defined by $\varphi(x)$. Let

$$A = \{(x, y) \in \mathrm{WO}^2 \mid \alpha_y \text{ is least with } L_{\alpha_y} \models \text{“}\varphi \text{ defines } \alpha_x\text{”}\} \in \Pi^1_1.$$

Let $\xi > \delta$ be least with $L_\xi \models$ “φ defines δ”. Note that for any $(x, y) \in A$, we have $\alpha_x \leq \delta$ and $\alpha_y \leq \xi$. Since for each ξ, the set WO_ξ of codes for ξ is Borel, A is a countable union of Borel sets and thus Borel.

For any code y of ξ, we obtain the slice WO_δ. But WO_δ has Borel rank at least α (Stern). □

The converse, i.e. Borel ranks are all below τ, uses the Π^1_1-boundedness theorem.

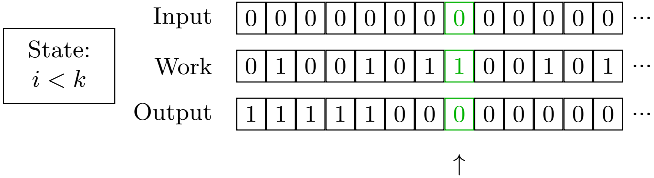
Similarly: There is a Σ^1_2 Borel set of Borel rank precisely τ.

Decision times

Ittm's

An infinite time Turing machine is a Turing machine with three tapes whose cells are indexed by natural numbers:

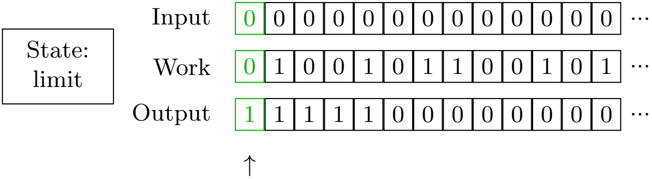
- The input tape
- The output tape
- The working tape



Ittm's

It behaves like a standard Turing machine at successor steps of a computation.
At limit steps of computation:

- The head goes back to the first cell.
- The machine goes into a“limit” state.
- The value of each cell equals the lim inf of the values at previous stages of computation.



Borel $\not\to$ decidable in countable time

Proposition (Welch, Carl, S.)

There is an open ittm-decidable set A that is not ittm-semidecidable in countable time.

Proof.

Let $\vec{\varphi} = \langle \varphi_n \mid n \in \omega \rangle$ be a computable enumeration of all Σ_1-sentences.

Let B denote the set consisting of 0^∞ and all $0^n{}^\frown 1^\frown x$, where x is the L-least code for the least L_α where φ_n holds. B is a countable closed set.

Let p denote an algorithm that semidecides B as follows: test if the input is of the form $0^n{}^\frown 1^\frown x$, run a wellfoundedness test for x (which takes at least α steps for codes for L_α), and then test whether α is least such that φ_n holds in L_α.

Thus p's decision time is at least σ. It is countable since B is countable.

□

Borel $\not\rightarrow$ decidable in countable time

Proposition (Welch, Carl, S.)

There is an open ittm-decidable set A that is not ittm-semidecidable in countable time.

Proof, continued.

Let A denote the complement of B. Towards a contradiction, suppose that A is semidecidable in countable time by an ittm-program q.

Let r be the decision algorithm for B that runs p and q simultaneously. Then r has a countable decision time α and by Σ^1_2-reflection, we have $\alpha < \sigma$. But this is clearly false, since p's decision time is at least σ.

□

Decision times for singletons

Theorem (Welch, Carl, S.)

The suprema of decision times for the following sets equal σ:

1. *Singletons*
2. *Complements of singletons.*

Some open problems

- In ZFC, does every Δ^1_2 Borel set have a Borel code in L?
- Is a countable Π^1_2 set contained in L if and only if its complements admits a countable Σ^1_2-rank?

References

- Kechris, Marker, Sami,
  Π^1_1 Borel sets,
  J. Symb. Log. 54 (1989), no. 3, 915–920.
- Stern,
  On Lusin's restricted continuum problem,
  Annals Math. 120 (1984), no. 1, 7–37.
- Louveau,
  A separation theorem for analytic sets,
  Trans. Americ. Math. Soc. 260 (1980), no. 2, 363–378
- Carl, Schlicht, Welch,
  Preprint on Borel sets in effective descriptive set theory,
  In preparation