Weak and Strong Versions of Effective Transfinite Recursion

Leeds-Ghent Logic Seminar

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March 16, 2022

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Recursively defined family

Given a well-order X and a formula $\varphi(n, x, Z)$, we define

$$H_\varphi(X, Y) :\Longleftrightarrow Y = \{(x, n) \in X \times \mathbb{N} \mid \varphi(n, x, Y^x)\},$$

where $Y^x := \{(y, n) \in Y \mid y <_X x\}$.

Intuition: φ computes $Y_x := \{n \mid (x, n) \in Y\}$ using Y_y for $y <_X x$.

Arithmetical Transfinite Recursion (ATR)

For any well-order X and arithmetical $\varphi(n, x, Z)$, there exists a set Y with $H_\varphi(X, Y)$.

Question: Can we apply recursion in weaker systems?

Arithmetical comprehension along a well-order	$\widehat{=}$	ATR
Δ^0_1-comprehension along a well-order	$\widehat{=}$	ETR

Effective Transfinite Recursion originates from recursion theory (Church, Kleene, Rogers).

Weak Effective Transfinite Recursion (WETR**)**
For any well-order X and Σ^0_1-formulas $\varphi(n, x, Z)$ and $\psi(n, x, Z)$ with

$$\forall n \in \mathbb{N}\ \forall x \in X\ \forall Z \subseteq \mathbb{N}\ \big(\varphi(n, x, Z) \leftrightarrow \neg\psi(n, x, Z)\big),$$

there exists a set Y with $H_\varphi(X, Y)$.

Proposition (Dzhafarov, Flood, Solomon, Westrick 2017)
ACA_0 *proves* WETR.

Weak Effective Transfinite Recursion is too restrictive.

Strong Effective Transfinite Recursion (SETR**)**
For any well-order X and Σ^0_1-formulas $\varphi(n, x, Z)$ and $\psi(n, x, Z)$ with

$$H_\varphi(X \restriction x, Z) \to \forall n \in \mathbb{N} \left(\varphi(n, x, Z) \leftrightarrow \neg\psi(n, x, Z)\right)$$

for all $x \in X$ and $Z \subseteq \mathbb{N}$, there exists a set Y with $H_\varphi(X, Y)$.

Proposition (Freund 2021)
ACA_0 *proves* SETR*.*

Introduction: Comparison of Recursion principles

Principle	$\varphi(n, x, Z)$	**Strength**
ATR	arithmetical	ATR_0
SETR	Δ^0_1 if $H_\varphi(X \restriction x, Z)$	$\leq \mathsf{ACA}_0$
WETR	Δ^0_1	$\leq \mathsf{ACA}_0$

Questions

- Is WETR properly weaker than SETR?
- What are the precise strengths of SETR and WETR?
- Can we say something about SETR_X and WETR_X for fixed well-orders X?

Strong
Effective Transfinite Recursion

SETR: Main results

Theorem (RCA_0)
For any well-order X, the following are equivalent:

- SETR_X
- Π^0_2*-induction along* X
- $\alpha \mapsto \alpha^X$ *preserves well-orders*

Corollary (RCA_0)
SETR *is equivalent to* ACA_0.

SETR: Recursion implies Induction

$$\mathsf{SETR}_X \quad \Longrightarrow \quad \Pi^0_2\text{-induction along } X$$

Progressive $Q(x) := \forall n\, \exists m\, P(x,n,m)$ for a Δ^0_0-formula $P(x,n,m)$

$$\varphi(n,x,Z) := \exists m\, P(x,n,m)$$
$$\neg\psi(n,x,Z) := ?$$

Given Y with $H_\varphi(X,Y)$, we have $(x,n) \in Y \longleftrightarrow \exists m\, P(x,n,m)$.

SETR: Recursion implies Induction

$$\mathsf{SETR}_X \quad \Longrightarrow \quad \Pi^0_2\text{-induction along } X$$

Progressive $Q(x) := \forall n\ \exists m\ P(x,n,m)$ for a Δ^0_0-formula $P(x,n,m)$

$$\varphi(n,x,Z) := \exists m\ (P(x,n,m) \land \forall (y,n') \leq_{\mathbb{N}} m\ (y <_X x \to (y,n') \in Z))$$
$$\neg\psi(n,x,Z) := \forall y <_X x\ \forall n'\ (\forall m <_{\mathbb{N}} (y,n')\ \neg P(x,n,m) \to (y,n') \in Z)$$

Show that $H_\varphi(X \restriction x, Z)$ implies $\varphi(n,x,Z) \leftrightarrow \neg\psi(n,x,Z)$.

Given Y with $H_\varphi(X,Y)$ and $(y,l) \in Y$ for all $y <_X x$ and $l \in \mathbb{N}$,
we have $(x,n) \in Y \longleftrightarrow \exists m\ P(x,n,m)$.

SETR: Induction implies Well-ordering principle

Π^0_2-induction along $X \implies \alpha \mapsto \alpha^X$ preserves well-orders

- Assume infinite descending sequence $(f_i)_{i\in\mathbb{N}}$ in α^X (and that X has a top element $\top$).
- For any $x \in X$, restrict f_i to $\alpha^{X_{\geq x}}$ with $X_{\geq x} := \{x' \in X | x' \geq x\}$.
- Show (using Π^0_2-induction) that the resulting sequences descend infinitely often.
- Consider the restriction to $\alpha^{X_{\geq \top}} \equiv \alpha$.

SETR: Well-ordering principle implies Recursion (1/3)

$$\alpha \mapsto \alpha^X \text{ preserves well-orders} \implies \mathsf{SETR}_X$$

Idea: Compute Y with $H_\varphi(X, Y)$ using term-evaluation

Terms:

- 0, 1, $P(n, x, s)$ for $n \in \mathbb{N}$, $x \in X$, and 0-1-sequences s that may have another term as last member

Transition rules:

- $P(n, x, s)$ for sequence s:
  - If $\varphi_0(n, x, s) \vee \psi_0(n, x, s)$, then $P(n, x, s) \mapsto b$ with $b = 1$ iff $\varphi_0(n, x, s)$.
  - Otherwise, $P(n, x, s) \mapsto P(n, x, s * \langle t \rangle)$ with $t = P(m, y, \langle\rangle)$ if $|s| = (m, y)$ for $y <_X x$, else $t = 0$.
- $P(n, x, s * \langle t \rangle)$ for sequence s and term $t \notin \{0, 1\}$:
  $P(n, x, s * \langle t \rangle) \mapsto P(n, x, s * \langle t' \rangle)$ for $t \mapsto t'$.

SETR: Well-ordering principle implies Recursion (2/3)

$$\alpha \mapsto \alpha^X \text{ preserves well-orders} \implies \mathsf{SETR}_X$$

Define $(n, x) \in Y$ iff $P(n, x, \langle\rangle) \mapsto 1$

Use the preserving $\alpha \mapsto \alpha^X$ to show that this is decidable: Define $T := \{0_T, 1_T, (n, x, t, b) \mid n \in \mathbb{N}, x \in X, t \in T(n, x), b \in 2\}$.

Map each term t to an element $\beta(t)$ in T^X:

- $t \in \{0, 1\}$, then $\beta(t) = 0_{T^X}$.
- $t = P(n, x, s)$ for sequence s:
  - If $\varphi_0(n, x, s) \vee \psi_0(n, x, s)$, then $\beta(t) := 1_{T^X}$.
  - Otherwise, $\beta(t) := T^x \cdot (n, x, s, 1)$.
- $t = P(n, x, s * \langle t' \rangle)$ for sequence s and term $t' \notin \{0, 1\}$: $\beta(t) := T^x \cdot (n, x, s, 0) + \beta(t')$.

Evaluation of $P(n, x, \langle\rangle)$ results in descending sequence in T^X.

SETR: Well-ordering principle implies Recursion (3/3)

$$\alpha \mapsto \alpha^X \text{ preserves well-orders} \implies \mathsf{SETR}_X$$

Tiny caveat: T is far from a well-order:

- Not linear because of $T(n, x)$
  (can be solved immediately using Kleene-Brouwer-order)
- Not well-founded if $\varphi(n, x, Z) \vee \psi(n, x, Z)$ does not hold for all Z

$\Longrightarrow$ descending sequence in T^X not an (immediate) contradiction

Solution:

- Let $(g_i)_{i \in \mathbb{N}}$ with $g_0 := P(n, x, \langle\rangle)$ and $g_i \mapsto g_{i+1}$.
- Define T' as restriction of T to terms occurring in $(\beta(g_i))_{i \in \mathbb{N}}$.
- Show that T' is well-order
- Contradiction via descending $(g_i)_{i \in \mathbb{N}}$ in T'^X

Weak
Effective Transfinite Recursion

WETR: Main results

Theorem (RCA_0)
For any well-order X, the following are equivalent:

- WETR_X
- *The disjunction of* WKL *and* Π^0_2*-induction along* X

Corollary (RCA_0)
WETR *is equivalent to* WKL_0.

WETR: WKL_0 implies Recursion (1/2)

$$\mathsf{WKL} \implies \mathsf{WETR}$$

Use WKL to define $t : \mathbb{N} \times X \to \mathbb{N}$ with

$$\varphi(n, x, Z) \longleftrightarrow \varphi_0(n, x, Z[t(n, x)]).$$

Compute $\varphi(n, x, Z)$ (for Z with $H_\varphi(X \restriction x, Z)$) using a program $e :\subseteq \mathbb{N} \times X \to \{0, 1\}$:

- Compute $e(m, y)$ for all $m \in \mathbb{N}$, $y <_X x$ with $(m, y) <_{\mathbb{N}} t(n, x)$.
- Define $s \in \{0, 1\}^*$ with $|s| = t(n, x)$ and $s_i = e(m, y)$ if (m, y) has code i (otherwise $s_i = 0$).
- Return 1 if and only if $\varphi_0(n, x, s)$ holds.

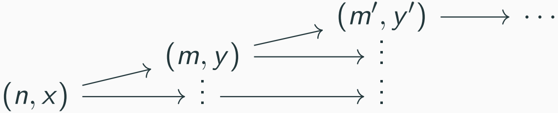
If e is total, then $H_\varphi(X, Y)$ holds for Y with

$$(n, x) \in Y :\longleftrightarrow e(n, x) = 1.$$

WETR: WKL_0 implies Recursion (2/2)

$$\mathsf{WKL} \implies \mathsf{WETR}$$

Prove that $e :\subseteq \mathbb{N} \times X \to \{0,1\}$ is total. Consider the tree:



Show that tree is finite:

- Using linear order: Define sequence $(x_i)_{i \in \mathbb{N}}$ s.t. x_i is the maximal element in X occurring on height i.
  Show $x_i >_X x_{i+1}$ for all $i \in \mathbb{N}$.
- Using WKL: Infinite path corresponds to sequence $(n_i, x_i)_{i \in \mathbb{N}}$ with $x_i >_X x_{i+1}$ for all $i \in \mathbb{N}$.

Starting from the leaves, compute $e(n, x)$ in finitely many steps.

WETR: RCA_0 admits Recursion Rule

$$\mathsf{RCA}_0 \vdash \varphi \text{ is } \Delta^0_1 \implies \mathsf{RCA}_0 \vdash \text{Recursion on } \varphi$$

Idea:

- Prove $t(n, x)\downarrow$ in WKL_0
- Convert this into a proof in RCA_0 using conservation theorem (Harrington)

Also: Conservation theorem holds for Π^1_1-sentences

$\implies$ Allow arbitrary Σ^1_1-formula as premise (on both sides)

WETR: Recursion implies WKL or Strong Recursion (1/3)

$$\mathsf{WETR}_X \ \wedge \neg\,\mathsf{WKL} \quad \Longrightarrow \quad \Pi^0_2\text{-induction along } X$$

Idea: Reuse the proof for "$\mathsf{SETR}_X \to \Pi^0_2$-induction along X"

Problem: We can only prove $\varphi(n, x, Z) \leftrightarrow \neg\psi(n, x, Z)$ using $H_\varphi(X \restriction x, Z)$.

Reason: $H_\varphi(X \restriction x, Z)$ and $(y, n) \in Z$ for $y < x$ imply existence of m with $P(y, n, m)$.

Solution: Code witness m for $P(y, n, m)$ directly into Z.

Requirement: (Continuous) mapping from $2^{\mathbb{N}}$ to $\mathbb{N}$.

WETR: Recursion implies WKL or Strong Recursion (2/3)

$$\mathsf{WETR}_X \wedge \neg\,\mathsf{WKL} \implies \Pi^0_2\text{-induction along } X$$

Requirement: (Continuous) mapping from $2^{\mathbb{N}}$ to $\mathbb{N}$.

Assume $\neg\,$WKL $\implies$ infinite tree T without path

Collect all lengths in L such that there exists sequence s with $(*)$

- $s \notin T$
- $s' \in T$ for all proper initial segments s' of s

Set L is infinite $\Rightarrow$ surjective $f : L \to \mathbb{N}$

Define:

$$Z \mapsto m$$

iff there is an $l \in L$ with $f(l) = m$ and $Z[l]$ satisfies $(*)$.

WETR: Recursion implies WKL or Strong Recursion (3/3)

$$\mathsf{WETR}_X \;\wedge \neg\,\mathsf{WKL} \quad\Longrightarrow\quad \Pi^0_2\text{-induction along } X$$

Final step: Adapt proof of $\mathsf{SETR}_X \Longrightarrow \Pi^0_2$-induction along X

Before:

$$\varphi(n, x, Z) := \; \exists m \, (P(x, n, m) \wedge \\ \forall (y, n') \leq_{\mathbb{N}} m \, (y <_X x \rightarrow (y, n') \in Z))$$

After:

$$\varphi((n, i), x, Z) := \exists m \, \big(P(x, n, m) \wedge seq(m)_i = 1 \wedge \\ \forall (y, n') \leq_{\mathbb{N}} m \, \big(y <_X x \rightarrow \exists m' \, \big(Z_{y,n'} \mapsto m' \wedge P(y, n', m')\big)\big)\big)$$

Rest of the proof: Analogous but a bit more complex.

Conclusion

Revisiting the table:

Principle	$\varphi(n, x, Z)$	**Strength**
ATR	arithmetical	ATR_0
SETR	Δ^0_1 if $H_\varphi(X \restriction x, Z)$	ACA_0
SETR_X		Π^0_2-induction along X
WETR	Δ^0_1	WKL_0 (rule version: RCA_0)
WETR_X		$\mathsf{WKL}_0 \vee \mathsf{SETR}_X$

Additional results:

- $\mathsf{RCA}_0 \vdash \Pi^0_2$-induction along X
  $\longleftrightarrow \alpha \mapsto \alpha^X$ preserves well-orders
- $\mathsf{RCA}_0 \nvdash \forall n \in \mathbb{N}\ \ \mathsf{WETR}_n$

Preprint: arXiv:2202.05611