Talk in Leeds

May 6^{th} 2021

Something special about + and ·

Gödel incompleteness for elementary school

(Part of joint work with Harvey Friedman)

Terms

$1 \in T$

$t, u \in T \Rightarrow t + u \in T$

$t, u \in T \Rightarrow t \cdot u \in T \quad$ if $t \neq 1 \neq u$

$t \in T \Rightarrow t^{\mathbb{N}} \in \mathbb{N}$

$(1+1)^{\mathbb{N}} = 2 \qquad ((1+1)\cdot(1+1))^{\mathbb{N}} = 4$

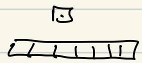
$s \leq t \Leftrightarrow s^{\mathbb{N}} \leq t^{\mathbb{N}}$

$1+1 \leq (1+1)\cdot(1+1) \qquad 2 \leq 4$

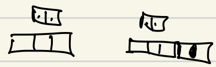
$t \trianglelefteq u \iff t \le u$ is obvious

2nd def.

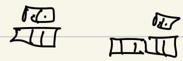
$1 \trianglelefteq t$



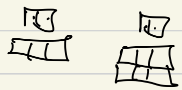
$s \trianglelefteq t \Rightarrow s \trianglelefteq t + u$



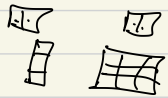
$s \trianglelefteq t \Rightarrow s \trianglelefteq u + t$



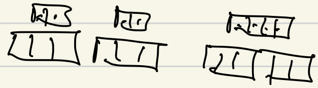
$s \trianglelefteq t \Rightarrow s \trianglelefteq t \cdot u$



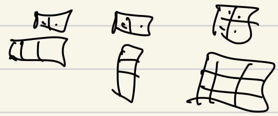
$s \trianglelefteq t \Rightarrow s \trianglelefteq u \cdot t$



$u \trianglelefteq u'$ and $t \trianglelefteq t' \Rightarrow u + t \trianglelefteq u' + t'$



$u \trianglelefteq u'$ and $t \trianglelefteq t' \Rightarrow u \cdot t \trianglelefteq u' \cdot t'$



Let $\preceq_T$ be a binary relation on T

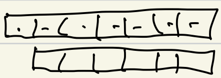
$$FAM(\preceq_T) \equiv \forall K \, \exists M \, \forall t_0, \dots, t_M \in T$$
$$\forall i \le M \left(t_i^{\mathbb{N}} \le K+i \right)$$
$$\to \exists i < j \le M \left(t_i \preceq_T t_j \right)$$

$\preceq_T \,:=\, \leq$

$FAM(\leq)$ is true: K given $M := K$.

$t_0, \dots, t_M \in T$ $\left(t_i^{\mathbb{N}} \le K+i \;\; \forall i \le M \right)$

$t_0^{\mathbb{N}} \le K$ some $t_i^{\mathbb{N}} \ge K$ V. o.w. pigeonhole p.



FAM($\trianglelefteq$) is natural

Thm: 1) FAM($\trianglelefteq$) is true.

2) $PA \nvdash$ FAM($\trianglelefteq$) $\quad ATR_0 \nvdash$ FAM($\trianglelefteq$)

FAM($\trianglelefteq$) remains strong when terms are considered mod comm.

$t+u \equiv u+t \quad t\cdot u \equiv u\cdot t$

Extend def of $\trianglelefteq$ by $\trianglelefteq'$

$t \trianglelefteq' u'$ and $u \trianglelefteq t' \Rightarrow u+t \trianglelefteq' u'+t'$
$\quad u\cdot t \trianglelefteq' u'\cdot t'$

Thm: FAM($\trianglelefteq'$) is true $\quad PA \nvdash$ FAM($\trianglelefteq'$)

$ATR_0 \nvdash$ FAM($\trianglelefteq'$).

If

$s+(t+u) \equiv (s+t)+u$

$s\cdot(t\cdot u) \equiv (s\cdot t)\cdot u$

then FAM remains true and strong.

$\mathrm{FAM}(f) \equiv \forall K \exists M \; \forall t_0, \dots, t_M \in T$

$\forall i \leq M \; t_i^W \leq K + f(i) \Longrightarrow \exists i < j \leq M \, (t_i \trianglelefteq t_j)$

$f_r(i) = r \cdot \log_2(i)$

<u>Thm</u>: Let $c = 4.076561785276046\ldots$

and $\rho := \frac{1}{\log_2(c)}$

1) $r \leq c \Rightarrow \mathrm{PRA} \vdash \mathrm{FAM}(f_r)$

2) $r > c \Rightarrow \mathrm{PA} \nvdash \mathrm{FAM}(f_r)$

What about terms with $+$ only

$\rightarrow$ independence result for PA

What about $+, \cdot, x^y$, Ack, $(F_\alpha)_{\alpha<\varepsilon_0}$

Results will emerge naturally

Avoiding $\trianglelefteq$

$0 \in EXP$

$a, b \in EXP \Rightarrow x^{a} + b \in EXP$

$a \in EXP \quad k \in \mathbb{N} \quad a(k) = a[x := k]^{\mathbb{N}}$

$2_k(\ell) = \left.2^{\cdot^{\cdot^{2^{\ell}}}}\right\} k$

$$SWO \equiv \forall K\, \exists M\, \forall a_0, \dots, a_M \in EXP$$
$$\forall i \leq M\, (a_i(2) \leq 2_K(i))$$
$$\longrightarrow \exists i < M \;\; a_i(2_K(M)) \leq a_{i+1}(2_K(M))$$

SWO is natural.

Thm: 1) SWO is true

2) $PA \nvdash SWO$

Integrating Ackermann function $\longrightarrow$ ATR_0 independence

Integrating $(F_\alpha)_{\alpha < \varepsilon_0}$ $\longrightarrow$ $(\Pi^1_1\text{-}CA)^-_0 \equiv ID_1$

Thank you!